Measurable combinatorics in hyperfinite graphs

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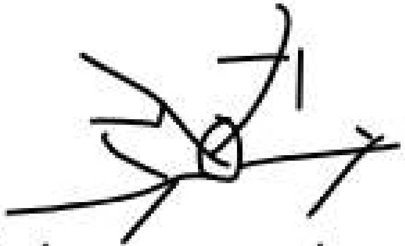
Based on joint work with Kun and Sabok, Weilacher, and upcoming work with Poulin and Zomback

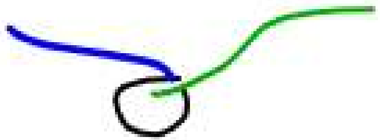
Classical results

Let G be a graph with maximum degree $\Delta(G)$.

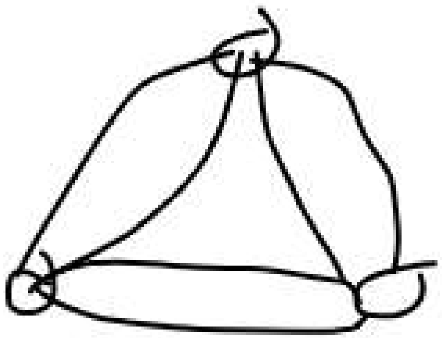
- (Euler 17xx) If every vertex of G has even degree then G admits a **balanced orientation**.



- (Kőnig 1917) If G is bipartite and degree regular then it admits a perfect matching and thus $\chi'(G) = \Delta(G)$.



- (Shannon 1949) If G is a **multigraph** then $\chi'(G) \leq \lfloor \frac{3\Delta(G)}{2} \rfloor$.



Borel graphs

Fix from now on a Polish space (X, τ) with Borel probability measure μ.

- A graph G with $V(G) = X$ is a **Borel graph** if $E(G) \subset X^2$ is Borel.
- The **Borel edge chromatic number**, $\chi'_B(G)$, is the smallest n such that G has a Borel proper n-coloring of its edges
- The μ **measurable (Baire measurable) edge chromatic number** is $\chi'_\mu(G)$ $(\chi'_{BM}) = \min(\chi'_B(G|C))$, where C ranges over conull (comeagre) G-invariant Borel sets.
- G is **hyperfinite** if there are component finite Borel graphs $F_0 \subset F_1 \subset ...$ with $E(G) = \bigcup_{i \in \omega} E(F_i)$

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- (Thornton '20) If G is pmp and $2d$-regular, it admits a Borel orientation with outdegree in $\{d-1, d, d+1\}$ a.e.
- (Marks, Unger '16) A bipartite G has a Borel perfect matching generically if $|N(S)| \geq (1+\epsilon)|S|$ for every finite $S \subset X$ and some $\epsilon > 0$.

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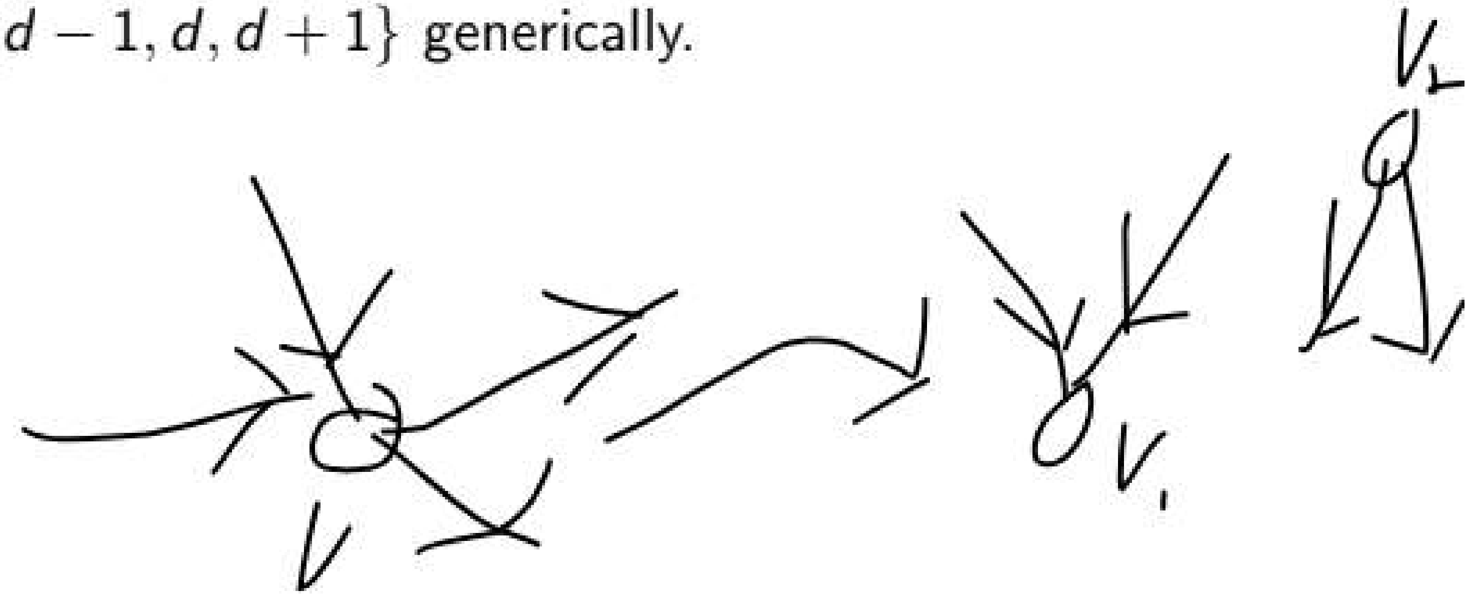

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The same holds for $\chi'_{\mu}(G)$ if G is μ-hyperfinite, and for $\chi'_{B}(G)$ if $asi(G) = 1$ and G has subexponential growth..

Better results for one-ended graphs

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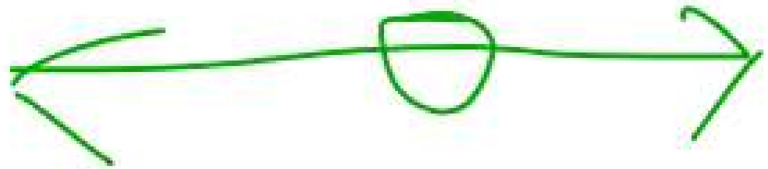
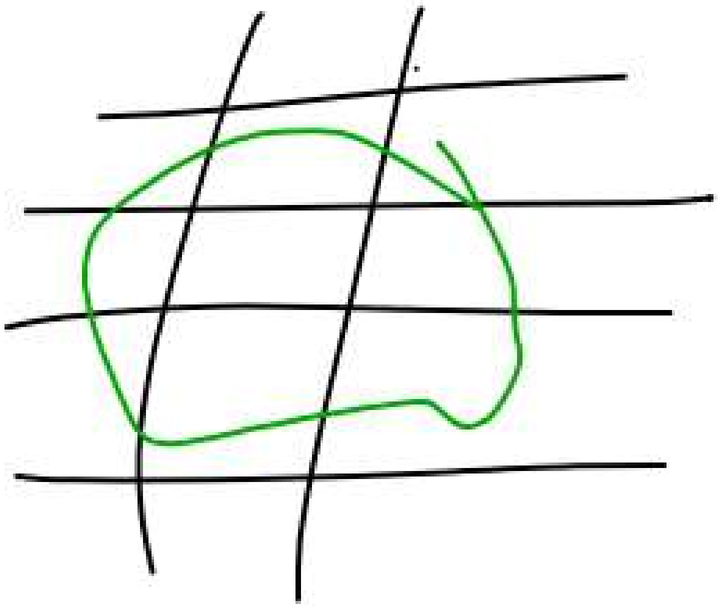
A Borel graph is one-ended if each of its connected components is.





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- (B., Poulin, Zomback '22+) Any d-regular, bipartite one-ended Borel graph admits a Borel perfect matching generically.
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Lemma

Any d-regular bipartite Borel graph admits a Baire measurable matching that covers all but an r-discrete set of vertices.

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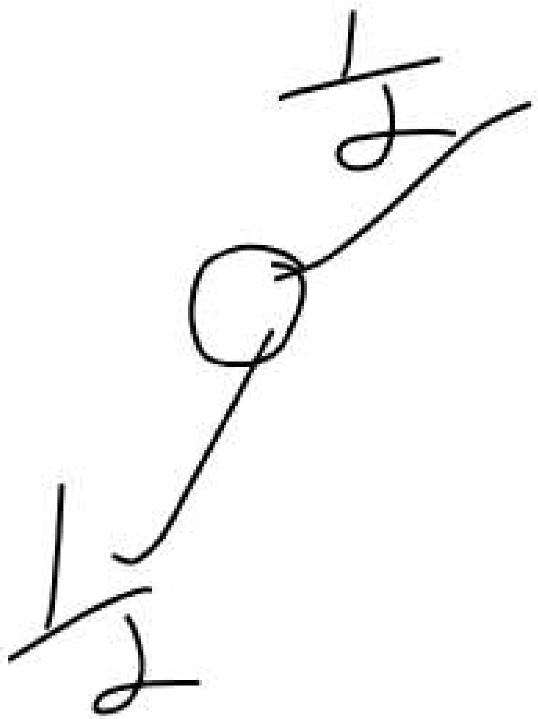
A **fractional perfect matching** is a function $\sigma : E(G) \to [0,1]$ such that $\sum_{v \in e} \sigma(e) = 1$ for all $v \in V(G)$. Given such a σ, let $F(\sigma) = \sigma^{-1}(0,1)$.



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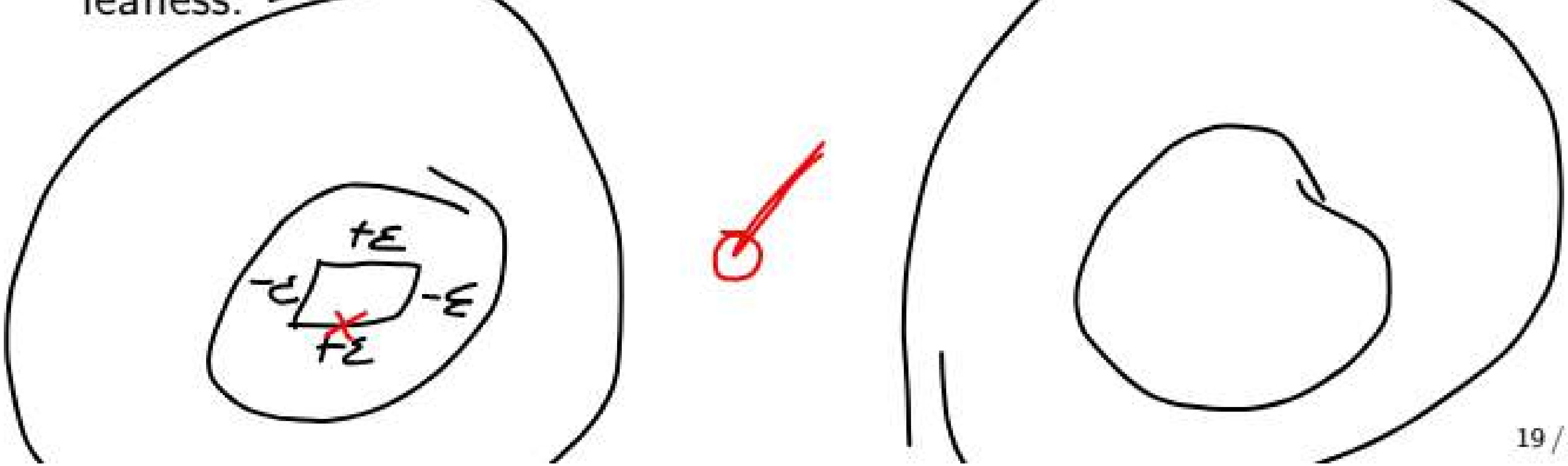
We can prove the lemma in three steps:

1. Starting with $\sigma = \frac{1}{d}$, round until we find a σ' with $F(\sigma')$ acyclic and leafless.



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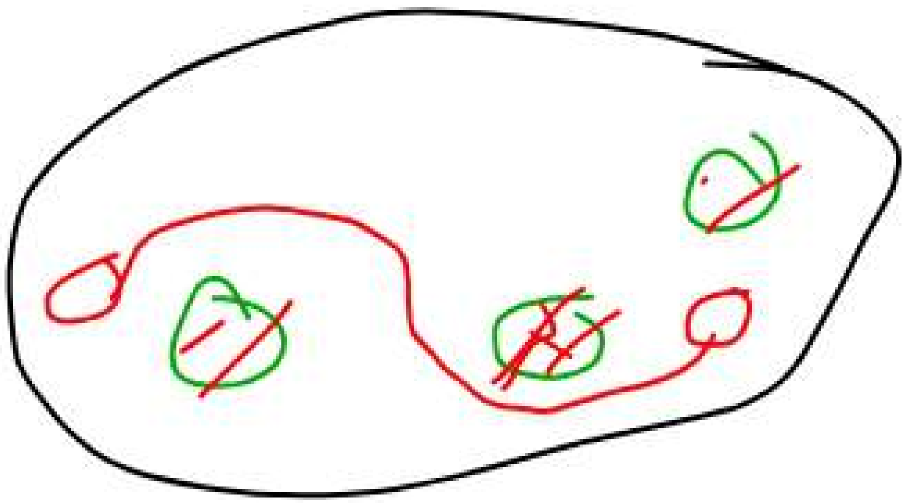
3. Use a toast to find an r-discrete set of points that hits generically many lines.

connected toasts

Definition

A borel family of sets $\mathcal{T} \subset V(G)^{<\infty}$ is a **toast** if it satisfies properties (1) and (2) of the below definition, and it is a **connected toast** if it also satisfies property 3:

1. $\bigcup_{K \in \mathcal{T}} E(K) = E(G)$,
2. for every pair $K, L \in \mathcal{T}$ either $(N(K) \cup K) \cap L = \emptyset$ or $K \cup N(K) \subseteq L$, or $L \cup N(L) \subseteq K$,
3. for every $K \in \mathcal{T}$ the induced subgraph on $K \setminus \bigcup_{K \supsetneq L \in \mathcal{T}} L$ is connected.



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1. $\sigma'(e) \in \{0, 1\}$ for all $e \in E(L)$.
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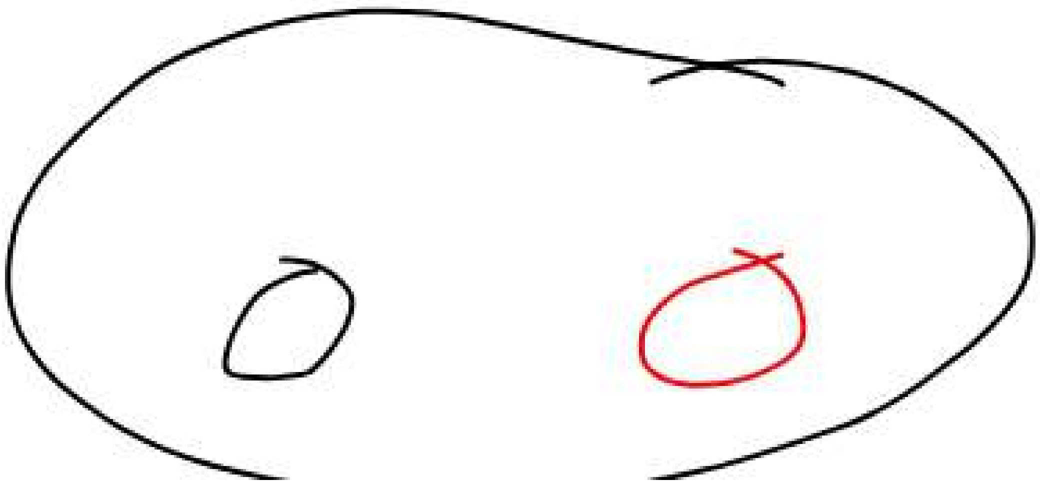

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For every $e \in L$ and $L \subset K \in \mathcal{T}$ there's a cycle in $F(\sigma)$ that's a subset of K and contains e.