Constructing Wadge classes and Describing Wadge quasi-orders.

Raphaël Carroy

Università di Torino

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Continuous reducibility, also known as Wadge reducibility

Let X and Y be two topological spaces.

Definition

For $A \subseteq X$ and $B \subseteq Y$, we say that A **continuously reduces** to B if there is a continuous function $f : X \to Y$ such that $f^{-1}(B) = A$.
We also say that A **Wadge reduces** to B, and we write it $A \leq_W^{X,Y} B$.

Note that this is reflexive and transitive, it is a **quasi-order**.
When $X = Y$ we write $\leq_W^X$ instead of $\leq_W^{X,X}$, and $\equiv_W^X$ stands for the associated equivalence relation, so $A \equiv_W^X B$ iff $A \leq_W^X B$ and $B \leq_W^X A$.
Note that $\leq_W^X$ induces a partial order on $\mathcal{P}(X)/\equiv_W^X$.

Wadge's Lemma

A motivation for looking at continuous reducibility originally was:

Question

Let ξ be a countable ordinal, and assume that A is a $\mathbf{\Sigma}^0_\xi \setminus \mathbf{\Delta}^0_\xi$ subset of a Polish space X.
Is A $\mathbf{\Sigma}^0_\xi$**-complete**, that is: does it reduce continuously all $\mathbf{\Sigma}^0_\xi$ subsets of X?

Continuous reducibility became associated to Wadge's name because of:

Lemma (Wadge's Lemma)

Assume that A and B are Borel subsets of the Baire space $\mathbb{N}^\mathbb{N}$. Then we have $A \leq_W^{\mathbb{N}^\mathbb{N}} B$ or $B \leq_W^{\mathbb{N}^\mathbb{N}} \mathbb{N}^\mathbb{N} \setminus A := A^c$.

This solves positively the problem for $X = \mathbb{N}^\mathbb{N}$.
We call this the **semi-linear ordering** principle, or **SLO** for $\leq_W^{\mathbb{N}^\mathbb{N}}$.

Consequences of Wadge's lemma

Pointclass versus Wadge pointclass:

- A **pointclass** is only supposed to be closed under continuous reducibility, $\mathbf{\Sigma}^0_\xi(\mathbb{N}^\mathbb{N})$ admits a complete set, we call it a **Wadge pointclass**.
- for $A \subseteq X$, $A\downarrow_X$ is the Wadge pointclass generated by A, and
- $[A]_X$ is the **Wadge degree** of A in X (*i.e.* all $B \equiv^X_W A$).
- By considering Wadge pointclasses, we can relate pointclasses to complete sets.

$\leq^{\mathbb{N}^\mathbb{N}}_W$ on Borel sets has antichains (for comparability) of size at most 2:

- If $A \equiv^{\mathbb{N}^\mathbb{N}}_W A^c$ then all Borel sets of $\mathbb{N}^\mathbb{N}$ are comparable with A.
- Otherwise, if B Borel is incomparable with A then $B \equiv^{\mathbb{N}^\mathbb{N}}_W A^c$.
- In the first case, we say that A is **self-dual**, in the second that it is **non-self-dual**, or **NSD**.

Martin-Monk's Theorem: put ordinals in the mix!

Theorem (Martin-Monk)

The quasi-order $\leq_W^{\mathbb{N}^\mathbb{N}}$ is well-founded on Borel subsets of $\mathbb{N}^\mathbb{N}$.

So, look at **coarse degrees**: $[A]_{\mathbb{N}^\mathbb{N}} \cup [A^c]_{\mathbb{N}^\mathbb{N}}$.
Since antichains are witnesses by NSD sets, $\leq_W^{\mathbb{N}^\mathbb{N}}$ induces a well-order on coarse degrees.
There is thus an interplay between classes of sets (coarse degrees), sets, and ordinals.

Many possible directions of research involving this interplay have been explored, we focus on

- Which Wadge pointclasses can we construct from other ones strictly below in the Wadge order? How?
- Can we describe the partial order on Wadge degrees up to isomorphism?

Framing the question: spaces

On which class of spaces can we work?
Using universality of $\mathbb{N}^{\mathbb{N}}$ among Polish 0-dimensional spaces,

Theorem (Wadge, Martin-Monk)

Assume Z is a Polish 0-dimensional space. Then $\leq_W^Z$ satisfies SLO and is well-founded on Borel subsets of Z.

What about other spaces?

Theorem

- *The Wadge quasi-order has infinite antichains and is ill-founded:*
  - *(Schlicht-Ikegami-Tanaka) On the real line $\mathbb{R}$,*
  - *(Duparc-Vuilleumier) On the Scott domain $\mathcal{P}(\omega)$.*
- *(Schlicht) $\leq_W^Z$ has infinite antichains for any Z Polish non 0-dimensional.*

So we **focus on Polish 0-dimensional spaces**.

Framing the question: classes of subsets

Wadge's Lemma and the Martin-Monk Theorem relies on determinacy of the Wadge game.

But if $\mathbf{\Lambda}$ is a pointclass in $\mathbb{N}^{\mathbb{N}}$ among:

- the pointclass of Borel sets,
- one the projective classes $\mathbf{\Sigma}^1_n$ or $\mathbf{\Pi}^1_n$ ($n \in \mathbb{N}$),
- The projective sets $\bigcup_{n\in\mathbb{N}} \mathbf{\Sigma}^1_n$,
- all subsets of $\mathbb{N}^{\mathbb{N}}$,

then determinacy of $\mathbf{\Lambda}$ ensures the determinacy of the Wadge game for pairs of sets in $\mathbf{\Lambda}$.

In the second case, this is using recent results of Müller, Schindler, and Woodin.

Theorem (Wadge, Martin-Monk)

Let $\mathbf{\Lambda}$ be one of the above classes. Assume determinacy of $\mathbf{\Lambda}$. Then $\leq_W^{\mathbb{N}^{\mathbb{N}}}$ satisfies SLO and is well-founded on $\mathbf{\Lambda}$.

For simplicity results are stated in ZF+DC or under AD.

On the first question: self-dual sets

So the first question becomes:
Under AD, which Wadge pointclasses of a Polish 0-dimensional space can we construct from other ones strictly below in the Wadge order? How?
This part is a joint work with Andrea Medini and Sandra Müller.

For $A, B \subseteq Z$ note $A <_W^Z B$ when $A \leq_W^Z B$ but $B \not\leq_W^Z A$.
There is a direct answer for self-dual sets.

Theorem (Wadge, van Wesep)

Let Z be Polish 0-dimensional, $A \subseteq Z$ be not clopen, and assume AD.
Then $A \subseteq Z$ is self-dual iff there is a clopen partition $\mathcal{U}$ of Z such that for all $U \in \mathcal{U}$ we have $A \cap U <_W^Z A$.

So let's focus on NSD sets and NSD Wadge pointclasses.
And first, Louveau's analysis in the Baire space.

An intuition: Louveau's trick.

For a pointclass $\mathbf{\Gamma}$ in Z, note $\check{\mathbf{\Gamma}} = \{A^c \mid A \in \mathbf{\Gamma}\}$, and $\mathbf{\Delta}(\mathbf{\Gamma}) = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$.

- Suppose that $A \subseteq \mathbb{N}^{\mathbb{N}}$ is in $\mathbf{\Delta}(\mathrm{D}_2(\mathbf{\Sigma}^0_2))$.
- Then $A = B \cap C = B' \cup C'$,with B, B' in $\mathbf{\Sigma}^0_2$ and C, C' in $\mathbf{\Pi}^0_2$.
- There is a $\mathbf{\Delta}^0_2$ set D separating C' from B^c.
- Then $A = (C \cap D) \cup (B' \setminus D)$.

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So, what did we do?

- D can be of any complexity in $\mathbf{\Delta}^0_2$, so any of the ω_1 levels of the difference hierarchy below $\mathbf{\Sigma}^0_2$.
- A has a $\mathbf{\Pi}^0_2$ trace in D and a $\mathbf{\Sigma}^0_2$ trace outside, but anyway strictly simpler.

This explains how to solve the question for Wadge pointclasses in $\mathbf{\Delta}(\mathrm{D}_2(\mathbf{\Sigma}^0_2))$. The same trick works more generally to construct $\mathbf{\Delta}(\mathrm{D}_2(\mathbf{\Gamma}))$, for a class $\mathbf{\Gamma}$ with the separation property.

What happens further up: $\boldsymbol{\Delta}^0_\omega$

How can we build sets and pointclasses describing $\boldsymbol{\Delta}^0_\omega \setminus \bigcup_n \boldsymbol{\Sigma}^0_n$?

- First, directly inspired by the analysis of self-dual sets. Take a clopen partition $(C_n)_{n\in\mathbb{N}}$ of $\mathbb{N}^\mathbb{N}$, and take $A_n \subseteq C_n$ that is in $\boldsymbol{\Sigma}^0_n$, and look at $A = \bigcup_n A_n = \bigsqcup_n A_n$.
- Second idea: we could fix a closed set $F \subseteq \mathbb{N}^\mathbb{N}$, and do the same construction in the complement of F.
- Third idea: same as before, adding a $\bigcup_n \boldsymbol{\Sigma}^0_n$-subset of F.

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Let's generalize to depth $\eta < \omega_1$

- For all $\xi < \eta$ take (pairwise disjoint open) sets $(V_{\xi,n})_n$
- Sets $(A_{\xi,n})_{\xi<\eta,n\in\mathbb{N}}$ and A^* in $\bigcup_n \mathbf{\Sigma}^0_n$, and look at
- $A = \left(\bigcup_{\xi,m} A_{\xi,m} \cap (V_{\xi,m} \setminus \bigcup_{\xi'<\xi,n} V_{\xi',n})\right) \cup A^* \setminus (\bigcup_{\xi,n} V_{\xi,n})$

We call this operation a **separated difference**, and write $A = \mathrm{SD}_\eta((V_{\xi,n}),(A_{\xi,n}),A^*)$.

Separated differences and open vs full parametrizations

If $\mathbf{\Delta}$ and $\mathbf{\Gamma}^*$ are pointclasses, $\mathrm{SD}_\eta(\mathbf{\Delta}, \mathbf{\Gamma}^*)$ is the class of all possible separated differences $A = \mathrm{SD}_\eta((V_{\xi,n}), (A_{\xi,n}), A^*)$ with $A_{\xi,n} \in \mathbf{\Delta}$ and $A^* \in \mathbf{\Gamma}^*$.
Observe that we did not parametrize the complexity of the sets $V_{\xi,n}$, they are always open.
Let's call this an **open parametrization** (of SD).

So, why not parametrizing also the class of the sets $V_{\xi,n}$?
Louveau's original version was **fully parametrized**...
But it uses strongly that the classes $\mathbf{\Sigma}^0_\alpha$ are cofinal in Borel sets!
This is something that we cannot afford in general, so we need a version that uses only open parametrization.

How to do without full parametrization on an example:

- Let $\mathbf{\Gamma}$ be the unions of a $\mathbf{\Sigma}^0_3$ and a $\mathbf{\Pi}^0_3$ separated by a $\mathbf{\Sigma}^0_2$ set.
- We have $\mathbf{\Gamma}_0$, the unions of $\mathbf{\Sigma}^0_2$ and $\mathbf{\Pi}^0_2$ separated by $\mathbf{\Sigma}^0_1$.
- A set in $\mathbf{\Gamma}$ is the preimage of one in $\mathbf{\Gamma}_0$ by a Baire 1 function.

Expansions and Louveau's theorem

Expansions

Given a pointclass $\mathbf{\Gamma}$ and a countable ordinal α, the α-expansion $\mathbf{\Gamma}^{(\alpha)}$ is the class of all preimages of elements of $\mathbf{\Gamma}$ by $\mathbf{\Sigma}^0_{1+\alpha}$-measurable functions.

Theorem (Louveau)

The collection of all Wadge pointclasses of Borel sets in $\mathbb{N}^{\mathbb{N}}$ is equal to Lo, *where* Lo *is the smallest pointclass satisfying*

- $\{\mathbb{N}^{\mathbb{N}}\}$ *and* $\{\emptyset\}$ *are in* Lo,
- Lo *is stable by α-expansions for all $\alpha < \omega_1$,*
- Lo *is stable by separated differences.*

Let us see now how the proof goes, and how we can generalize it.

A sketch of the proof of Louveau's theorem

Let $\mathbf{\Gamma}$ be a NSD Wadge pointclass of Borel sets in $\mathbb{N}^{\mathbb{N}}$.
Call $\mathrm{PU}_\alpha(\mathbf{\Gamma})$ the class of all sets of the form $\bigcup A_n \cap B_n$, where $A_n \in \mathbf{\Gamma}$, and $(B_n)_n$ is a $\mathbf{\Delta}^0_{1+\alpha}$ partition.

Definition (Louveau - Saint Raymond)

The **level** of a pointclass $\mathbf{\Gamma}$ is $\ell(\mathbf{\Gamma}) = \sup\{\alpha < \omega_1 \mid \mathbf{\Gamma} = \mathrm{PU}_\alpha(\mathbf{\Gamma})\}$.

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Using Louveau's theorem with fully parametrized SD, we have:

Proposition (Louveau - Saint Raymond)

If $\boldsymbol{\Gamma} \neq \{\mathbb{N}^{\mathbb{N}}\}, \{\emptyset\}$, then $\ell(\boldsymbol{\Gamma})$ is a maximum.

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Theorem (Expansion Theorem & Analysis of level 0 classes)

- *(Saint Raymond) If $\ell(\mathbf{\Gamma}) = \alpha$ then $\mathbf{\Gamma} = \mathbf{\Gamma}_0^{(\alpha)}$, with $\ell(\mathbf{\Gamma}_0) = 0$.*
- *(Louveau) If $\ell(\mathbf{\Gamma}) = 0$ then there is $\eta < \omega_1$, $\mathbf{\Delta} \subsetneq \mathbf{\Gamma}$, and $\mathbf{\Gamma}^* \subseteq \mathbf{\Delta}$ satisfying $\mathbf{\Gamma} = \mathrm{SD}_\eta(\mathbf{\Delta}, \mathbf{\Gamma}^*)$.*

Generalizing naively won't work directly...

Assume AD, and let $\mathbf{\Gamma}$ be a NSD Wadge pointclass in $\mathbb{N}^{\mathbb{N}}$.
Call $\mathrm{PU}_\alpha(\mathbf{\Gamma})$ the class of all sets of the form $\bigcup A_n \cap B_n$, where $A_n \in \mathbf{\Gamma}$, and $(B_n)_n$ is a $\mathbf{\Delta}^0_{1+\alpha}$ partition.

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Using Louveau's theorem with fully parametrized SD, we have:

Here is what we need!

If $\ell(\mathbf{\Gamma}) < \omega_1$, then $\ell(\mathbf{\Gamma})$ is a maximum.

Theorem

- *(Saint Raymond) If $\ell(\mathbf{\Gamma}) = \alpha$ then $\mathbf{\Gamma} = \mathbf{\Gamma}_0^{(\alpha)}$, with $\ell(\mathbf{\Gamma}_0) = 0$.*
- *(Louveau) If $\ell(\mathbf{\Gamma}) = 0$ then there is $\eta < \omega_1$, $\mathbf{\Delta} \subsetneq \mathbf{\Gamma}$, and $\mathbf{\Gamma}^* \subseteq \mathbf{\Delta}$ satisfying $\mathbf{\Gamma} = \mathrm{SD}_\eta(\mathbf{\Delta}, \mathbf{\Gamma}^*)$.*

Relativization

We found a proof of what we need, but it requires to be able to talk about $\boldsymbol{\Gamma}(Z)$, for $\boldsymbol{\Gamma}$ a NSD Wadge pointclass in $\mathbb{N}^{\mathbb{N}}$, and Z Borel 0-dimensional.

Definition (Louveau - Saint Raymond)

$\boldsymbol{\Gamma}(Z) = \{A \subseteq Z \mid g^{-1}(A) \in \boldsymbol{\Gamma}$ for every continuous $g : \mathbb{N}^{\mathbb{N}} \to Z\}$.

Lemma (Louveau - Saint Raymond)

Assume AD, then $A \in \boldsymbol{\Gamma}(Z)$ iff for all embeddings $j : Z \to \mathbb{N}^{\mathbb{N}}$ there is $B \in \boldsymbol{\Gamma}$ with $A = j^{-1}(B)$.

And the crucial

Lemma (Relativization Lemma)

Assume AD. If W is Borel subspace of Z then $A \in \boldsymbol{\Gamma}(W)$ if and only if there is $\tilde{A} \in \boldsymbol{\Gamma}(Z)$ such that $A = \tilde{A} \cap W$.

Consequences

Call $\mathrm{NSD}(Z)$ the set of NSD Wadge pointclasses of a 0-dimensional space Z. Using the relativization lemma,

Theorem (Carroy - Medini - Müller)

Assume AD. Let Z and W be Polish 0-dimensional spaces.

- *If $\mathbf{\Lambda} \in \mathrm{NSD}(Z)$, then there is a unique $\mathbf{\Gamma} \in \mathrm{NSD}(\mathbb{N}^{\mathbb{N}})$ such that $\mathbf{\Gamma}(Z) = \mathbf{\Lambda}$.*
- *If Z and W are uncountable, and $\mathbf{\Gamma}, \mathbf{\Lambda}$ are in $\mathrm{NSD}(\mathbb{N}^{\mathbb{N}})$ then $\mathbf{\Gamma}(Z) \subseteq \mathbf{\Lambda}(Z)$ iff $\mathbf{\Gamma}(W) \subseteq \mathbf{\Lambda}(W)$.*
- *If Z is uncountable then $\mathrm{NSD}(Z) = \{\mathbf{\Gamma}(Z) \mid \mathbf{\Gamma} \in \mathrm{NSD}(\mathbb{N}^{\mathbb{N}})\}$.*

The generalized Louveau analysis

Let Z be a Polish 0-dimensional space, and $\mathbf{\Gamma} \in \mathsf{NSD}(Z)$.

Proposition (Andretta-Martin)

If $\ell(\mathbf{\Gamma}) = \omega_1$, then $\mathbf{\Gamma}$ is closed under preimages by Borel functions.

Let $\mathsf{Lo}(Z)$ is the smallest collection satisfying

- If $\ell(\mathbf{\Gamma}) = \omega_1$ then $\mathbf{\Gamma} \in \mathsf{Lo}(Z)$,
- $\mathsf{Lo}(Z)$ is stable by α-expansions for all $\alpha < \omega_1$,
- if $\mathsf{Lo}(Z)$ is stable by separated differences.

Theorem (Carroy - Medini - Müller)

Assume AD. $\mathsf{Lo}(Z)$ *is the collection of all Wadge pointclasses in Z.*

An first application: Hausdorff operations

We also obtain another proof of a theorem of van Wesep concerning Hausdorff operations.

Given $D \subset 2^\omega$ and a sequence of sets $\vec{A} = A_0, A_1, \cdots$ in a Polish 0-dimensional space X, define a set $\mathcal{H}_D(\vec{A})$ as follows:

$$x \in \mathcal{H}_D(\vec{A}) \leftrightarrow \{i \in \omega \mid x \in A_i\} \in D$$

We call $\mathcal{H}_D$ a **Hausdorff operation**, define $\Gamma_D(X)$ as all subsets of X that are the result of applying $\mathcal{H}_D$ on open sets of X.

Theorem (Wadge, van Wesep)

(AD) Every non-self-dual set is the result of a Hausdorff operation on open sets.

A second application: Homogeneity vs strong homogeneity

- A space X is **homogeneous** if for all x and y in X there is a homeomorphism $f : X \to X$ satisfying $h(x) = y$.
- A 0-dimensional space X is **strongly homogeneous** if all clopen subsets U, V of X are homeomorphic.

Theorem (Carroy - Medini - Müller)

(AD) A non-locally-compact subset of $2^{\mathbb{N}}$ is homogeneous if and only if it is strongly homogeneous.

A third application: Describing Wadge partial orders

This part of the talk is a joint work with Luca Motto Ros and Salvatore Scamperti.

Assuming AD, let $\Theta(Z)$ (resp. Θ) be the length of the Wadge hierarchy on Z (resp. $\mathbb{N}^{\mathbb{N}}$). To describe up to isomorphism the Wadge partial order on a Polish 0-dimensional space Z, it is enough to know if for $\alpha < \Theta$ the αth coarse Wadge class is self-dual or not.

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Note $\alpha \in \mathsf{SD}(Z)$ in the former case, $\alpha \in \mathsf{NSD}(Z)$ in the latter.

Here's what we know so far:

- (Wadge, Martin-Monk) If Z is Polish 0-dimensional, then $\leq_W^Z$ satisfies SLO and is well-founded
- If Z is Polish 0-dimensional and $\alpha < \Theta(Z)$ has uncountable cofinality then $\alpha \in \mathsf{NSD}(Z)$
- $\leq_W^{\mathbb{N}^\mathbb{N}}$ satisfies the **alternating duality** property: for all $\alpha < \Theta$, $\alpha \in \mathsf{SD}(\mathbb{N}^\mathbb{N})$ iff $\alpha + 1 \in \mathsf{NSD}(\mathbb{N}^\mathbb{N})$.
- For all $\alpha < \Theta$ of countable cofinality we have $\alpha \in \mathsf{SD}(\mathbb{N}^\mathbb{N})$.

Alternating duality and non-compact perfect kernel

Theorem (Carroy - Motto Ros - Scamperti)

Assume AD, and let Z be Polish 0-dimensional. Then $\leq_W^Z$ satisfies the alternating duality.

So, in order to describe $\leq_W^Z$ up to isomorphism of partial orders, it is enough to determine when $\alpha \in \mathsf{SD}(Z)$ for $\alpha < \Theta(Z)$ of countable cofinality.

The **perfect kernel** of a space Z is the set of its accumulation points.

Theorem (Carroy - Motto Ros - Scamperti)

Assume AD, and let Z be Polish 0-dimensional with a non-compact perfect kernel. Then for all $\alpha < \Theta$ of countable cofinality we have $\alpha \in \mathsf{SD}(Z)$.

So in all these cases the Wadge partial order is exactly the same as on the Baire space.

Spaces with compact perfect kernel

What happens when the perfect kernel of a Polish 0-dimensional Z is compact?
Then define an ordinal $\mathrm{Comp}(Z) < \omega_1$ as the least ordinal α such that the αth Cantor-Bendixson derivative is compact.

Theorem (Carroy - Motto Ros - Scamperti)

Assume AD, and let Z be Polish 0-dimensional with a compact perfect kernel. Let $\alpha < \Theta(Z)$ be of countable cofinality.
If $\alpha < \mathrm{Comp}(Z)$ then $\alpha \in \mathrm{SD}(Z)$, and
if $\alpha > \mathrm{Comp}(Z)$ then $\alpha \in \mathrm{NSD}(Z)$.

What about $\alpha = \mathrm{Comp}(Z)$ when it is of countable cofinality? We can't say!
There are two uncountable Polish 0-dimensional spaces with a compact perfect kernel Z and W such that $\alpha = \mathrm{Comp}(Z) = \mathrm{Comp}(W)$ has countable cofinality, $\alpha \in \mathrm{SD}(Z)$ and $\alpha \in \mathrm{NSD}(W)$.

Open questions

With Motto Ros and Scamperti we have started investigating:

Question

(AD) Is 1-1 continuous reducibility a well-quasi-order on $\mathcal{P}(\mathbb{N}^{\mathbb{N}})$?

Question

What could be an $\omega_1$ary operation on Wadge pointclasses? What does it even mean?

More precisely,

Question

Can we find a set in some Z that is the $\leq_W^Z$-supremum of $\mathrm{D}_\alpha(\boldsymbol{\Sigma}^1_1)$ for all $\alpha < \omega_1$?

Thank you!