Most(?) theories have Borel complete reducts and expansions

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Spaces of countable models

Fix L a countable language. Let

$$Str_L = \{L\text{-structures } M \text{ with universe } \omega\}$$

Topologize: Basic open sets

$$U_{\varphi(n_1,\ldots,n_k)} = \{M \in Str_L : M \models \varphi(n_1,\ldots,n_k)\}$$

Str_L is a standard Borel space (separable, completely metrizable of size continuum).
For $\Phi \in L_{\omega_1,\omega}$, $Mod(\Phi) = \{M \in Str_L : M \models \Phi\}$ is a Borel subset.

$Sym(\omega)$ acts on Str_L by $\sigma.M =$ the L-structure M' formed by permuting ω. Note: $(Mod(\Phi), \cong)$ is invariant under this action.
Lopez-Escobar: The only invariant Borel subsets of a standard Borel space are $Mod(\Phi)$ for some $\Phi \in L_{\omega_1,\omega}$.

How complicated is $(Mod(\Phi), \cong)$?

Friedman-Stanley Given two sentences Φ, Ψ (possibly in different countable languages L, L') we say $(Mod(\Phi), \cong)$ is Borel reducible to $(Mod(\Psi), \cong)$, $\Phi \leq_B \Psi$, if there is a Borel $f : Str_L \to Str_{L'}$ such that for all $M, N \in Mod(\Phi)$, $f(M), f(N) \in Mod(\Psi)$ and

$$M \cong N \iff f(M) \cong f(N)$$

There is a maximal $\equiv_B$-class, containing graphs, linear orders, RCF, DCF. Φ is Borel complete if $(Mod(\Phi), \cong)$ is in this maximal class.

$(Mod(\Phi), \cong)$ Borel or properly analytic?

For any $\Phi \in L_{\omega_1,\omega}$ the equivalence relation $\cong$ on $Mod(\Phi)$ is always analytic, but sometimes is Borel.

Fact: If Φ is Borel complete, then $\cong$ is not Borel.

Example

$T =$ "Independent unary predicates" (model completion of empty theory in $L = \{U_n : n \in \omega\}$) has Borel isomorphism relation. So does $Th(\mathbb{Z}, +)$.

Hjorth-Kechris-Louveau, building on Friedman-Stanley, give a good understanding to the possible behaviors of $(Mod(\Phi), \cong)$ when $\cong$ is Borel.

For many years, little was known about non-Borel complete theories with $\cong$ properly analytic [no first order examples were known].

Ulrich-Rast-L: $REF(bin)$ 'binary splitting, refining equivalence relations' is not Borel complete, but $\cong$ is not Borel.

Thesis: This region is vast.

Will see: There are reducts of 'Independent unary predicates' and $Th(\mathbb{Z}, +)$ that are Borel complete, and reducts that are not Borel complete, yet $\cong$ is not Borel.

$CC(h)$: Cross-cutting equivalence relations, indexed by h.

Let $L = \{E_n : n \in \omega\}$ and $h : \omega \to (\omega \setminus \{0, 1\})$.

$CC(h)$ asserts:

- Each E_n is an equivalence relation with $h(n)$ classes; and
- The $\{E_n\}$ cross-cut: For any finite $F \subseteq \omega$, $E_F := \bigwedge_{n \in F} E_n$ has $\prod_{n \in F} h(n)$ classes.

$CC(h)$ is complete, admits QE, is weakly minimal, trivial (i.e., mutually algebraic).

There is a unique 1-type, but $2^{\aleph_0}$ 2-types.

Is $\cong$ Borel in $Mod(CC(h))$?

- $CC(2)$ (binary splitting, cross-cutting) has Borel isomorphism; BUT
- $CC(3)$ (tertiary splitting, cross-cutting) does not.
- If $h(n) \geq 3$ for infinitely many n, then $Mod(CC(h))$ has non-Borel isomorphism.

However, there are major differences between $Mod(CC(h))$, even among ones without Borel isomorphism.

The complexity of $Mod(CC(h))$ is built into the group $G(h)$ of elementary permutations of $\mathrm{acl}^{eq}(\emptyset)$, namely

$$G(h) := \Pi_{n \in \omega} \mathbb{Z} / h(n) \mathbb{Z}$$

One first observation: The group $G(h)$ has bounded exponent if and only if $\{h(n)\}$ is bounded.

On one hand:

Theorem (L-Ulrich)

If h is unbounded, (i.e., for all m, $h(n) \geq m$ for some n) then $CC(h)$ is Borel complete.

Proof.

Find a 'sufficiently indiscernible' countable subset $Y \subseteq G(h)$ and use this to code graphs using "hybrids" $y_m * y_n$ whose projections on even coordinates is y_m, and on odd coordinates by y_n. □

Reducts

- An L'-structure M' is a reduct of an L-structure M if they have the same universes, and every L'-definable set in M' is definable in M.
- An L'-theory T' is a reduct of an L-theory T if some $M \models T$ has a reduct $M' \models T'$.

Example

Both $CC(2^n)$ and $CC(4)$ are reducts of $CC(2)$. Hence $CC(2)$ has a Borel complete reduct.

Proof.

For $CC(2^n)$, partition $\omega = \bigsqcup\{F_n : n \in \omega\}$ with $|F_n| = n$. Let $E_n^* := \bigwedge_{i \in F_n} E_i$. Then $\{E_n^*\}$ are cross-cutting, where E_n^* has 2^n classes. For $CC(4)$, partition ω into 2-element sets. □

Corollary

Let T be a complete theory in a countable language with $S_1(\emptyset)$ uncountable. Then $Mod(T)$ has Borel complete reduct.

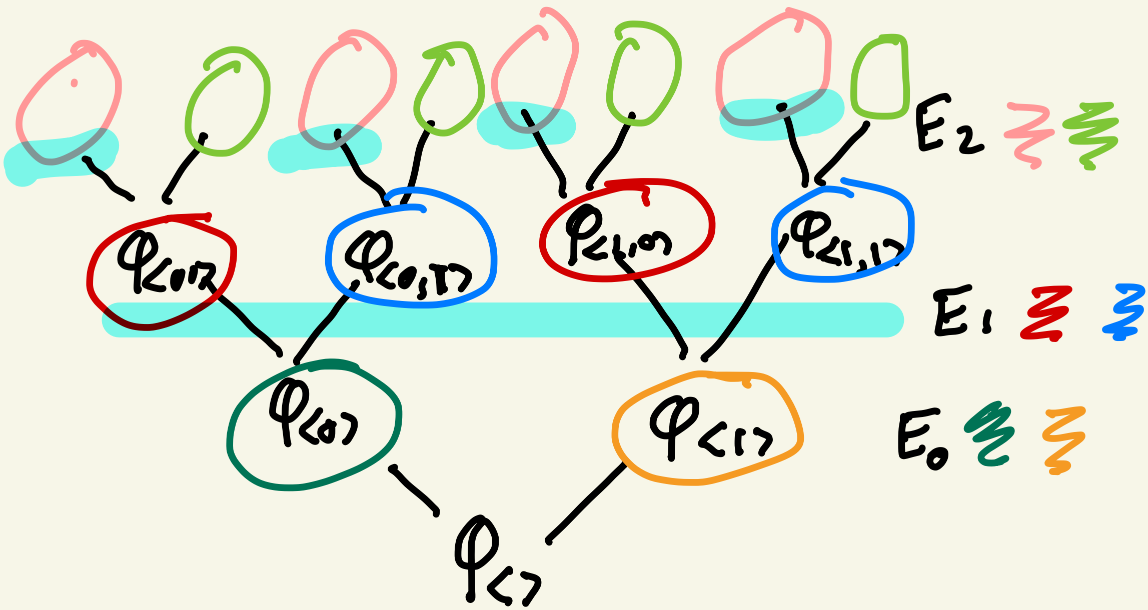
Proof.

It suffices to show $CC(2)$ is a reduct of T. Choose consistent formulas $\{\varphi_\eta(x) : \eta \in 2^{<\omega}\}$ satisfying:

- For $\nu \trianglelefteq \eta$, $\varphi_\eta(x) \vdash \varphi_\nu(x)$;
- For each $n \in \omega$, $\{\varphi_\eta(x) : \eta \in 2^n\}$ are pairwise contradictory.
- For each $n \in \omega$, $T \models \forall x(\bigvee_{\eta \in 2^n} \varphi_\eta(x))$.

Let $\delta_n^0(x) := \bigwedge_{\eta \in 2^n}[\varphi_\eta(x) \to \varphi_{\eta \wedge 0}(x)], \delta_n^1(x) := \bigwedge_{\eta \in 2^n}[\varphi_\eta(x) \to \varphi_{\eta \wedge 1}(x)]$ and let $E_n(x, y) := [\delta_n^0(x) \leftrightarrow \delta_n^0(y)]$ $\{E_n\}$ is a family of cross-cutting equivalence relations, each with 2 classes. □

Tree of formulas $\varphi_\eta(x)$ when $S_1(\emptyset)$ uncountable.



Each E_n has 2 classes, these cross-cut

$\therefore$ CC(2) is a reduct of MFT

Conclusions:

- 'Independent unary predicates' and $Th(\mathbb{Z}, +)$ have Borel complete reducts.
- If T is not ω-stable, then $ElDiag(M)$ has a Borel complete reduct for some $M \models T$.
- If T is not small (i.e., $S_n(\emptyset)$ is uncountable for some n) then T^{eq} has a Borel complete reduct.

On the other hand: If h is uniformly bounded, then $CC(h)$ is not Borel complete.

For any $\Phi \in L_{\omega_1,\omega}$, $CSS_{\rm ptl}(\Phi)$ is the class (possibly proper) of potential canonical Scott sentences, i.e., sentences $\varphi \in L_{\infty,\omega}$ such that for some forcing extension $\mathbb{V}[G] \supseteq \mathbb{V}$, there is a countable $M \models \Phi$ whose canonical Scott sentence is φ.

Theorem (Ulrich,Rast,L)

If $\Phi \leq_B \Psi$, then $|CSS_{\rm ptl}(\Phi)| \leq |CSS_{\rm ptl}(\Psi)|$.

In practice, $CSS_{\rm ptl}(\Phi)$ can be hard to count.

On expansions

For $\Phi \in L_{\omega_1,\omega}$, an expansion of Φ is some $\Phi^* \in (L^*)_{\omega_1,\omega}$ with $L^* \supseteq L$ such that $\Phi^* \vdash \Phi$.

Theorem (L-Ulrich)

$\Phi \in L_{\omega_1,\omega}$ has a Borel complete expansion if and only if S_∞ divides $Aut(M)$ for some countable $M \models \Phi$.

Corollary

Every first order theory T admitting an infinite model has a Borel complete expansion.

Not true for sentences $\Phi \in L_{\omega_1,\omega}$.

The first example(??) of a distinction between first order and infinitary sentences involving Borel completeness??

$CC(h)$ when h uniformly bounded

Let $M \models CC(h)$ be countable. Let $E_\infty(x,y) := \bigwedge_{n\in\omega} E_n(x,y)$. Let Φ assert

- $CC(h) + \forall x \forall y (E_\infty(x,y) \rightarrow x = y)$.

Note: Every model $M \models \Phi$ has size $\leq 2^{\aleph_0}$, and $Aut(M)$ is a subgroup of $ElPerm(\mathrm{acl}^{eq}(\emptyset)) \cong \Pi_{n\in\omega} \mathbb{Z} / h(n)\mathbb{Z}$.

Since $\Pi_{n\in\omega} \mathbb{Z} / h(n)\mathbb{Z}$ has bounded exponent (but S_∞ does not), Φ does not have a Borel complete expansion.

Theorem

When h is uniformly bounded, $CC(h)$ is not Borel complete.

Proof.

Given $M \models CC(h)$ countable, let $M^* := (M/E_\infty, E_n, U_m)_{n,m\in\omega}$, where for $m \geq 1$, $U_m(a/E_\infty)$ iff $|a/E_\infty| = m$ and $U_0(a/E_\infty)$ iff a/E_∞ is infinite. This is a Borel embedding of $CC(h)$ into an expansion of Φ. Since Φ has no Borel complete expansions, $CC(h)$ is not Borel complete. □

Corollary

If $S_1(T)$ is uncountable for some complete T, then T has a reduct that is not Borel complete, yet has non-Borel isomorphism.

Proof.

$CC(4)$ is a reduct of $CC(2)$, which is a reduct of $Mod(T)$. □

How to prove the expansion theorem?

Idea: Given $\Phi \in L_{\omega_1,\omega}$, describe a language $L^\flat$ and an $(L^\flat)_{\omega_1,\omega}$ sentence $\Phi^\flat$ so that "Models of $\Phi^\flat$ code canonical Scott sentences of expansions of models of Φ."

Real Theorem (L-Ulrich)

TFAE for $\Phi \in L_{\omega_1,\omega}$.

1. Φ *has a Borel complete expansion.*
2. $\Phi^\flat$ *has arbitrarily large models.*
3. $\Phi^\flat$ *admits Ehrenfeucht-Mostowski models.*
4. S_∞ *divides* $Aut(M)$ *for some countable* $M \models \Phi$.

Thank you!

- M. C. Laskowski and D. Ulrich, Most(?) theories have Borel complete reducts, *Journal of Symbolic Logic* (to appear). arXiv:2103.09724
- M. C. Laskowski and D. Ulrich, Characterizing the existence of a Borel complete expansion (submitted) arXiv:2109.06140
- D. Ulrich, R. Rast, and M.C. Laskowski, Borel complexity and potential canonical Scott sentences, *Fundamenta Mathematicae* **239** (2017), no. 2, 101–147. arXiv:1510.05679

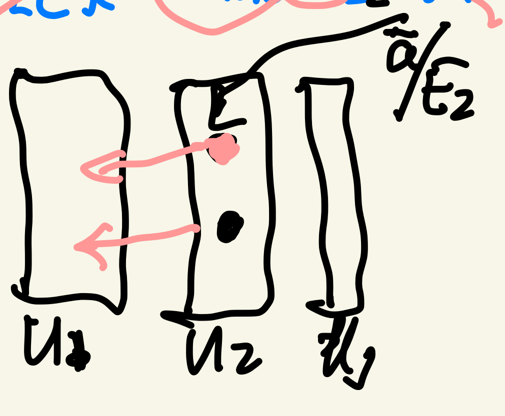
$(M, \bar{\mathcal{A}})$

$(M, E_n)_{new}$

$E_n(\bar{a}, \bar{b})$ iff $\mathrm{tp}_{qr}(\bar{a}) = \mathrm{tp}_{qr}(\bar{b})$

$(M, \bar{\mathcal{A}})$ $\bar{\mathcal{A}}$ describes an arbitrary back-and-forth system on M.

$(M, E_n)_{new} \mapsto$



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