Extending the Reach of the Point-to-Set Principle

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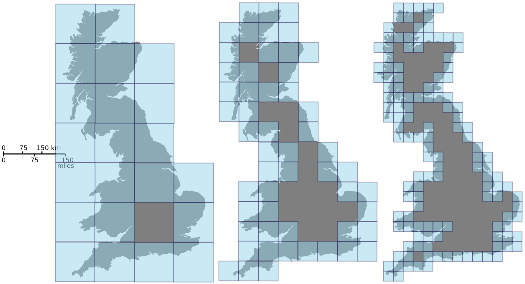
Two Approaches to Fractal Dimension

The point-to-set principle is a bridge between these two that lets us apply the theory of computation to geometric measure theory.

1. Measure-theoretic ("classical"): How strongly does granularity affect the size of covers?



2. Algorithmic information-theoretic: How strongly does precision affect the length of programs?

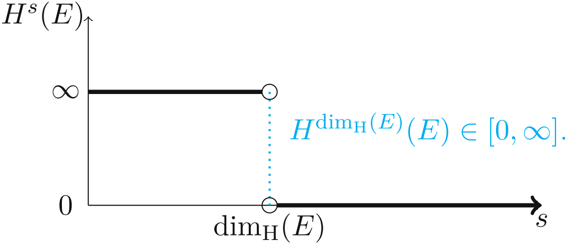
Image credit: Alexis Monnerot-Dumaine

Fractal Dimension: Measure Theoretic Approach

The s-dimensional Hausdorff outer measure of $E \subseteq \mathbb{R}^n$ is

$$H^s(E) = \lim_{\delta \to 0} \inf \left\{ \sum_{i \in \mathbb{N}} \operatorname{diam}(U_i)^s : E \subseteq \bigcup_{i \in \mathbb{N}} U_i \text{ and } \forall i \ \operatorname{diam}(U_i) < \delta \right\}.$$

The Hausdorff dimension of E is $\dim_{\mathrm{H}}(E) = \inf\{s : H^s(E) = 0\}$.



The packing dimension $\dim_{\mathrm{P}}$ is a dual version that is defined similarly, but the U_i form a packing instead of a cover.

Fractal Dimension: Algorithmic Information Approach

The Kolmogorov complexity of a bit string $\sigma \in \{0,1\}^*$ is the length of the shortest binary program that outputs σ:

$$K(\sigma) = \min\left\{|\pi| : U(\pi) = \sigma\right\},$$

where U is a universal Turing machine.

- $K(\sigma) =$ amount of algorithmic information in σ.
- $K(\sigma) \leq |\sigma| + o(|\sigma|)$.
- Extends naturally to other discrete domains, like $\mathbb{Q}^n$.

The Kolmogorov complexity of $x \in \mathbb{R}^n$ at precision r is

$$K_r(x) = \min\{K(p) : p \in \mathbb{Q}^n \cap B_{2^{-r}}(x)\}.$$

Algorithmic Dimension

J. Lutz and Mayordomo: The dimension of an individual point $x \in \mathbb{R}^n$ is

$$\dim(x) = \liminf_{r \to \infty} \frac{K_r(x)}{r}\,.$$

Athreya, Hitchcock, J. Lutz, and Mayordomo: The strong dimension of an individual point $x \in \mathbb{R}^n$ is

$$\mathrm{Dim}(x) = \limsup_{r \to \infty} \frac{K_r(x)}{r}\,.$$

Examples:

- x computable $\implies \mathrm{Dim}(x) = 0 \implies \dim(x) = 0$.
- x random $\implies \dim(x) = n \implies \mathrm{Dim}(x) = n$.

The converses do not hold.

Relative Dimension

The Kolmogorov complexity of a bitstring $\sigma \in \{0,1\}^*$ relative to an oracle $w \in \{0,1\}^\infty$ is

$$K^w(\sigma) = \min\left\{|\pi| : U^w(\pi) = \sigma\right\}.$$

The dimension of a point $x \in \mathbb{R}^n$ relative to oracle w is

$$\dim^w(x) = \liminf_{r\to\infty} \frac{K^w_r(x)}{r}.$$

- The oracle can encode a point in $\mathbb{R}^n$.
- For all $x \in \mathbb{R}^n$, $\dim^x(x) = 0$.

Point-to-Set Principle (J. Lutz and N. Lutz 2018)

For every set $E \subseteq \mathbb{R}^n$,

$$\dim_{\mathrm{H}}(E) = \min_{w} \sup_{x \in E} \dim^{w}(x)\,.$$

classical Hausdorff dimension

dimensions of individual points

$$\dim_{\mathrm{P}}(E) = \min_{w} \sup_{x \in E} \mathrm{Dim}^{w}(x)\,.$$

classical packing dimension

strong dimensions of individual points

Some Applications

- Intersections (N. Lutz 2017): For all ~~analytic~~ sets $E, F \subseteq \mathbb{R}^n$ and almost every $z \in \mathbb{R}^n$,

$$\dim_{\mathrm{H}}(E \cap (F + z)) \leq \max\{0, \dim_{\mathrm{H}}(E \times F) - n\}.$$

- Projections (N. Lutz and Stull 2018): For every ~~analytic~~ set $E \subseteq \mathbb{R}^n$ and almost every $z \in \mathbb{R}^n$,

$$\dim_{\mathrm{P}}(\operatorname{proj}_z E) \geq \min\{1, \dim_{\mathrm{H}}(E)\}.$$

- Products (N. Lutz 2020): For every ~~analytic~~ set $E \subseteq \mathbb{R}^n$,

$$\dim_{\mathrm{P}}(E) = \sup_{F \subseteq \mathbb{R}^n} \left(\dim_{\mathrm{H}}(E \times F) - \dim_{\mathrm{H}}(F) \right).$$

Extending the Reach

Prior work in algorithmic fractal dimensions has mostly been in Euclidean spaces or sequence spaces over finite alphabets, but geometric measure theory explores many other domains. In order to apply the point-to-set principle to more general settings:

- We extend it to arbitrary separable metric spaces — metric spaces that have a countable dense set.
- We extend it to gauged dimensions, which are particularly useful in infinite-dimensional spaces.

Gauged Classical Dimensions

Standard classical fractal dimensions are some variation on:

The smallest value s such that

$$\sum_i \mathrm{diam}(U_i)^s$$

can be small when the cover elements are required to be small.

Given any family $\varphi = \{\varphi_s\}_{s>0}$ of continuous, non-decreasing functions, where $\varphi_s(\delta) = o(\varphi_t(\delta))$ as $\delta \to 0^+$ for all $s > t$, dimensions can be defined with respect to φ as:

The smallest value s such that

$$\sum_i \varphi_s(\mathrm{diam}(U_i))$$

can be small when the cover elements are required to be small.

Gauged Algorithmic Dimensions in Separable Metric Spaces

Let (X, ρ) be a separable metric space with countable dense set D.

$$K_\delta(x) := \min\{K(d) : d \in D \text{ and } \rho(x, d) < \delta\}$$

The φ-gauged algorithmic dimension of a point x is

$$\dim^\varphi(x) = \inf\left\{s : \liminf_{\delta \to 0^+} 2^{K_\delta(x)} \varphi_s(\delta) = 0\right\},$$

and the φ-gauged strong algorithmic dimension of a point x is

$$\mathrm{Dim}^\varphi(x) = \inf\left\{s : \limsup_{\delta \to 0^+} 2^{K_\delta(x)} \varphi_s(\delta) = 0\right\}.$$

General Point-to-Set Principle (LLM 2022)

Let (X, ρ) be a separable metric space, φ a gauge family (with some modest conditions), and $E \subseteq X$. Then,

$$\dim_{\mathrm{H}}^{\varphi}(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \dim^{\varphi, A}(x)$$

and

$$\dim_{\mathrm{P}}^{\varphi}(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \mathrm{Dim}^{\varphi, A}(x).$$

Application: The Hyperspace of Compact Sets

Given any metric space (X, ρ), the space $\mathcal{K}(X)$ of all non-empty compact sets of X is a metric space under the Hausdorff metric

$$\rho_{\mathrm{H}}(E, F) = \max \left\{ \sup_{x \in E} \inf_{y \in F} \rho(x, y), \ \sup_{y \in F} \inf_{x \in E} \rho(x, y) \right\}.$$

To meaningfully quantify the dimension of sets in the hyperspace, we will need a very aggressive gauge family!

Measuring the Dimension of the Hyperspace

Let ψ be the gauge family given by $\psi_s(\delta) = 2^{1/\delta^s}$.

Theorem (McClure 1996): If E is self-similar, then

$$\dim_{\mathrm{H}}^{\psi}(\mathcal{K}(E)) = \dim_{\mathrm{H}}(E)\,.$$

More generally, given any gauge function φ_s, define

$$\widetilde{\varphi}_s(\delta) = 2^{-1/\varphi_s(\delta)}\,,$$

and extend this to define $\widetilde{\varphi}$ ("jump phi") for any gauge family φ.

E.g., For the "standard" gauge $\theta_s(\delta) = \delta^{-s}$,

$$\widetilde{\theta}_s(\delta) = 2^{-1/-\delta^s} = \psi_s(\delta)$$

Minkowski Dimensions

- More "primitive" fractal dimension notions that require the cover elements to all be the same size.
- Equivalent to Hausdorff and packing dimensions for "well-behaved" sets, including self-similar sets.
- Not countably stable.
- Do not readily admit a point-to-set principle.

Image source: Kenneth J. Falconer. *Fractal Geometry: Mathematical Foundations and Applications.* Wiley, third edition, 2014.

Hyperspace Minkowski Dimension Theorem (LLM 2022)

Let (X, ρ) be a separable metric space, φ a gauge family, and $E \subseteq X$. Then the lower and upper φ-gauged Minkowski dimensions of E coincide with the lower and upper $\widetilde{\varphi}$-gauged Minkowski dimensions of $\mathcal{K}(E)$:

$$\underline{\dim}_{\mathcal{M}}^{\varphi}(E) = \underline{\dim}_{\mathcal{M}}^{\widetilde{\varphi}}(\mathcal{K}(E))$$

and

$$\overline{\dim}_{\mathcal{M}}^{\varphi}(E) = \overline{\dim}_{\mathcal{M}}^{\widetilde{\varphi}}(\mathcal{K}(E)).$$

McClure's theorem follows as a corollary.

Hyperspace Packing Dimension Theorems (LLM 2022)

Let (X, ρ) be a separable metric space, φ a gauge family (with some modest conditions), and $E \subseteq X$.

- If E is compact, then $\dim_{\mathrm{P}}^{\widetilde{\varphi}}(\mathcal{K}(E)) = \overline{\dim}_{\mathcal{M}}^{\widetilde{\varphi}}(\mathcal{K}(E))$.
- If E is analytic, then $\dim_{\mathrm{P}}^{\widetilde{\varphi}}(\mathcal{K}(E)) \geq \dim_{\mathrm{P}}^{\varphi}(E)$.
  - This inequality can be strict.
  - There also exist sets such that $\dim_{\mathrm{H}}^{\widetilde{\varphi}}(\mathcal{K}(E)) > \dim_{\mathrm{H}}^{\varphi}(E)$.

Conclusions

These results exhibit and amplify the power of the theory of computing to make unexpected contributions to other areas of the mathematical sciences. We hope and expect to see more such results in the near future!

Three specific open questions:

- Does E need to be analytic in the hyperspace packing dimension theorem?
- Is there a corresponding hyperspace Hausdorff dimension theorem?
- How can we characterize algorithmic dimensions in separable metric spaces in terms of martingales or more general gales?