An effective strengthening of Mathias' theorem

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Main theorem

Theorem (M. – Törnquist)

For every infinite Σ^1_1 *almost disjoint family* $\mathcal{A}$ *there is a* Δ^1_1 *witness to non-maximality, i.e., there is some* $x \in \Delta^1_1([\mathbb{N}]^\infty)$ *so that* $x \cap z$ *is finite for all* $z \in \mathcal{A}$*.*

Preliminaries

We identify the power set $\mathcal{P}(\mathbb{N})$ with the set $2^{\mathbb{N}}$ of characteristic functions and denote the set of infinite subsets of $\mathbb{N}$ as $[\mathbb{N}]^{\infty}$.

For a set X, we denote by $X^{<\infty}$ the set of finite sequences of elements of X. A *tree* on X is a subset $T \subseteq X^{<\infty}$, which is closed under initial segments. For a tree T, we denote by $[T]$ the set of *infinite branches* through T, which are such $x \in X^{\mathbb{N}}$, so that $x \restriction n \in T$ for every $n \in \mathbb{N}$. We will be most interested in trees on $2 \times \mathbb{N}$.

A subset $\mathcal{A} \subseteq [\mathbb{N}]^{\infty}$ is *(boldface) analytic*, if there is a tree $T \subseteq (2 \times \mathbb{N})^{<\infty}$ so that $\mathcal{A} = p[T]$, where $p : 2^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}$ is the projection on the first component.

Almost disjoint families

For $x, y \in [\mathbb{N}]^\infty$ we say that x and y are *almost disjoint*, if $x \cap y$ is finite. A family $\mathcal{A} \subseteq [\mathbb{N}]^\infty$ is *almost disjoint* (a.d.) if any two $x \neq y \in \mathcal{A}$ are almost disjoint. Such a family is *maximal almost disjoint* (mad) if there is no a.d. $\mathcal{B} \supsetneq \mathcal{A}$.

We focus on infinite mad families, since any finite partition of $\mathbb{N}$ into infinite sets is mad. Note that using a diagonalisation argument there is no countably infinite mad family. Invoking AC in the form of Zorn's lemma it is easy to show that there are infinite mad families.

Question

Are there definable infinite mad families? Are there analytic infinite mad families?

Definable mad families

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Theorem (Mathias)

There is no infinite analytic mad family.

This was first proven by Mathias in the 70s using forcing. Almost four decades later Törnquist found a proof using a derivative argument on a tree.

It was shown by A. W. Miller that if $V = L$ then there is an infinite Π^1_1 mad family.

A shorter proof of Mathias' theorem

We found a shorter version of Asger's proof. The main idea is the same; the new thing is that we don't resort to using *diagonal sequences*. But this means that we have to be more careful when removing parts of the tree.

For $x, y \in [\mathbb{N}]^\infty$ write $x \subseteq^* y$ for $|y \setminus x| < \infty$. If $s \in (2 \times \mathbb{N})^{<\infty}$, then let $s_* \in 2^{<\infty}$ be the first component of s, i.e., if $s = (s_0, s_1)$, then $s_* = s_0$.

For a tree T on $2 \times \mathbb{N}$ and $s \in T$ let $T_{[s]} := \{t \in T : s \leq t \vee t \leq s\}$ be the tree of nodes comparable to s.

Diagonalising lemma

Lemma (Diagonalising lemma)

Suppose that $\mathcal{A}$ is a family of subsets of $\mathbb{N}$ and $\mathcal{B} = (B_n)_{n\in\mathbb{N}}$ is a countable sequence of subsets of $\mathbb{N}$ so that:

(1) $(\forall z \in \mathcal{A})\,(\exists m \in \mathbb{N})\, z \subseteq^* \cup_{n<m} B_n$;

(2) $(\forall m \in \mathbb{N})\, |\mathbb{N} \setminus \cup_{n<m} B_n| = \infty$.

Then there is some $x \in [\mathbb{N}]^\infty$ so that $|x \cap z| < \infty$ for all $z \in \mathcal{A}$.

Proof.

Inductively define $x \in [\mathbb{N}]^\infty$ by picking $x_n \in \mathbb{N} \setminus \cup_{k<n} B_k$, larger than all previous $x_0, \ldots, x_{n-1}$. It is always possible to pick such x_n by condition (2). Setting $x = \{x_n : n \in \mathbb{N}\}$, condition (1) implies that $|x \cap z| < \infty$ for all $z \in \mathcal{A}$. □

Branching lemma

Now fix a countable collection $\mathcal{B}$ of infinite subsets of $\mathbb{N}$. Then define

$$T^{\mathcal{B}} := \{t \in T : (\exists w \in [T_{[t]}])\,(\forall n \in \mathbb{N})\,(\forall B_0, \ldots, B_{n-1} \in \mathcal{B})\, p(w) \not\subseteq^* \cup_{k \in n} B_k\}.$$

Lemma (Branching lemma)

Suppose T is a tree on $2 \times \mathbb{N}$ such that $p[T]$ is almost disjoint and $\mathcal{B}$ a countable family of infinite subsets of $\mathbb{N}$. Suppose $s, t \in T^{\mathcal{B}}$ are incompatible in the first component. Then there are $s' \in T^{\mathcal{B}}$, $s \leq s'$ and $t' \in T^{\mathcal{B}}$, $t \leq t'$ so that for all $s'' \in T^{\mathcal{B}}$, $s' \leq s''$ and all $t'' \in T^{\mathcal{B}}$, $t' \leq t''$ we have $s''_* \cap t''_* = s'_* \cap t'_*$.

Proof. Suppose for contradiction that the lemma fails for $s, t \in T^{\mathcal{B}}$. Set $s_0 := s$ and $t_0 := t$. Then we inductively use the negation of the statement of the lemma on s_n, t_n to get $s_{n+1} \geq s_n$ and $t_{n+1} \geq t_n$ with $(s_n)_* \cap (t_n)_* \subsetneq (s_{n+1})_* \cap (t_{n+1})_*$. Setting $x := \cup_n s_n$ and $y := \cup_n t_n$, we get that $x, y \in [T]$, but $p(x) \cap p(y)$ is infinite, which is a contradiction. □

There are no infinite analytic mad families

Note that for $s, t \in T^{\mathcal{B}}$, we can express the fact that there are no $s', t' \in T^{\mathcal{B}}$ with $s \le s'$, $t \le t'$ and $s_* \cap t_* \subsetneq s'_* \cap t'_*$ as $(\cup p(T^{\mathcal{B}}_{[s]})) \cap (\cup p(T^{\mathcal{B}}_{[t]})) = s_* \cap t_*$.

We denote by Fin the ideal of finite subsets of $\mathbb{N}$ and for $\mathcal{A} \subseteq [\mathbb{N}]^\infty$ we write $\mathcal{I}_{\mathcal{A}}$ for the ideal generated by $\mathcal{A}$ and Fin.

Proof of Mathias' theorem.

Let T be a tree on $2 \times \mathbb{N}$ so that $\mathcal{A} := p[T]$ is an infinite almost disjoint family. We will recursively define a countable family $\mathcal{B}$ of infinite subsets of $\mathbb{N}$, which will satisfy conditions of the Diagonalising lemma. Set $\mathcal{B}_0 := \emptyset$. Suppose we have defined $\mathcal{B}_\alpha$ for $\alpha \le \gamma$ so that

(1) if $\alpha \le \beta \le \gamma$ then $\mathcal{B}_\alpha \subseteq \mathcal{B}_\beta$ and both are countable;

(2) for all $\alpha \le \gamma$, for all $n \in \mathbb{N}$ and all $B_0, \ldots, B_{n-1} \in \mathcal{B}_\alpha$ we have that $\mathbb{N} \setminus \cup_{k<n} B_k \notin \mathcal{I}_{\mathcal{A}}$.

We will now define $\mathcal{B}_{\gamma+1}$. If there are $s, t \in T^{\mathcal{B}_\gamma}$ so that s_*, t_* are incompatible and so that $(\cup p(T^{\mathcal{B}_\gamma}_{[s]})) \cap (\cup p(T^{\mathcal{B}_\gamma}_{[t]})) = s_* \cap t_*$, then consider the following cases:

(1) if $(\forall n \in \mathbb{N})\,(\forall B_0, \ldots, B_{n-1} \in \mathcal{B}_\gamma)\,(\cup p(T^{\mathcal{B}_\gamma}_{[s]})) \setminus \cup_{k<n} B_k \notin \mathcal{I}_\mathcal{A}$, then put $\mathcal{B}_{\gamma+1} := \mathcal{B}_\gamma \cup \{\cup p(T^{\mathcal{B}_\gamma}_{[t]})\}$;

(2) else if $(\forall n \in \mathbb{N})\,(\forall B_0, \ldots, B_{n-1} \in \mathcal{B}_\gamma)\,(\cup p(T^{\mathcal{B}_\gamma}_{[t]})) \setminus \cup_{k<n} B_k \notin \mathcal{I}_\mathcal{A}$, then put $\mathcal{B}_{\gamma+1} := \mathcal{B}_\gamma \cup \{\cup p(T^{\mathcal{B}_\gamma}_{[s]})\}$;

(3) else put $\mathcal{B}_{\gamma+1} := \mathcal{B}_\gamma \cup \{\cup p(T^{\mathcal{B}_\gamma}_{[s]}), \cup p(T^{\mathcal{B}_\gamma}_{[t]})\}$.

It is clear that in all three cases $\mathcal{B}_{\gamma+1}$ still satisfies that for all $n \in \mathbb{N}$ and any $B_0, \ldots, B_{n-1} \in \mathcal{B}_{\gamma+1}$ we have that $\mathbb{N} \setminus \cup_{k<n} B_k \notin \mathcal{I}_\mathcal{A}$, since the intersection of the two potential new sets (the cones above s' and t') is finite and since the condition held for $\mathcal{B}_\gamma$.

If there are no such s, t stop the process and set $\alpha^* := \gamma$ and $\mathcal{B}^* := \mathcal{B}^\gamma$.

Suppose we have defined $\mathcal{B}_\alpha$ for all $\alpha < \lambda$, where λ is countable limit, so that the above conditions (1) and (2) hold. Then let $\mathcal{B}_\lambda := \cup_{\alpha<\lambda}\mathcal{B}_\alpha$. Clearly the conditions are preserved.

The process clearly stops at a countable α^*, since at each step we use some pair s, t which hasn't been used up to that point. Since there are only countably many pairs, the process cannot last for uncountably many steps.

Claim

Any two $s, t \in T^{\mathcal{B}^*}$ are compatible in the first component.

Proof.

Suppose for contradiction that the process stopped, but that there are $s, t \in T^{\mathcal{B}^*}$ which are incompatible in the first component. Applying the Branch lemma, we get some $s' \in T^{\mathcal{B}^*}_{[s]}$ and $t' \in T^{\mathcal{B}^*}_{[t]}$ which satisfy that $(\cup p(T^{\mathcal{B}^*}_{[s']})) \cap (\cup p(T^{\mathcal{B}^*}_{[t']})) = s'_* \cap t'_*$. This is clearly a contradiction. □

Now, consider two further cases:

(1) if $\cup p(T^{\mathcal{B}^*}) \notin \mathcal{I}_{\mathcal{A}}$, then it must hold that $p[T^{\mathcal{B}^*}] = \emptyset$. In this case let $\mathcal{B} := \mathcal{B}^*$.

(2) if $\cup p(T^{\mathcal{B}^*}) \in \mathcal{I}_{\mathcal{A}}$, then let $\mathcal{B} := \mathcal{B}^* \cup \{\cup p(T^{\mathcal{B}^*})\}$.

Note that $\mathcal{B}$ satisfies that for all $n \in \mathbb{N}$ and all $B_0, \ldots, B_{n-1} \in \mathcal{B}$ we have that $\mathbb{N} \setminus \cup_{k<n} B_k \notin \mathcal{I}_{\mathcal{A}}$.

Claim

For all $z \in \mathcal{A}$ there are $n \in \mathbb{N}$ and $B_0, \ldots, B_{n-1} \in \mathcal{B}$ so that $z \subseteq^* \cup_{k<n} B_k$.

Proof.

In case $z \in p[T^{\mathcal{B}^*}]$, we have that $z \in \mathcal{B}$. So suppose that $z \notin p[T^{\mathcal{B}^*}]$. Then there is some $\alpha < \alpha^*$ so that $z \in p[T^{\mathcal{B}_\alpha}] \setminus p[T^{\mathcal{B}_{\alpha+1}}]$. But this means that there are $n \in \mathbb{N}$ and $B_0, \ldots, B_{n-1} \in \mathcal{B}_\alpha$ so that $z \subseteq^* \cup_{k<n} B_k$. □

Finally, observe that $\mathcal{B}$ is countable; if it is infinite, we can end the proof by application of the Diagonalising lemma. If it is finite, just take $x := \mathbb{N} \setminus \cup\mathcal{B}$. Then it is clear that for all $z \in \mathcal{A}$ we have that $x \cap z$ is finite. $\square$

Preliminaries on effectiveness

An element $x \in 2^{\mathbb{N}}$ (or in $2^{(2\times\mathbb{N})^{<\infty}}$, etc.) is *recursive*, denoted by Δ^0_1, if there is a Turing machine deciding the membership of x. A subset $\mathcal{A} \subseteq 2^{\mathbb{N}}$ is Σ^1_1 (*lightface analytic*) if there is a recursive tree $T \subseteq (2\times\mathbb{N})^{<\infty}$ so that $\mathcal{A} = p[T]$. For $a \in \mathbb{N}^{\mathbb{N}}$, we analogously define $\Delta^0_1[a]$ and $\Sigma^1_1[a]$. Then note that analytic sets are exactly $\Sigma^1_1[a]$ sets for a ranging over $\mathbb{N}^{\mathbb{N}}$.

One defines similarly what it means for $x \in 2^{\mathbb{N}}$ to be Σ^1_1. If both x and $\mathbb{N}\setminus x$ are Σ^1_1, we say that x is Δ^1_1 (*hyperarithmetic*). By $\Delta^1_1(2^{\mathbb{N}})$ we denote the set of all $x \in 2^{\mathbb{N}}$ which are Δ^1_1. This set is Π^1_1, i.e., $2^{\mathbb{N}} \setminus \Delta^1_1(2^{\mathbb{N}})$ is Σ^1_1.

Kripke-Platek set theory

Kripke-Platek set theory (KP) is a fragment of ZF, where we omit the power set axiom and restrict separation and collection axioms to Δ_0 formulas. Then one proves that the KP axioms imply Δ_1-separation, Σ_1-collection, (strong) Σ_1-replacement and Σ_1-recursion.

Let ω_1^{CK} denote the least non-recursive ordinal. Then $L_{\omega_1^{\mathsf{CK}}}$, the initial segment of Gödel's constructible universe L, is the smallest ω-model of KP. Moreover, it holds that Δ^1_1 elements of $2^{\mathbb{N}}$ are precisely $L_{\omega_1^{\mathsf{CK}}} \cap 2^{\mathbb{N}}$.

So if we work in $L_{\omega_1^{\mathsf{CK}}}$ and produce $x \in 2^{\mathbb{N}}$, then x is automatically Δ^1_1!

Useful theorems about effectiveness

Theorem (Effective perfect set theorem)

If $A \subseteq 2^{\mathbb{N}}$ *is* Σ^1_1 *and contains* x *not in* Δ^1_1*, then* A *contains a perfect subset.*

Theorem (Spector – Gandy)

The quantifier $\exists x \in \Delta^1_1(2^{\mathbb{N}})$ *may be considered universal.*

Effective theorem

Theorem (M. – Törnquist, 2020)

For every infinite Σ^1_1 almost disjoint family $\mathcal{A}$ there is a Δ^1_1 witness to non-maximality, i.e., there is some $x \in \Delta^1_1([\mathbb{N}]^\infty)$ so that $x \cap z$ is finite for all $z \in \mathcal{A}$.

We adapt some definitions to $L_{\omega_1^{\mathsf{CK}}}$, so that we don't use infinite branches. Let T be a tree on $2 \times \mathbb{N}$ so that $p[T]$ is almost disjoint, let $\mathcal{B}$ be a countable family of subsets of $\mathbb{N}$ and let $s \in T$. Then set

$$x_s^T := \cup\{t_* : t \in T_{[s]}\}$$

and

$$T^{\mathcal{B}} := \{s \in T : (\forall n \in \omega)\,(\forall B_0, \dots, B_{n-1} \in \mathcal{B})\, x_s^T \setminus \cup_{k<n} B_k \notin \mathsf{Fin}\}.$$

Sketch of Proof.

Assume for contradiction that there were a recursive tree T on $2 \times \mathbb{N}$ such that $\mathcal{A} := p[T]$ is an infinite almost disjoint family, so that for every $x \in \Delta^1_1([\mathbb{N}]^\infty)$ there is some $z \in \mathcal{A}$ with $x \cap z$ infinite. Then we have that for any $x \in \Delta^1_1([\mathbb{N}]^\infty)$, the question "$x \in \mathcal{A}$" is Δ^1_1.

Claim

There is a Π^1_1 predicate φ such that if $x \in \Delta^1_1([\mathbb{N}]^\infty)$ then $x \in \mathcal{I}_\mathcal{A}$ iff $\varphi(x)$. In particular, for $x \in \Delta^1_1([\mathbb{N}]^\infty)$, it is Δ^1_1 to say "$x \in \mathcal{I}_\mathcal{A}$".

Proof.

Let $x \in \Delta^1_1([\mathbb{N}]^\infty)$, and assume that $x \in \mathcal{I}_\mathcal{A}$. Then

$$\{z \in \mathcal{A} : |x \cap z| = \infty\}$$

is a finite Σ^1_1 set, so by the Effective perfect set theorem it consists of finitely many Δ^1_1 reals.

It follows that for $x \in \Delta^1_1([\mathbb{N}]^\infty)$,

$$x \in \mathcal{I}_{\mathcal{A}} \iff (\exists \vec{z} \in \Delta^1_1(([\mathbb{N}]^\infty)^{<\infty}))\ (\forall i < \mathsf{lh}(\vec{z}))\ \vec{z}_i \in \mathcal{A} \wedge x \subseteq^* \bigcup_{i<\mathsf{lh}(\vec{z})} \vec{z}_i,$$

and the right hand side is a Π^1_1 predicate by the Spector – Gandy theorem and the observation that "$\vec{z}_i \in \mathcal{A}$" is Δ^1_1 since $\vec{z}_i \in \Delta^1_1([\mathbb{N}]^\infty)$. □

We work in $L_{\omega_1^{\mathsf{CK}}}$, and by the above, we can use the question "$x \in \mathcal{I}_{\mathcal{A}}$" as a part of our derivative process. But why does the process stop before ω_1^{CK}?

At successor steps, instead of taking arbitrary s, t so that s_*, t_* are incompatible, we have to be more careful. There will be certain steps, when we will enumerate all such s, t. Then we will use the least pair, not yet used and still not covered. When we run out of such pairs, we make a new enumeration. Let β_α denote the step, when we enumerate the pairs for the αth time.

Define the relation $\prec$ on $T \times T$ by

$$(s,t) \prec (s',t') \iff s' \leq s \wedge t' \leq t \wedge s'_* \cap t'_* \subsetneq s_* \cap t_*.$$

Since $p[T]$ is almost disjoint in V, we get that $\prec$ is well founded in V. Hence $L_{\omega_1^{\mathsf{CK}}}$ also thinks that $\prec$ is well-founded and correctly calculates its rank. Let $\Gamma := \mathsf{rk}(\prec) < \omega_1^{\mathsf{CK}}$.

Claim

Suppose that (s,t) was listed at step β_γ. Then $\mathsf{rk}(\prec \restriction \{(s',t') : (s',t') \prec (s,t)\}) \geq \gamma$.

Proof.

By induction on γ. □

Suppose for contradiction that the process doesn't stop before ω_1^{CK}. Then $\beta_{\Gamma+1}$ is defined and less than ω_1^{CK}. Let (s,t) be some pair listed at step $\beta_{\Gamma+1}$. Then $\mathsf{rk}(\prec \restriction \{(s',t') : (s',t') \prec (s,t)\}) \geq \Gamma + 1$, which is a contradiction with $\mathsf{rk}(\prec) = \Gamma$. Moreover, this tells us that the process stops before $\omega \cdot (\Gamma + 1) < \omega_1^{\mathsf{CK}}$.

The rest of the proof is the same. □

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The same argument can be used with minor changes to give new proofs of the following:

Theorem (Törnquist)

Let $a \in \mathbb{N}^{\mathbb{N}}$ and suppose that $\aleph_1^{L[a]} < \aleph_1$. Then there are no infinite $\Sigma^1_2[a]$ mad families.

Note that $\Sigma^1_2[a]$ subsets of $2^{\mathbb{N}}$ are of the form $p[T]$, where $T \in L$ is a tree on $2 \times \omega_1$.

Theorem (Törnquist)

If $\mathsf{MA}(\kappa)$ *holds for some $\kappa < 2^{\aleph_0}$ then there are no infinite κ-Suslin mad families.*

Recall that $\mathcal{A} \subseteq 2^{\mathbb{N}}$ is κ-Suslin if it is of the form $p[T]$, for T a tree on $2 \times \kappa$.

Ongoing work

We conclude with some open problems, which we are work in progress.

Theorem (Haga – Schrittesser – Törnquist)

For any $\alpha < \omega_1$, there are no infinite analytic Fin^α-mad families.

Question

Is there a derivative argument for "there are no analytic Fin^α mad families"? If yes, does the derivative argument effectivise?

Question

Does it hold that there is a Δ^1_1 witness to non-maximality for each infinite Σ^1_1 Fin^α-mad family, for all $\alpha < \omega_1^{\mathsf{CK}}$?

Something to take home

(1) Simplified proof of "there are no infinite analytic mad families" using a derivative argument on a tree.

(2) The derivative argument is effective enough so that it can be carried out in $L_{\omega_1^{\mathsf{CK}}}$.

(3) Moreover, the process stops before ω_1^{CK}, thanks to well-foundedness.

(4) So we have an effective strengthening of Mathias' theorem.

(5) The same argument works with minor changes to give new proofs of facts about Σ^1_2 mad families and κ-Suslin mad families.

THANK YOU!

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