Logic Seminar (Caltech)

Orbit equivalences of multidimensional Borel flows

Konstantin Slutsky
Feb 16th, 2022

Iowa State University
A multidimensional Borel flow is a Borel action $\mathbb{R}^d \curvearrowright \Omega$ on a standard Borel space. The action of $\vec{r} \in \mathbb{R}^d$ upon $x \in \Omega$ will be denoted additively: $x + \vec{r}$.

Irrational rotation on the torus is a classical example of an $\mathbb{R}$ flow.
A multidimensional **Borel flow** is a Borel action $\mathbb{R}^d \curvearrowright \Omega$ on a standard Borel space. The action of $\vec{r} \in \mathbb{R}^d$ upon $x \in \Omega$ will be denoted additively: $x + \vec{r}$.

Irrational rotation on the torus is a classical example of an $\mathbb{R}$ flow.
A multidimensional **Borel flow** is a Borel action $\mathbb{R}^d \curvearrowright \Omega$ on a standard Borel space. The action of $\vec{r} \in \mathbb{R}^d$ upon $x \in \Omega$ will be denoted additively: $x + \vec{r}$.

Irrational rotation on the torus is a classical example of an $\mathbb{R}$ flow.
A multidimensional **Borel flow** is a Borel action $\mathbb{R}^d \rightarrow \Omega$ on a standard Borel space. The action of $\vec{r} \in \mathbb{R}^d$ upon $x \in \Omega$ will be denoted additively: $x + \vec{r}$.

Irrational rotation on the torus is a classical example of an $\mathbb{R}$ flow.

![Diagram of irrational rotation on the torus]
Irrational Rotation on a Torus
An **orbit equivalence** between flows $\mathbb{R}^d \curvearrowright \Omega_1$ and $\mathbb{R}^d \curvearrowright \Omega_2$ is a Borel bijection $\phi : \Omega_1 \rightarrow \Omega_2$ that sends orbits onto orbits:

$$\phi(x + \mathbb{R}^d) = \phi(x) + \mathbb{R}^d.$$
When the flow is \textbf{free}

- any orbit of the action can be identified with the affine space $\mathbb{R}^d$;
When the flow is **free**

- any orbit of the action can be identified with the affine space $\mathbb{R}^d$;
- one can **transfer any translation-invariant structure** from the Euclidean space onto every orbit of the flow;
When the flow is **free**

- any orbit of the action can be identified with the affine space $\mathbb{R}^d$;
- one can **transfer any translation-invariant structure** from the Euclidean space onto every orbit of the flow;
- each point considers itself to be the origin, and transfers the structure via the corresponding bijection.
Transferring topology:

For $x \in \Omega$

$$\mathcal{O}_x = \left\{ A \subseteq x + \mathbb{R}^d : \{ \vec{r} \in \mathbb{R}^d \mid x + \vec{r} \in A \} \text{ is open} \right\}. $$

Note that $\mathcal{O}_x = \mathcal{O}_y$ whenever $x + \mathbb{R}^d = y + \mathbb{R}^d$. 
Transferring topology:

For \( x \in \Omega \)

\[
O_x = \left\{ A \subseteq x + \mathbb{R}^d : \{ \vec{r} \in \mathbb{R}^d | x + \vec{r} \in A \} \text{ is open} \right\}.
\]

Note that \( O_x = O_y \) whenever \( x + \mathbb{R}^d = y + \mathbb{R}^d \).

Two flows are **smoothly equivalent** if there exists an orbit equivalence between the phase spaces which is an *orientation preserving diffeomorphism* between orbits.
Smooth equivalence of one dimensional flows is known under the name of *time change equivalence*. 
Smooth equivalence of one dimensional flows is known under the name of time change equivalence.

It has been studied extensively in ergodic theory, where

- phase space $\Omega$ is endowed with a probability measure;
Smooth equivalence of one dimensional flows is known under the name of time change equivalence.

It has been studied extensively in ergodic theory, where

- phase space $\Omega$ is endowed with a probability measure;
- flows are assumed to be (quasi) measure preserving;
Smooth equivalence of one dimensional flows is known under the name of time change equivalence.

It has been studied extensively in ergodic theory, where

- phase space $\Omega$ is endowed with a probability measure;
- flows are assumed to be (quasi) measure preserving;
- all orbit equivalence maps must be (quasi) measure preserving;
Smooth equivalence of one dimensional flows is known under the name of time change equivalence.

It has been studied extensively in ergodic theory, where

- phase space $\Omega$ is endowed with a probability measure;
- flows are assumed to be (quasi) measure preserving;
- all orbit equivalence maps must be (quasi) measure preserving;
- and may be defined up to a null set.
Theorem (Feldman–Rudolph, Ornstein–Weiss)

There are continuumly many pairwise time change inequivalent measure preserving ergodic $\mathbb{R}$ flows.
### Theorem (Feldman–Rudolph, Ornstein–Weiss)

There are continuumly many pairwise time change inequivalent measure preserving ergodic $\mathbb{R}$ flows.

### Theorem (Rudolph)

All ergodic measure preserving $\mathbb{R}^d$ flows, $d \geq 2$, are smoothly equivalent.

### Theorem (Feldman)

All ergodic *quasi* measure preserving $\mathbb{R}^d$ flows, $d \geq 2$, are smoothly equivalent.
Descriptive set theoretical framework is both **more restrictive** (one has to define equivalence on each and every orbit, flows may not preserve any measure) and **more relaxed** (orbit equivalence maps don’t need to be measure preserving).
Descriptive set theoretical framework is both more restrictive (one has to define equivalence on each and every orbit, flows may not preserve any measure) and more relaxed (orbit equivalence maps don’t need to be measure preserving).

Recall that a flow $\mathbb{R}^d \curvearrowright \Omega$ is tame if it admits a Borel transversal — a Borel set $S \subset \Omega$ that chooses one point from each orbit.
Descriptive set theoretical framework is both more restrictive (one has to define equivalence on each and every orbit, flows may not preserve any measure) and more relaxed (orbit equivalence maps don’t need to be measure preserving).

Recall that a flow $\mathbb{R}^d \curvearrowright \Omega$ is tame if it admits a Borel transversal — a Borel set $S \subset \Omega$ that chooses one point from each orbit.

**Theorem (Miller–Rosendal)**

*All free non tame $\mathbb{R}$ flows are time change equivalent.*
## Summary of Results on Smooth Equivalence

<table>
<thead>
<tr>
<th>Dimension</th>
<th>$d = 1$</th>
<th>$d \geq 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ergodic Theory</td>
<td>Many</td>
<td>One</td>
</tr>
<tr>
<td>Borel Dynamics</td>
<td>One</td>
<td>One</td>
</tr>
</tbody>
</table>
## Summary of Results on Smooth Equivalence

<table>
<thead>
<tr>
<th>Dimension</th>
<th>$d = 1$</th>
<th>$d \geq 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ergodic Theory</td>
<td>Many</td>
<td>One</td>
</tr>
<tr>
<td>Borel Dynamics</td>
<td>One</td>
<td>?</td>
</tr>
</tbody>
</table>
### Summary of Results on Smooth Equivalence

<table>
<thead>
<tr>
<th>Dimension</th>
<th>$d = 1$</th>
<th>$d \geq 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ergodic Theory</td>
<td>Many</td>
<td>One</td>
</tr>
<tr>
<td>Borel Dynamics</td>
<td>One</td>
<td><strong>One</strong></td>
</tr>
</tbody>
</table>

#### Theorem (S.)

All free non tame $\mathbb{R}^d$ flows, $d \geq 2$, are smoothly equivalent.
Transferring metric:

For $x, y \in \Omega$ within the same orbit there exists a unique $\rho(x, y) \in \mathbb{R}^d$ such that $x + \rho(x, y) = y$.

$$d(x, y) = \|\rho(x, y)\|$$
A cross section for an action $\mathbb{R}^d \curvearrowright \Omega$ is a Borel set $C \subseteq \Omega$ which intersects all orbits of the action and is lacunary: for some open ball $B_r \subseteq \mathbb{R}^d$ around the origin 

$$(x + B_r) \cap (y + B_r) = \emptyset \text{ whenever } x, y \in C, \ x \neq y.$$
Suspension Flow

With a (bi-infinite, lacunary) cross section $C$ of an $\mathbb{R}$ flow one associates a suspension flow.

---

**Gap function** $f_C(x) = \min\{r > 0 : x + r \in C\}$

**Induced automorphism** $T : C \to C$ given by $T(x) = x + f_C(x)$
With a (bi-infinite, lacunary) cross section $C$ of an $\mathbb{R}$ flow one associates a suspension flow.

**Gap function** $f_C(x) = \min\{r > 0 : x + r \in C\}$

**Induced automorphism** $T : C \to C$ given by $T(x) = x + f_C(x)$
With a (bi-infinite, lacunary) cross section $C$ of an $\mathbb{R}$ flow one associates a **suspension** flow.

**Gap function** $f_C(x) = \min\{r > 0 : x + r \in C\}$

**Induced automorphism** $T : C \to C$ given by $T(x) = x + f_C(x)$
With a (bi-infinite, lacunary) cross section $C$ of an $\mathbb{R}$ flow one associates a suspension flow.

**Gap function** $f_C(x) = \min\{r > 0 : x + r \in C\}$

**Induced automorphism** $T : C \to C$ given by $T(x) = x + f_C(x)$
The flow can be modeled on the space $\Omega$ under the graph of the gap function, by flowing upward within a fiber, and then jumping to the next one as determined by $T$.

$$\Omega = \{(x, t) \in X \times \mathbb{R} : 0 \leq x < f(x)\}$$
The following cross section for the irrational rotation has a constant gap function.
The following cross section for the irrational rotation has a constant gap function.

Such a representation is very special — many flows do not admit a cross section with a constant gap function.
Katok’s Representation Theorem

Geometrically appealing suspension flow construction is one-dimensional.

An action $\mathbb{Z}^d \curvearrowleft X$ can be turned into an $\mathbb{R}^d$ flow on the space $\Omega = X \times [0, 1)^d$. For $(x, \vec{s}) \in X \times [0, 1)^d$ and $\vec{r} \in \mathbb{R}^d$ let $\vec{n} \in \mathbb{Z}^d$ be such that $\vec{s} + \vec{r} - \vec{n} \in [0, 1)^d$; set

$$(x, \vec{s}) + \vec{r} = (x + \vec{n}, \vec{s} + \vec{r} - \vec{n}).$$
Katok’s Representation Theorem

Geometrically appealing suspension flow construction is one-dimensional.

An action \( \mathbb{Z}^d \curvearrowright X \) can be turned into an \( \mathbb{R}^d \) flow on the space \( \Omega = X \times [0, 1)^d \). For \((x, \vec{s}) \in X \times [0, 1)^d\) and \(\vec{r} \in \mathbb{R}^d\) let \(\vec{n} \in \mathbb{Z}^d\) be such that \(\vec{s} + \vec{r} - \vec{n} \in [0, 1)^d\); set

\[(x, \vec{s}) + \vec{r} = (x + \vec{n}, \vec{s} + \vec{r} - \vec{n}).\]
Katok’s Representation Theorem

Geometrically appealing suspension flow construction is one-dimensional.

An action $\mathbb{Z}^d \curvearrowright X$ can be turned into an $\mathbb{R}^d$ flow on the space $\Omega = X \times [0, 1)^d$. For $(x, \vec{s}) \in X \times [0, 1)^d$ and $\vec{r} \in \mathbb{R}^d$ let $\vec{n} \in \mathbb{Z}^d$ be such that $\vec{s} + \vec{r} - \vec{n} \in [0, 1)^d$; set

$$(x, \vec{s}) + \vec{r} = (x + \vec{n}, \vec{s} + \vec{r} - \vec{n}).$$

Not every $\mathbb{R}^d$ flow is of this form, just as not every $\mathbb{R}$ flow has cross sections with constant gaps.
Let $x \mapsto x + \vec{r}$ and $x \mapsto x \oplus \vec{r}$ be two flows that have the same orbits. The associated cocycle $h : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is given by the condition

$$x \oplus \vec{r} = x + h(x, \vec{r})$$

for all $x \in X$ and $\vec{r} \in \mathbb{R}^d$. 

$x \oplus \vec{r} = x + h(x, \vec{r})$

$x$
Katok’s Representation Theorem

**Theorem (Katok)**

For every free ergodic measure preserving $\mathbb{R}^d$ flow $x \mapsto x + \vec{r}$ and any $\epsilon > 0$ there exists a flow $\mathbb{R}^d \curvearrowright X \times [0, 1)^d$ arising from an ergodic action $\mathbb{Z}^d \curvearrowright X$ such that the corresponding cocycle $h : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is $(1 - \epsilon, 1 + \epsilon)$-bi-Lipschitz:

$$1 - \epsilon \leq \frac{\|h(x, \vec{r})\|}{\|\vec{r}\|} \leq 1 + \epsilon.$$
For every free ergodic measure preserving $\mathbb{R}^d$ flow $x \mapsto x + \vec{r}$ and any $\epsilon > 0$ there exists a flow $\mathbb{R}^d \curvearrowright X \times [0, 1)^d$ arising from an ergodic action $\mathbb{Z}^d \curvearrowright X$ such that the corresponding cocycle $h : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is $(1 - \epsilon, 1 + \epsilon)$-bi-Lipschitz:

\[
1 - \epsilon \leq \frac{\|h(x, \vec{r})\|}{\|\vec{r}\|} \leq 1 + \epsilon.
\]

The statement of Katok’s Theorem holds within the framework of Borel dynamics.
Layered and Unlayered Toasts
A common pattern is to start with a cross section \( C \subseteq \Omega \) and run a construction within disjoint regions around the points of \( C \).
Properties of Regions

- **Exhaustive:** $\bigcup_n R_n = \Omega$. 

![Diagram showing layered toasts with regions $R_1$, $R_2$, and $R_3$.](image-url)
Layered Toasts in Ergodic Theory

Properties of Regions

- **Exhaustive:** $\bigcup_n R_n = \Omega$.
- **Layered:** $R_n \subseteq R_{n+1}$.
Layered Toasts in Ergodic Theory

Properties of Regions

- **Exhaustive**: $\bigcup_n R_n = \Omega$.
- **Layered**: $R_n \subseteq R_{n+1}$.
- **Shape**: rectangles.
Properties of Regions

- **Exhaustive:** $\bigcup_n R_n = \Omega$.
- **Layered:** $R_n \subseteq R_{n+1}$.
- **Shape:** rectangles.
- **Boundary:** $b_n \to \infty$. 
Properties of Regions

- **Exhaustive**: $\bigcup_n R_n = \Omega$. 
Properties of Regions

- **Exhaustive:** $\bigcup_n R_n = \Omega$.
- **Unlayered:** $R_m \subseteq R_n$. 
Properties of Regions

- **Exhaustive:** $\bigcup_n R_n = \Omega$.
- **Unlayered:** $R_m \subseteq R_n$.
- **Shape:** non-convex.
Properties of Regions

- **Exhaustive**: $\bigcup_n R_n = \Omega$.
- **Unlayered**: $R_m \subseteq R_n$.
- **Shape**: non-convex.
- **Boundary**: $b_n \geq \text{const.}$
Construction of an Orbit Equivalent Flow

Let $R$ be a region of an unlayered toast. A Borel injection $\phi_R : R \to \mathbb{R}^d$ defines a partial action on $R$

$$x \oplus \vec{r} = \phi_R^{-1}(\phi_R(x) + \vec{r}).$$
Construction of an Orbit Equivalent Flow

Let $R$ be a region of an unlayered toast. A Borel injection $\phi_R : R \to \mathbb{R}^d$ defines a partial action on $R$

$$x \oplus \vec{r} = \phi_R^{-1}(\phi_R(x) + \vec{r}).$$
Construction of an Orbit Equivalent Flow

The action is **partial** in the sense that $x \oplus (\vec{r} + \vec{s}) = (x \oplus \vec{r}) \oplus \vec{s}$ whenever both sides are defined.
Construction of an Orbit Equivalent Flow

The action is **partial** in the sense that \( x \oplus (\vec{r} + \vec{s}) = (x \oplus \vec{r}) \oplus \vec{s} \) whenever both sides are defined.

**NB:** Shifted map \( x \mapsto \phi_R(x) + \vec{s} \) defines the **same** partial action for any \( \vec{s} \in \mathbb{R}^d \).
When constructing $\phi_{R_2}$, we take into account partial actions given by $\phi_{R_1}$ for the regions $R_1 \subseteq R_2$. 
Construction of an Orbit Equivalent Flow

When constructing $\phi_{R_2}$, we take into account partial actions given by $\phi_{R_1}$ for the regions $R_1 \subseteq R_2$. However, $\phi_{R_2}$ is not an extension of $\phi_{R_1}$. 
When constructing $\phi_{R_2}$, we take into account partial actions given by $\phi_{R_1}$ for the regions $R_1 \subseteq R_2$. However, $\phi_{R_2}$ is not an extension of $\phi_{R_1}$. Instead, $\phi_{R_2}$ extends shifts of $\phi_{R_1}$. 
Proof of Borel Version of Katok’s Theorem
Let $\vec{v} \in \mathbb{R}^d$ be of norm $||\vec{v}|| \leq 1$, $K > 1$, and $A \subset \mathbb{R}^d$ be a closed bounded set. Define $f_{A,K} : A \rightarrow A$ by

$$f_{A,K} (\vec{r}) = \vec{r} + \frac{d(\vec{r}, \partial A)}{K} \vec{v}.$$
Lipschitz Maps

Let \( \vec{v} \in \mathbb{R}^d \) be of norm \( ||\vec{v}|| \leq 1 \), \( K > 1 \), and \( A \subset \mathbb{R}^d \) be a closed bounded set. Define \( f_{A,K} : A \to A \) by

\[
f_{A,K}(\vec{r}) = \vec{r} + \frac{d(\vec{r}, \partial A)}{K} \vec{v}.
\]

\( f_{A,K} \) is \((1 - K^{-1}, 1 + K^{-1})\)-bi-Lipschitz and \( f_{A,K}(A) = A \).
Lipschitz Maps

Let $A^L = \{ \vec{r} \in A : d(\vec{r}, \partial A) \geq L \}$ and observe that $f_{A,K}|_{\partial A^L} = \vec{r} + L/K \cdot \vec{v}$. Define

$$g_{A,K,L}(\vec{r}) = \begin{cases} f_{A,K}(\vec{r}) & \text{if } \vec{r} \in A \setminus A^L; \\ \vec{r} + L/K \cdot \vec{v} & \text{if } \vec{r} \in A^L; \end{cases}$$
Lipschitz Maps

Let $A^L = \{ \vec{r} \in A : d(\vec{r}, \partial A) \geq L \}$ and observe that $f_{A,K}|_{\partial A^L} = \vec{r} + L/K \cdot \vec{v}$. Define

$$g_{A,K,L}(\vec{r}) = \begin{cases} f_{A,K}(\vec{r}) & \text{if } \vec{r} \in A \setminus A^L; \\ \vec{r} + L/K \cdot \vec{v} & \text{if } \vec{r} \in A^L; \end{cases}$$

$g_{A,K,L}$ is $(1 - K^{-1}, 1 + K^{-1})$-bi-Lipschitz and $g_{A,K,L}(A) = A$. 
Pick a sequence of unlayered toasts whose boundaries are $K$-separated for some sufficiently large $K = K(\epsilon)$. We construct a grid that is bi-Lipschitz equivalent to the standard $\mathbb{Z}^d$ grid.

The first step of the construction consists of “identity” maps.

\[ x = c + \rho(c, x) \]
Pick a sequence of unlayered toasts whose boundaries are $K$-separated for some sufficiently large $K = K(\epsilon)$. We construct a grid that is bi-Lipschitz equivalent to the standard $\mathbb{Z}^d$ grid.

The first step of the construction consists of “identity” maps.
Construction of an Orbit Equivalent Flow

Pick a sequence of unlayered toasts whose boundaries are $K$-separated for some sufficiently large $K = K(\epsilon)$. We construct a grid that is bi-Lipschitz equivalent to the standard $\mathbb{Z}^d$ grid.

The first step of the construction consists of “identity” maps.
Step of Induction

\[ c \]

\[ 0 \]
Step of Induction
Step of Induction

- $c$
- $0$
Step of Induction
Step of Induction
Step of Induction

\( \bullet C \)

\( \bullet 0 \)
Step of Induction
Step of Induction
Proof of Borel Version of Rudolph’s Theorem
The plan is to reduce the multidimensional case to $d = 1$ by proving the following.

**Theorem (S.)**

*Every free $\mathbb{R}^d$ flow on $\Omega$ is **smoothly orbit equivalent** to a flow $\mathbb{R} \times \mathbb{R}^{d-1} \curvearrowright L \times \mathbb{R}^{d-1}$, where $\mathbb{R} \curvearrowright L$ is one-dimensional, and $\mathbb{R}^{d-1} \curvearrowright \mathbb{R}^{d-1}$ acts by translation.*
The plan is to reduce the multidimensional case to $d = 1$ by proving the following.

**Theorem (S.)**

Every free $\mathbb{R}^d$ flow on $\Omega$ is **smoothly orbit equivalent** to a flow $\mathbb{R} \times \mathbb{R}^{d-1} \curvearrowright L \times \mathbb{R}^{d-1}$, where $\mathbb{R} \curvearrowright L$ is one-dimensional, and $\mathbb{R}^{d-1} \curvearrowright \mathbb{R}^{d-1}$ acts by translation.

Note that $L \times \vec{0}$ picks a line out of every orbit upon which the $\mathbb{R}$ flow acts.
Regions $R_1$ can be chosen to be diffeomorphic to a unit disk $B_1 \subset \mathbb{R}^d$, so we may pick such a diffeomorphism and pull the line segment $[-1, 1] \times \tilde{0}$ into the region $R_1$ to be part of the line.
Regions $R_1$ can be chosen to be diffeomorphic to a unit disk $B_1 \subset \mathbb{R}^d$, so we may pick such a diffeomorphism and pull the line segment $[-1, 1] \times \vec{0}$ into the region $R_1$ to be part of the line.
Regions $R_1$ can be chosen to be diffeomorphic to a unit disk $B_1 \subset \mathbb{R}^d$, so we may pick such a diffeomorphism and pull the line segment $[-1, 1] \times \tilde{0}$ into the region $R_1$ to be part of the line.
The following basic fact from differential topology is used in the construction.

**Theorem**

Let $F$, $F'$ and $D_i \subset F$, $D'_i \subset F'$, $1 \leq i \leq n$, be smooth disks. Suppose that $D_i$ are pairwise disjoint, and so are $D'_i$. Any family $\phi_i : D_i \to D'_i$ of orientation preserving smooth diffeomorphisms admits a common extension to a diffeomorphism $\psi : F \to F'$. 
This lets us extend partial actions on $R_1$ regions to a $R_2$ region.
The result of such an extension is a partial action on $R_2$, which extends partial actions on $R_1$. 
Let $\mathbb{R}^d \curvearrowright \Omega_1$ and $\mathbb{R}^d \curvearrowright \Omega_2$ be free non tame Borel flows. By the argument above, each of them is smoothly equivalent to a product flow on $L_i \times \mathbb{R}^{d-1}$. The "first coordinate flows" are time change equivalent by the Miller–Rosendal theorem. If $\xi : L_1 \to L_2$ is such a time change equivalence, then

$$L_1 \times \mathbb{R}^{d-1} \ni (y, \vec{r}) \mapsto (\xi(y), \vec{r}) \in L_2 \times \mathbb{R}^{d-1}.$$ 

is a smooth equivalence of the multidimensional flows.
Summary of the Results on Smooth Equivalence

This concludes our sketch of the argument.

<table>
<thead>
<tr>
<th>Dimension</th>
<th>$d = 1$</th>
<th>$d \geq 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ergodic Theory</td>
<td>Many</td>
<td>One</td>
</tr>
<tr>
<td>Borel Dynamics</td>
<td>One</td>
<td><strong>One</strong></td>
</tr>
</tbody>
</table>
Thank you!