

A dichotomy for σ -Baire class α functions

Andrew Marks (UCLA), joint with Antonio Montalbán (Berkeley)

Work in progress

Dec 8, 2021

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- ▶ Some of these types of priority argument proofs can be made simpler using recent ideas from computability theory. In particular we show that Montalbán's recent (2020) “game metatheorem” can be adapted to descriptive set theory. This “topological metatheorem” is about the existence of a strategy in certain game so that result of playing the game for every input yields a continuous function.

Piecewise continuous functions

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Let $x = \bigoplus_n z_n$. Then for every n , $x \geq_T g_n(x)$. Since $x' = j(x) = g_n(x)$ for some n , we therefore have $x \geq_T x'$. Contradiction! \square

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It is a theorem of Solecki-Zapletal that the Turing jump is a basis for all non σ -continuous Borel functions.

Continuous reducibility (Hertling-Weihrauch '94)

If $f: X \rightarrow Y$ and $g: Z \rightarrow W$, say that g is **continuously reducible** to f noted $g \leq_c f$ if there are a continuous $\phi: Z \rightarrow X$ and partial continuous $\psi: Y \rightarrow W$ so that $g = \psi \circ f \circ \phi$.

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The Solecki-Zapletal dichotomy

The Turing jump is the basis for all non σ -continuous functions!

Theorem (Solecki 98, Zapletal 04)

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This theorem can be sharpened so that the continuous reduction becomes a topological embedding. The dichotomy in Solecki and Zapletal is stated in terms of the Pawlikowski function, but the Pawlikowski function is continuously equivalent to the Turing jump.

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A proof sketch of Solecki-Zapletal

Assume $f: \omega^\omega \rightarrow \omega^\omega$. By relativization, we may assume that f is lightface Δ_1^1 . By a standard reflection argument $C = \bigcup \{A: A \text{ is } \Sigma_1^1 \text{ and } f \upharpoonright A \text{ is continuous}\}$ is a Π_1^1 set. Hence, its complement C^c is a Σ_1^1 set and $f \upharpoonright A$ is not continuous for any Σ_1^1 set $A \subseteq C^c$

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The key idea is that if $f \upharpoonright A$ is not continuous, we can find a basic open neighborhood N_t so that $(f \upharpoonright A)^{-1}(N_t)$ is not relatively open in A . Let $A^* = (f \upharpoonright A)^{-1}(N_t) \setminus \{N_r: f(A \cap N_r) \subseteq N_t\}$. Suppose we are building some $\phi(x) \in 2^\omega$ to be inside A^* so that $f(\phi(x)) \subseteq N_t$. The point is that even if we have committed to $\phi(x) \subseteq N_r$ for an arbitrarily long r , since $N_r \cap (f \upharpoonright A)^{-1}(2^\mathbb{N} \setminus N_t)$ is nonempty we can at any point switch to building an element of this set instead so that $\phi(x) \subseteq 2^\mathbb{N} \setminus N_t$.



Generalizing the Solecki-Zapletal dichotomy

A function is Baire class 0 iff it is continuous and f is Baire class α if f is the pointwise limit of a sequence of functions f_n where each f_n is Baire class α_n for some $\alpha_n < \alpha$. In terms of the Borel hierarchy, f is Baire class α if the preimage of every open set is $\Sigma_{\alpha+1}^0$.

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Recall for every α there is a complete Baire class α function under continuous reducibility, and any complete Baire class $\alpha + 1$ function is not σ -Baire class α (a countable union of Baire class α partial functions).

Theorem (M.-Montalbán)

Suppose $f : X \rightarrow Y$ is a Borel function between Polish spaces. Then exactly one of the following holds:

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The proof is similar in spirit to the $\alpha = 0$ (Solecki-Zapletal) case, except it uses a $0^{(\alpha+1)}$ -injury priority argument.

Priority arguments in descriptive set theory

Recently the field of descriptive set theory has been making some serious use of priority arguments. Computability theory uses priority arguments to construct simple objects (e.g. c.e. sets) with complicated properties (e.g. controlling what they can compute). Likewise the uses of priority arguments in descriptive set theory build simple objects (e.g. continuous functions) with complicated properties (e.g. being a Wadge reduction between two complicated sets).

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These new uses are more than simply relativizing constructions from computability. These arguments often construct a tree of “approximations” like in the Solecki-Zapletal argument and along each branch a priority argument is taking place. However, additional complexity comes from how the outcomes along different branches now interact with each other (the construction is more “global”). These constructions also use deep results of effective descriptive set theory (e.g. Louveau’s analysis of $\Sigma_1^1 \cap \Pi_\alpha^0$) to find the correct types of approximations that can appropriately respond to the flow of the construction.

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We adapt a new tool for conducting priority arguments: we prove a topological adaptation of Montalbán's recent game metatheorem.

Choquet games

Suppose $\mathbb{A} \subseteq \mathcal{P}(X)$. The **strong Choquet game** $G_{\mathbb{A}}$ is the two player game where the players alternate playing sets $A_i, B_i \in \mathbb{A}$:

$$\begin{array}{rcccc} \text{I} & x_0, A_0 & & x_1, A_1 & \dots \\ \text{II} & & B_0 & & B_1 \end{array}$$

where $A_0 \supseteq B_0 \supseteq A_1 \supseteq B_1 \dots$ and $x_i \in B_i$. We say that player II wins the game if $\bigcap_i A_i = \bigcap_i B_i \neq \emptyset$.

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where $A_0 \supseteq B_0 \supseteq A_1 \supseteq B_1 \dots$ and $x_i \in B_i$. We say that player II wins the game if $\bigcap_i A_i = \bigcap_i B_i \neq \emptyset$.

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- ▶ The Σ_1^1 subsets of ω^{ω} (Gandy).

A warning

We will not use any determinacy in this talk!

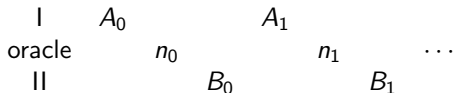
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The point of Choquet games is to simplify dealing with issues of Baire category, density arguments, etc.

The game $G_{\mathbb{A}, \mathbb{B}}$

Given sets $\mathbb{A}, \mathbb{B} \subseteq \mathcal{P}(\omega^\omega)$, consider the game:



We think of the moves of the oracle as fixed, and not responding to players I and II.

So on turn i ,

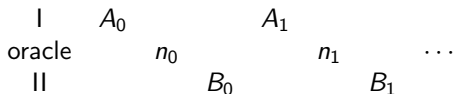
- ▶ Player I plays $A_i \in \mathbb{A}$ where $A_i \subseteq B_{i-1}$
- ▶ The oracle plays $n_i \in \mathbb{N}$
- ▶ Player II plays $B_i \in \mathbb{B}$ where $B_i \cap A_i \neq \emptyset$ and $B_i \subseteq B_{i-1}$.

Player II wins if $\bigcap_i B_i$ is a singleton.

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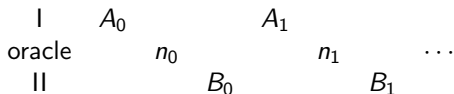
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If σ is a strategy for I, $y \in \omega^\omega$ are the moves of the oracle, and τ is a winning strategy for II, let $\sigma * y * \tau$ be the outcome of the game when it is played using these strategies. That is, the real z where $\{z\} = \bigcap_i B_i$.

Topological adaptation of Montalbán's game metatheorem.

$$G_{\mathbb{A}, \mathbb{B}} : \begin{array}{ccccccc} & I & A_0 & & A_1 & & \\ \text{oracle} & & n_0 & & n_1 & & \dots \\ & II & & & B_0 & & B_1 \end{array}$$

Theorem (M.-Montalbán)

Fix $\alpha < \omega_1$. Let \mathbb{A} be all Σ_1^1 sets and \mathbb{B} be all Π_α^0 (HYP) sets. There is a complete Baire class α function $T_\alpha : \omega^\omega \rightarrow \omega^\omega$ so:

for every strategy σ for I, there is a winning strategy τ for II so $x \mapsto \sigma * T_\alpha(x) * \tau$ is continuous.

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It is trivial to find a winning strategy for II so that $x \mapsto \sigma * T_\alpha(x) * \tau$ is Baire class α . The magic of the theorem is making this map continuous. $T_\alpha: \omega^\omega \rightarrow \omega^\omega$ is a "Skolemized" version of an appropriate iterate of the Turing jump. (It is the set of α -true stages of x).

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More generally, the topological metatheorem is true when $\mathbb{B} \subseteq \mathbb{A} \subseteq \mathcal{P}(X)$ are countable collections of subsets of a Polish space X , \mathbb{B} consists of Π_α^0 sets, and \mathbb{A}, \mathbb{B} satisfy certain closure properties.

An example application: Borel Wadge determinacy

The following theorem is a trivial consequence of Borel determinacy, but it was a long open question whether this theorem was provable without the full power of Borel determinacy.

Theorem (Louveau and Saint-Raymond '88 in second order arithmetic)

Suppose X is a Polish space and $C \subseteq X$ is Borel and not $\Sigma_{\alpha+1}^0$. Then a complete $\Pi_{\alpha+1}^0$ set is Wadge reducible to C .

Proof. By relativization, we may assume that C is Δ_1^1 . Let

$$A_0 = C \setminus \{B : B \text{ is } \Pi_{\alpha}^{0,\text{hyp}} \text{ and } B \subseteq C\}.$$

Note that A_0 is Σ_1^1 . We must have $A_0 \neq \emptyset$ since otherwise B would be $\Sigma_{\alpha+1}^0$.

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If $W_e^{s \uparrow}$, then on turn n I plays a set $A_n \subseteq B_{n-1} \cap A_{n-1}$, using a winning strategy for II in the Choquet game on Σ_1^1 sets to ensure that if

$W_e^{T_\alpha(x) \uparrow}$, then the moves of player I $A_0 \supseteq A_1 \supseteq A_2 \dots$ have nonempty intersection. Hence, the outcome of the game is in $\bigcap_i A_i$ and hence in A_0 and thus in C .

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If $W_e^s \downarrow$, and s is the string of shortest length so this is true, then player I plays the set $A_n = B_{n-1} \setminus C$. This set is nonempty since $B_{n-1} \cap A_{n-1} \neq \emptyset$, $A_{n-1} \subseteq \dots \subseteq A_0$, and by the definition of A_0 . On subsequent moves I plays according to a winning strategy for the Choquet game on Σ_1^1 sets so that $A_n \supseteq A_{n+1} \supseteq \dots$ have nonempty intersection.

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$$W_e^{T_\alpha(x)} \downarrow \iff f(x) \in C.$$

Hence, f is a Wadge reduction from a complete $\Pi_{\alpha+1}^0$ set to C . □

Finishing the dichotomy for Baire class α functions

Our proof of the dichotomy for σ -Baire class α functions is done using Montalbán's full true stages machinery. The true stages machine iteratively defines partial orders $(\leq_\beta)_{\beta \leq \alpha}$ which approximate the truth of Σ_β^0 sentences by defining trees with unique branches through them.

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So the full true stages proof is currently our best proof of the dichotomy theorem. A hybrid approach which may work is a "tree" game metatheorem that takes place more directly on a tree, instead of having an oracle in the game.

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- ▶ Is continuous reducibility on all functions on $2^\mathbb{N}$ a well-quasi-order assuming AD^+ ?
- ▶ Are there versions of Montalbán's game metatheorem beyond the Borel?

Thanks!