Ramsey and Hypersmoothness

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Let X, Y be Polish spaces, assume that E is an equivalence relation on X and F is an equivalence relation on Y. A Borel *reduction* of E to F is a Borel map $f : X \to Y$ with

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Write $E \leq_B F$ if such a reduction exists.

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Write $E \leq_B F$ if such a reduction exists.

An equivalence relation E is called *finite (countable)* if each of its classes is finite (countable).

Hyperfiniteness

A countable Borel equivalence relation (CBER) is *hyperfinite*, if there are finite Borel equivalence relations $F_0 \subseteq F_1 \subseteq F_2 \ldots$ such that $E = \bigcup_{n \in \mathbb{N}} F_n$.

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Proposition. A CBER E is hyperfinite if and only if $E \leq_B \mathrm{E}_0$, where $x \mathrm{E}_0 y$ iff $\{n : x(n) \neq y(n)\}$ is finite.

Non-hyperfiniteness

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Then E_G is not hyperfinite. Here $\chi^{el}_{\mu}(G) > 3$ means that if B is Borel with $\mu(B) = 1$ and $B = \bigcup_{i \in 3} B_i$ then for some i we have $B_i^2 \cap H_j \neq \emptyset$ for each j.

A sketch

Let $E_G = \bigcup_{n \in \mathbb{N}} F_n$ with F_n finite.

Define $f_n(x)(i) = \frac{|D_{i,x} \cap [x]_{F_n}|}{|[x]_{F_n}|}$, where $D_{i,x}$ is the i colored direction in G from x.

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Mazur's lemma: Let μ be a Borel probability measure on X. For any sequence of Borel functions $f_n : X \to [0,1]$ there is a Borel set B with $\mu(B) = 1$ and $g_n \in conv(f_n, f_{n+1}, \dots)$ such that $(g_n)_{n \in \mathbb{N}}$ pointwise converges on B.

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Assume that $f, g : X \to X$ are countable-to-1, $G_{f,g}$ is acyclic, $\chi_B^{el}(G_{f,g}) = \aleph_0$. Is $E_{f,g}$ necessarily not hyperfinite?

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Are there such functions?

The shift graph

Let us denote by $[S]^{\mathbb{N}}$ the collection of countably infinite subsets of the set S.

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Theorem. (Galvin-Prikry) Let $[\mathbb{N}]^{\mathbb{N}} = B_0 \cup \cdots \cup B_n$ be a Borel covering. Then there exists some $i \leq n$ and $x \subset \mathbb{N}$ infinite with $[x]^{\mathbb{N}} \subset B_i$.

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Let $S_0 : [\mathbb{N}]^{\mathbb{N}} \to [\mathbb{N}]^{\mathbb{N}}$ be the *shift-map*, defined by

$$S_0(\{x_0, x_1, \dots\}) = \{x_1, x_2, \dots\}.$$

Define the shift-graph $\mathcal{G}_{S_0}$ by letting $x\mathcal{G}_{S_0}y$ iff $y = S_0(x)$.

Theorem. (Kechris-Solecki-Todorčević) $\chi_B(G_{S_0}) = \aleph_0$.

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Question

Are there $f, g : [\mathbb{N}]^{\mathbb{N}} \to [\mathbb{N}]^{\mathbb{N}}$ Borel, countable-to-1, such that $G_{f,g}$ is acyclic and for every x we have $\chi_B^{el}(G_{f,g} \restriction [x]^{\mathbb{N}}) = \aleph_0$?

Ramsey and hyperfiniteness of CBERs

Theorem (Mathias, Soare; Kanovei-Sabok-Zapletal)

Let $(g_n)_{n\in\mathbb{N}} : [\mathbb{N}]^{\mathbb{N}} \to [\mathbb{N}]^{\mathbb{N}}$ *be a collection of Borel functions. There exists an* $x \in [\mathbb{N}]^{\mathbb{N}}$ *such that for every* $y \in [x]^{\mathbb{N}}$ *and for every* $n \in \mathbb{N}$ *we have that* $g_n(y) \in [x]^{\mathbb{N}}$ *implies that* $g_n(y) \setminus y$ *is finite.*

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Corollary. Assume that E is a CBER on $[\mathbb{N}]^{\mathbb{N}}$. Then there is some $x \in [\mathbb{N}]^{\mathbb{N}}$ with $E \upharpoonright [x]^{\mathbb{N}} \subseteq \mathsf{E}_0$.

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Proposition. If E is a BER, then E is hypersmooth iff $E \leq_B \mathsf{E}_1$, where E_1 is defined on $(2^{\mathbb{N}})^{\mathbb{N}}$ by $x \mathsf{E}_1 y \iff \{n : x_n \neq y_n\}$ is finite.

Theorem. Assume that E is an equivalence relation on $[\mathbb{N}]^{\mathbb{N}}$ such that for every $x \in [\mathbb{N}]^{\mathbb{N}}$

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Let S_0, S_1 be maps on $[\mathbb{N}]^{\mathbb{N}}$ defined by

$$S_0(\{x_0, x_1, x_2, \dots\}) = \{x_1, x_2, x_3 \dots\},$$

$$S_1(\{x_0, x_1, \dots\}) = \{x_1, x_3, x_5, \dots\}.$$

Let E_{S_0,S_1} be the connected component equivalence relation of G_{S_0,S_1}.

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Corollary. E_{S_0,S_1} is not hypersmooth.

Theorem

Assume that $(f_n)_{n\in\mathbb{N}} : [\mathbb{N}]^{\mathbb{N}} \to (2^{\mathbb{N}})^{\mathbb{N}}$ *are Borel. There exist an* $x \in [\mathbb{N}]^{\mathbb{N}}$ *and a countable set* C *such that for every* $y, z \in [x]^{\mathbb{N}}$ *almost disjoint and* $n \in \mathbb{N}$ *we have that* $f_n(z) = f_n(y)$ *implies* $f_n(y) \in C$*.*

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This implies the main theorem: assume that f reduces E to E_1, and let $f_n = \mathcal{S}^n \circ f$.

Theorem (Prömel-Voigt)

Assume that $f : [\mathbb{N}]^{\mathbb{N}} \to 2^{\mathbb{N}}$ *is a Borel function. There exist an* $x \in [\mathbb{N}]^{\mathbb{N}}$ *and a function* $\Gamma : [x]^{\mathbb{N}} \to [\mathbb{N}]^{\leq \mathbb{N}}$ *with the following properties:*

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5. Γ *is continuous*

More questions

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Let K be compact. Is $E_{S_0,S_1} \upharpoonright K$ hypersmooth?

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Does AD imply $\chi(\mathcal{G}_S) \geq \aleph_0$*?*

Thank you for your attention!