Descriptive complexity of Banach spaces and generic objects

Michal Doucha

Czech Academy of Sciences
joint work with Marek Cúth, Martin Doležal and Ondřej Kurka
and a work in progress with Marek Cúth and Noé de Rancourt

November 17, 2021

Plan of the talk

1. Polish spaces of separable Banach spaces and their topological properties
2. Characterization of the Hilbert space
3. L_p-spaces.
4. Fraïssé Banach spaces and generic Banach spaces

Standard Borel spaces of metric structures

Recall that a standard Borel space is a set with a σ-algebra with which it is isomorphic to an uncountable Polish space with its σ-algebra of Borel subsets.

Let $\mathcal{C}$ be a class of separable metric structures and suppose that $\mathcal{C}$ admits a universal element $X \in \mathcal{C}$. The hyperspace $F(X)$ of closed subsets of X usually doesn't admit a canonical topology, but it does admit a canonical standard Borel structure. It often happens that $\mathcal{C}$ can be identified with a (uncountable) Borel subset of $F(X)$, so it is a standard Borel space of its own.

Example

- Let $\mathbb{U}$ be the Urysohn universal metric space. Then $F(\mathbb{U})$ can serve as a standard Borel space of all Polish metric spaces.
- (Bossard) The subset $\mathbb{X} \subseteq F(C(2^{\mathbb{N}}))$ of linear subspaces is Borel, so it can serve as the standard Borel space of separable Banach spaces.

Banach spaces and descriptive set theory

Sample theorem (Szlenk, Bourgain)

There is no separable reflexive space containing an isometric copy of every separable reflexive space.

The set of subspaces of a fixed space is Borel, while the set of all reflexive spaces is coanalytic and non-Borel.
This stream of research has been extended recently by Argyros and Dodos and is presented in the monograph:
P. Dodos, *Banach spaces and descriptive set theory: selected topics*, vol. 1993 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2010.

Admissible topologies

Let X be a Polish space and $F(X)$ the standard Borel space of closed subsets of X. Althought there is no canonical topology on $F(X)$, there are many natural topologies there (e.g. the Vietoris topology inherited from $F(\hat{X})$, where $\hat{X}$ is a compactification of X, the Wijsman topology, etc.)

Admissible topologies (Godefroy, Saint-Raymond 2018)

They introduce certain axioms for topologies on $F(X)$, where X is typically a (isometrically universal) separable Banach space that guarantee

- All the natural topologies on $F(X)$ satisfy these axioms.
- The set of linear subspaces of $F(X)$ are G_δ in these topologies.
- The Borel complexities with respect to different admissible topologies 'vary little' (the identity function is of Baire class 1).

Polish spaces of (pseudo)norms

Let V be the countable infinite dimensional vector space over $\mathbb{Q}$ (imagine finitely supported functions from $\mathbb{N}$ to $\mathbb{Q}$). Let $(e_i)_{i\in\mathbb{N}}$ be a basis.

Definition

Let $\mathcal{P} \subseteq \mathbb{R}^V$ be the set of all pseudonorms on V. This is a closed subset of $\mathbb{R}^V$.
For every $\mu \in \mathcal{P}$ denote by X_μ the Banach space obtained by taking the completion of (V, μ) and taking a quotient by the subspace, where μ vanishes.

Facts

- Let $\mathcal{P}_\infty \subseteq \mathcal{P}$ be the subset $\{\mu \in \mathcal{P} \colon X_\mu \text{ is infinite-dimensional}\}$. $\mathcal{P}_\infty$ is G_δ.
- Let $\mathcal{B} \subseteq \mathcal{P}_\infty$ be the subset $\{\mu \in \mathcal{P}_\infty \colon \mu \text{ is a norm on } \mathbb{R} \otimes_{\mathbb{Q}} V\}$. $\mathcal{B}$ is G_δ.

Polish spaces of (pseudo)norms

Facts

- Let $\mathcal{B} \subseteq \mathcal{P}_\infty$ be the subset $\{\mu \in \mathcal{P}_\infty \colon \mu \text{ is a norm on } \mathbb{R} \otimes_{\mathbb{Q}} V\}$. That is, for every finite $I \subseteq \mathbb{N}$ and $(\alpha_i)_{i \in I} \subseteq \mathbb{R} \setminus \{0\}$, $\mu(\sum_{i \in I} \alpha_i e_i) \neq 0$.
  $\mathcal{B}$ is G_δ.

Fact

- For every separable Banach space X there is $\mu \in \mathcal{P}$ such that $X \equiv X_\mu$.
- For every infinite-dimensional separable Banach space X there is $\mu \in \mathcal{B}$ such that $X \equiv X_\mu$.

Pick $(f_i)_{i \in \mathbb{N}} \subseteq X$ of linearly independent vectors such that their linear span is dense in X. Then set

$$\mu(\sum_{i \in I} \alpha_i e_i) = \| \sum_{i \in I} \alpha_i f_i \|_X.$$

Comparison with admissible topologies

Theorem

1. For every isometrically universal Banach space X and an admissible topology τ on $SB(X) \subseteq F(X)$ there is a continuous reduction $\varphi : (SB(X), \tau) \to \mathcal{P}$. In particular, continuous reduction from $(SB_\infty(X), \tau)$ to $\mathcal{P}_\infty$.
2. There is a Baire class 1 reduction from $\mathcal{P}_\infty$ to $\mathcal{B}$.
3. For every isometrically universal X and admissible topology τ there is a Baire class 1 reduction $\varphi : \mathcal{B} \to (SB(X), \tau)$.

Topological properties of the Polish spaces

Definition

Let X and Y be Banach spaces. We say that X is *finitely representable* in Y if for every finite-dimensional subspace $E \subseteq X$ and every $\varepsilon > 0$ there exists $T : E \to Y$ which is an isomorphism with its range and $\|T\|\|T^{-1}\| \leq 1 + \varepsilon$. Equivalently, any non-trivial ultrapower of Y contains X isometrically.

If $\mathcal{C}$ is a family of Banach spaces, we say that X is finitely representable in $\mathcal{C}$ if for every finite-dimensional subspace $E \subseteq X$ and every $\varepsilon > 0$ there exist $Y \in \mathcal{C}$ and $T : E \to Y$ which is an isomorphism with its range such that $\|T\|\|T^{-1}\| \leq 1 + \varepsilon$.

Topological properties of the Polish spaces

Let $\mu \in \mathcal{B}$, denote by $\langle X_\mu \rangle_{\equiv}^{\mathcal{B}}$ the set $\{\nu \in \mathcal{B} \colon X_\nu \equiv X_\mu\}$.

Proposition

Let $\mathcal{C} \subseteq \mathcal{B}$ be such that $\langle X_\mu \rangle_{\equiv}^{\mathcal{B}}$ for every $\mu \in \mathcal{C}$. Then

$$\{\nu \in \mathcal{B} \colon X_\nu \text{ is finitely representable in } \mathcal{C}\} = \overline{\mathcal{C}}.$$

In particular, for any $\mu \in \mathcal{B}$,

$$\{\nu \in \mathcal{B} \colon X_\nu \text{ is finitely representable in } X_\mu\} = \overline{\langle X_\mu \rangle_{\equiv}^{\mathcal{B}}}.$$

Complexity of isometry classes and isomorphism classes

Fact 1

For each $\mu \in \mathcal{B}$, $\langle \mu \rangle_{\equiv}^{\mathcal{B}}$ is Borel in $\mathcal{B}$.

By Melleray, the relation of linear isometry of Banach spaces is Borel bireducible with an orbit equivalence relation. Orbit equivalence relations have Borel classes of equivalence.

Let $\simeq$ denote the relation of linear isomorphism and for each $\mu \in \mathcal{B}$ let $\langle \mu \rangle_{\simeq}^{\mathcal{B}}$ denote the set $\{\nu \in \mathcal{B} \colon X_\nu \simeq X_\mu\}$.

Fact 2

The isomorphism classes $\langle \mu \rangle_{\simeq}^{\mathcal{B}}$ are in general analytic and non-Borel.

By Ferenczi, Louveau and Rosendal, the relation of linear isomorphism of Banach spaces is complete analytic.

Hilbert space

Theorem

The separable infinite-dimensional Hilbert space is the unique Banach space such that $\langle X \rangle_{\equiv}$ is closed in $\mathcal{B}$.

One way is easy. Recall that the Hilbert space norm is the unique norm satisfying the *parallelogram law*. That is, for every $x, y \in X$

$$2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2.$$

Hilbert space

Theorem (Dvoretzky)

Hilbert spaces are finitely representable in every infinite-dimensional Banach space.

Suppose that X is an infinite-dimensional Banach space such that $\langle X \rangle_\equiv$ is closed in $\mathcal{B}$. But by Dvoretzky's theorem, $\ell_2(\mathbb{N}) \in \overline{\langle X \rangle_\equiv} = \langle X \rangle_\equiv$, so $\ell_2(\mathbb{N}) \equiv X$.

Hilbert space

Theorem

The separable infinite-dimensional Hilbert space is characterized as the unique Banach space whose isomorphism class, i.e. $\{\mu \in \mathcal{B}\colon X_\mu \simeq \ell_2(\mathbb{N})\}$, is F_σ.

Hilbert space

Definition

Let X be a Banach space.
Let $p \in [1,2]$. X has *type* p if there is $C > 0$ such that for every finite set $\{x_1, \ldots, x_n\} \subseteq X$ we have

$$\Big(\mathrm{Average}_{\pm} \|\sum_{i=1}^{n} \pm x_i\|^p\Big)^{1/p} \leq C\Big(\sum_{i=1}^{n} \|x_i\|^p\Big)^{1/p}.$$

Let $p \in [2,\infty)$. X has *cotype* p if there is $C > 0$ such that for every finite set $\{x_1, \ldots, x_n\} \subseteq X$ we have

$$\Big(\sum_{i=1}^{n} \|x_i\|^p\Big)^{1/p} \leq C\Big(\mathrm{Average}_{\pm} \|\sum_{i=1}^{n} \pm x_i\|^p\Big)^{1/p}.$$

Clearly, for fixed p, the spaces with type p, resp. cotype p, form an F_σ subset (in $\mathcal{B}$).

Hilbert space

Theorem (Kwapień)

Hilbert space is the unique Banach space (up to isomorphism) that has type and cotype 2.

Corollary

$\langle \ell_2(\mathbb{N}) \rangle^{\mathcal{B}}_{\simeq}$ is F_σ.

Hilbert space

The homogeneous subspace problem (Komorowski and Tomczak-Jaegermann, Gowers)

$\ell_2(\mathbb{N})$ is the unique infinite-dimensional Hilbert space (up to isomorphism) whose every infinite-dimensional closed subspace is isomorphic to itself.

Suppose that X is an infinite-dimensional Banach space such that $\langle X\rangle_{\simeq}^{\mathcal{B}} = \bigcup_n F_n$, where $F_n \subseteq \mathcal{B}$ are closed for each n. Assuming X is not isomorphic to $\ell_2(\mathbb{N})$ we may fix some infinite-dimensional subspace $Y \subseteq X$ not isomorphic to X.

Hilbert space

Let $\mathcal{T}$ denote the set of finite tuples (including empty) of natural numbers without repetition. For every $\gamma \in \mathcal{T}$ and every $\mu \in \mathcal{B}$ we put $\mathbb{M}_\mu^\gamma$ to be the set

$$\Big\{\nu \in \mathcal{B}\colon \text{for every } (a_i)_{i=1}^{|\gamma|} \in \mathbb{Q}^{|\gamma|} \text{ we have } \nu\Big(\sum_{i=1}^{|\gamma|} a_i e_i\Big) = \mu\Big(\sum_{i=1}^{|\gamma|} a_i e_{\gamma(i)}\Big)\Big\}$$

Claim

For every $\mu \in \mathcal{B}$ with $X_\mu \simeq X$ there are $\gamma \in \mathcal{T}$ and $m \in \mathbb{N}$ such that $\mathbb{M}_\mu^{\gamma'} \cap F_m \neq \emptyset$, for every $\gamma' \supseteq \gamma$.

Hilbert space

Let $(x_i)_{i\in\mathbb{N}}$ be a set of linearly independent vectors in X such that

- their span is dense in X,
- there exists an infinite subset $I \subseteq \mathbb{N}$ such that the span of $\{x_i \colon i \in I\}$ is a dense subspace of Y.

Define $\mu \in \mathcal{B}$ by

$$\mu(\sum_{j\in\mathbb{N}} \alpha_j e_j) = \|\sum_{j\in\mathbb{N}} \alpha_j x_j\|_X.$$

We have $X_\mu \equiv X$.
We apply the previous claim to this μ to get $\gamma \in \mathcal{T}$ and $m \in \mathbb{N}$ such that $\mathbb{M}_\mu^{\gamma'} \cap F_m \neq \emptyset$, for every $\gamma' \supseteq \gamma$.

Let $(x_i)_{i\in\mathbb{N}}$ be a set of linearly independent vectors in X such that

- their span is dense in X,
- there exists an infinite subset $I \subseteq \mathbb{N}$ such that the span of $\{x_i \colon i \in I\}$ is a dense subspace of Y.

We apply the previous claim to this μ to get $\gamma \in \mathcal{T}$ and $m \in \mathbb{N}$ such that $\mathbb{M}_\mu^{\gamma'} \cap F_m \neq \emptyset$, for every $\gamma' \supseteq \gamma$.

There exists $\mathrm{rng}(\gamma) \subseteq \tilde{I} \subseteq \mathbb{N}$ such that $|I \setminus \tilde{I}| = |\tilde{I} \setminus I| < \infty$. For the closed linear span of $\{x_i \colon i \in \tilde{I}\}$, denoted by Z, we have $Z \simeq Y$. Let $\varphi : \mathbb{N} \to \tilde{I}$ be a bijection with $\varphi \supseteq \gamma$. We define $\nu \in \mathcal{B}$ by

$$\nu\Big(\sum_{i=1}^{k} a_i e_i\Big) := \mu\Big(\sum_{i=1}^{k} a_i e_{\varphi(i)}\Big), \quad k \in \mathbb{N},\ (a_i)_{i=1}^{k} \in \mathbb{Q}^k.$$

Claim

We have $X_\nu \equiv Z$ and $\nu \in F_m$

That will be a contradiction since X_ν is not isomorphic to $X \equiv X_\mu$.
It suffices to show that for every $v_1, \ldots, v_l \in V$ and $\varepsilon > 0$ there is $\mu' \in F_m$ such that $|\mu'(v_j) - \nu(v_j)| < \varepsilon$ for every $j \leq l$.
Let $L \in \mathbb{N}$, $L \geq |\gamma|$, be such that $v_1, \ldots, v_l \in \mathrm{span}\{e_i \colon i \leq L\}$.
Since $\varphi|_{\{1,\ldots,L\}} \supseteq \gamma$, we may pick $\mu' \in \mathbb{M}_\mu^{\varphi|_{\{1,\ldots,L\}}} \cap F_m$. Then

$$\mu'\Big(\sum_{i=1}^{L} a_i e_i\Big) = \mu\Big(\sum_{i=1}^{L} a_i e_{\varphi(i)}\Big) = \nu\Big(\sum_{i=1}^{L} a_i e_i\Big), \quad (a_i)_{i=1}^{L} \in \mathbb{Q}^L.$$

Complexity of isomorphism classes

Fact

No Banach space can have a closed isomorphism class. The only potential candidate for a G_δ isomorphism class is the Gurarii space.

1. For every $\mu \in \mathcal{B}$, $\langle \mu \rangle_{\simeq}^{\mathcal{B}}$ is dense in $\mathcal{B}$.
2. The isometry class of the Gurarii space is dense G_δ in $\mathcal{B}$. Thus any space with G_δ isomorphism class must be isomorphic to the Gurarii space.

Theorem

Johnson, Lindenstrauss and Schechtman introduced the notion of a Banach space *determined by its pavings*. Any such space has $G_{\delta,\sigma}$ isomorphism class.

L_p spaces

Theorem

For every $1 \leq p < \infty$, $p \neq 2$, we have that

1. the isometry class of $L_p[0,1]$ is G_δ-complete;
2. the isometry class of ℓ_p is $F_{\sigma,\delta}$-complete.

L_p spaces

Definition

Fix $1 \leq p \leq \infty$. A Banach space X is called an $\mathscr{L}_{p,1+}$-space if for every finite-dimensional $E \subseteq X$ and $\varepsilon > 0$ there is a finite-dimensional $E \subseteq F \subseteq X$ and a linear isomorphism $T : F \to \ell_p^n$ with $\|T\|\|T^{-1}\| \leq 1 + \varepsilon$.

Theorem

For a fixed $1 \leq p \leq \infty$, the class of $\mathscr{L}_{p,1+}$-spaces is G_δ.

Theorem (Lindenstrauss, Pełczyński)

For $1 \leq p < \infty$, a separable $\mathscr{L}_{p,1+}$-space is isometric to $L_p(\mu)$, where μ is a σ-finite measure on a standard Borel space.

Theorem

Let $X = L_p(\mu)$ for a σ-finite measure.

1. $X \equiv L_p[0,1]$ if and only if

$$\forall x \in S_X\ \forall \varepsilon > 0\ \forall \delta > 0\ \exists x_1, x_2 \in X$$
$$\big((x_1, x_2) \sim_{1+\varepsilon} \ell_p^2 \text{ and } \|2^{1/p}x - x_1 - x_2\| < \delta\big).$$

2. $X \equiv \ell_p$ if and only if

$$\forall x \in S_X\ \forall \delta \in (0,1)\ \exists \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall x_1, \ldots, x_N \in X$$
$$\big((N^{1/p}x_i)_{i \leq N} \sim_{1+\varepsilon} \ell_p^N \Rightarrow \|x - \sum_{i=1}^{N} x_i\| > \delta\big).$$

G_δ isometry classes

Theorem

1. The Gurarii space has a dense G_δ isometry class.
2. No other isometrically universal Banach space can have G_δ isometry class.
3. No other Banach space with trivial cotype can have G_δ isometry class.

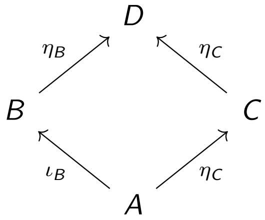
For (2), such X would have also dense isometry class.

For (3), by Maurey-Pisier, trivial cotype implies that ℓ_∞ is finitely representable in such a space X. Since any Banach space is finitely representable in ℓ_∞, it follows that X has a dense isometry class.

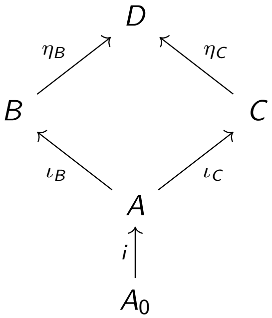
Fraïssé theory (of Banach spaces)

Let $\mathcal{K}$ be a class of finite relational structure in some fixed (finite) language. We say that $\mathcal{K}$ has the joint embedding property if for every $A, B \in \mathcal{K}$ there is $C \in \mathcal{K}$ and embeddings of A and B into C. Moreover, we say that $\mathcal{K}$ has

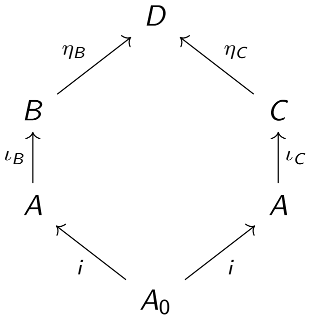
- **the amalgamation property** if for every $A, B, C \in \mathcal{K}$ and embeddings $\iota_B : A \to B$ and $\iota_C : A \to C$ there exist $D \in \mathcal{K}$ and embeddings $\eta_B : B \to D$ and $\eta_C : C \to D$ such that the following diagram commutes.



- **the cofinal amalgamation property** if for every $A_0 \in \mathcal{K}$ there is $A \in \mathcal{K}$ and an embedding $i : A_0 \to A$ so that for every $B, C \in \mathcal{K}$ and embeddings $\iota_B : A \to B$ and $\iota_C : A \to C$ there exist $D \in \mathcal{K}$ and embeddings $\eta_B : B \to D$ and $\eta_C : C \to D$ such that the following diagram commutes.



- **the weak amalgamation property** if for every $A_0 \in \mathcal{K}$ there is $A \in \mathcal{K}$ and an embedding $i : A_0 \to A$ so that for every $B, C \in \mathcal{K}$ and embeddings $\iota_B : A \to B$ and $\iota_C : A \to C$ there exist $D \in \mathcal{K}$ and embeddings $\eta_B : B \to D$ and $\eta_C : C \to D$ such that the following diagram commutes.



Fraïssé Banach spaces

Examples

- The class of all finite-dimensional Banach spaces is a Fraïssé class and the limit is the Gurarii space.

(Ferenczi, Lopez-Abad, Mbombo, Todorčević)

- For $p \in [1, \infty) \setminus \{2n \colon n \geq 2\}$, the set of all finite-dimensional subspaces of $L^p([0,1])$ is a Fraïssé class whose limit is $L^p([0,1])$.
- For $p \in \{2n \colon n \geq 2\}$, the set of all finite-dimensional subspaces of $L^p([0,1])$ has the cofinal amalgamation property (the set $\{\ell^p(n) \colon n \in \mathbb{N}\}$ has the amalgamation property).

Fraïssé Banach spaces

weak Fraïssé correspondence

There is a bijective correspondence between

1. weak Fraïssé classes $\mathcal{K}$ of finite-dimensional Banach spaces (closed with respect to the Banach-Mazur distance);
2. Banach spaces X whose $\overline{\mathrm{Age}(X)} = \mathcal{K}$, which satisfy *certain weak extension property*;
3. Banach spaces X whose $\overline{\mathrm{Age}(X)} = \mathcal{K}$, which satisfy *certain weak approximate ultrahomogeneity*.

We call such Banach spaces **guarded Fraïssé Banach spaces**.

Guarded Fraïssé Banach spaces

Theorem

Let X be a separable Banach space. Fix some $\mu \in \mathcal{B}$ such that $X_\mu \equiv X$. The following are equivalent:

1. X is guarded Fraïssé.
2. The isometry class $[\mu]_\equiv$ is G_δ.
3. The isometry class $[\mu]_\equiv$ is comeager in its closure $\overline{[\mu]_\equiv}$. That is, X is a generic Banach space among those Banach spaces finitely representable in X.

ω-categorical Banach spaces

Recall that a separable Banach space X is ω-categorical if every separable Banach space Y elementarily equivalent to X (equivalently, every Banach space Y such that for some ultrafilter $\mathcal{U}$, the ultrapowers $\prod_{\mathcal{U}} X$ and $\prod_{\mathcal{U}} Y$ are linearly isometric) is linearly isometric to X.

Theorem

Let X be a separable ω-categorical Banach space. Then $\mathrm{Age(X)}$ is a weak Fraïssé class and the unique limit is also ω-categorical.

In fact, if X is a separable Banach space whose theory has less than continuum pairwise non-isometric models, then $\mathrm{Age(X)}$ is a weak Fraïssé class.

ω-categorical Banach spaces

Examples of ω-categorical Banach spaces (known to us/communicated to us by C. W. Henson)

- $L^p([0,1])$, for $p \in [1,\infty)$.
- The Gurarii space.
- $C(2^{\mathbb{N}})$.
- $L_p([0,1], L_q([0,1]))$, for $p, q \in [1,\infty)$.

Fact (Ferenczi, Lopez-Abad, Mbombo, Todorčević)

$L_p([0,1], L_q([0,1]))$, for $p, q \in [1,\infty)$ is not a Fraïssé Banach space.

Remark

It follows, from the theorem above, that $\mathrm{Age}(L_p([0,1], L_q([0,1])))$ is a weak Fraïssé class. It is different from $\mathrm{Age}(L_p([0,1]))$, so its limit is a new guarded Fraïssé Banach space and a new generic Banach space (having G_δ isometry class).
Is it $L_p([0,1], L_q([0,1]))$ itself?