Real Determinacy in Admissible Sets

Caltech Logic Seminar

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Outline

1. Gale-Stewart Games
We will consider Gale-Stewart games in a general form. These are given as follows:

1. A set $X$ of possible moves.
2. A countable ordinal $\alpha$, the length of the game.
3. A set $A \subset X^\alpha$, the payoff set.

The game is played as follows: players I and II alternate turns playing elements of $X$. Player I plays on turns indexed by an even ordinal and Player II plays on turns indexed by an odd ordinal. After $\alpha$-many turns, a sequence $x \in X^\alpha$ has been produced. Player I wins if $x \in A$; otherwise Player II wins.

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Determinacy axioms

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Thus, there are three parameters in play: the length, the pool of possible moves, and the complexity of the games played.
A natural project is to compare these axioms and their consistency strength. This is often nontrivial.
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**Question**

What is the consistency strength of the assertion that all $F_\sigma$ games of length $\omega + \omega$ on $\mathbb{N}$ are determined?

Lower bound: one strong cardinal. Upper bound (strict): one Woodin cardinal.

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A transitive set $A$ is \textit{admissible} if $(A, \in) \models KP$. 

KP is Kripke-Platek set theory. Today (and only today), KP consists of the following axioms:
1. extensionality, pairing, union, infinity, foundation,
2. separation and collection for $\Delta^0_0$ formulas,
3. $\mathcal{R}$ exists.

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AD is the assertion that all games of length $\omega$ on $\mathbb{N}$ are determined.

Theorem (Woodin) 

The following are equiconsistent:

1. $ZF + DC + AD$,
2. $ZFC +$ there are infinitely many Woodin cardinals.
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What is the strength of $\text{KP} + \text{DC} + \text{AD},$ in terms of large cardinals?
This is open, however, it is easy to see that the strength is close to that of ZF + DC + AD.
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**Theorem (Martin-Steel, Woodin)**

*The following are equiconsistent:*

1. $\text{ZFC} + \text{Projective Determinacy}$,
2. $\text{ZFC} + \{ \text{there are } n \text{ Woodin cardinals: } n \in \mathbb{N} \}$. 
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Thus, KP and ZF have similar strength in the context of DC + AD (or just AD).
Another type of principle worth considering is the one asserting the existence of a transitive model of KP + DC + AD, in the context of ZFC.

Theorem

The following are equivalent over ZFC:

1. There is a transitive model of KP + AD containing $\mathbb{R}$;
2. All open games of length $\omega^2$ with moves in $\mathbb{R}$ are determined.

The proof is easy, but we will omit it.
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One would expect that, as before, the strength of $\text{KP} + \text{AD}_\mathbb{R}$ is similar to that of $\text{ZF} + \text{AD}_\mathbb{R}$. This is not the case:

**Theorem**

Suppose that there are $\omega^2$ Woodin cardinals. Then, there is a transitive model of $\text{KP} + \text{DC} + \text{AD}_\mathbb{R}$ containing $\mathbb{R}$. 
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This theorem implies the previous one, since open determinacy for games of length \(\omega^3\) follows from the existence of \(\omega^2\) Woodin cardinals. The proof is descriptive set theoretic.
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For some additional perspective, suppose there are $\omega^2$ Woodin cardinals and a measurable cardinal above them. Then, by a theorem of Steel, the model $M_{\omega^2}$ exists.
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By relativizing, we see that if there are \(\omega^2\) Woodin cardinals below a measurable, then all analytic games of length \(\omega^3\) are determined.
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The following are equivalent over ZFC:

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**Theorem (Martin, Woodin)**

Let $\alpha \geq \omega \cdot 2$ be a recursive wellordering which is provably wellfounded in ZFC. The following are equivalent over ZF + DC:

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Over KP, however, this equivalence is not true. Determinacy axioms for longer games form a proper hierarchy.
Theories and their consistency strength.

\( \mathsf{ZFC} + \text{"there are infinitely many Woodin cardinals"} \)

\( \Delta^1_2 \)-Determinacy for games of length \( \omega^2 \) on \( \mathbb{N} \)

\( \mathsf{ZC} + \text{"there are infinitely many Woodin cardinals"} \)

\( L_{\mathsf{K}^{	ext{on}}(\mathbb{R})} \vdash \mathsf{AD} \iff \Sigma^0_2 \text{-det for games of length } \omega^2 \text{ on } \mathbb{N} \)

\( L_{\mathsf{K}^{	ext{on}}(\mathbb{R})} \vdash \mathsf{AD} \iff \mathsf{\exists} \text{-proj. det} \iff \Delta^1_2 \text{-det for games of length } \omega^2 \text{ on } \mathbb{N} \)

\( L_{\mathsf{K}^{	ext{on}}(\mathbb{R})} \vdash \mathsf{AD} \iff \mathsf{PD} \text{(schem)} \iff \mathsf{ZFC} + \{ \text{there are n Woodins: mean?} \} \)

\( \Delta^1_2 \)-Determinacy \( \iff \Sigma^1_2 \)-Determinacy \( \iff \forall x \hspace{1em} M^x_1 \text{ exists} \)

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\( \mathsf{ZFC} \)

\( \mathsf{ZC} \iff \Sigma^1_2 \)-Determinacy

\( \mathsf{ZC} \)

Credits: Friedman, Martin, Horning, Woodin, Steel, M"uller, Schlicht, A.
\[ \text{ZFC + "there are } \omega^2 \text{ Woodin cardinals"} \]

\[ \Delta^1_1 \text{-cut of length } \omega^3 \]

There is a transitive model \( V \subseteq M \models \text{KP} + \text{AD} + \text{AD}^\text{neg} - \Sigma^0_1 \text{-cut of length } \omega^3 \)

\( \Pi^0_1 \)-cut of length \( \omega^2 \) - \( \Delta^0_1 \)-cut of length \( \omega^3 \)

\( \Sigma^1_1 \)-cut for games of length \( \omega^2 \)

\( \text{ZFC + "there are infinitely many Woodin cardinals"} \)

Credits: Trang, Müller, A.
Can add DC
For the remainder of the talk, let us sketch the proof of the following theorem:

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We first focus on the existence of a model of $\text{KP} + \text{AD}_R$ from open determinacy of length $\omega^3$. This requires reviewing the theory of Spector classes of relations and inductive definability.
Generalized quantifiers

- A quantifier on $\mathbb{R}$ is a non-empty collection of subsets of $\mathbb{R}$ closed under supersets but not equal to $\mathcal{P}(\mathbb{R})$. We write $Qx A(x)$ for $A \in Q$.

Example:

$\exists = \{ A \subseteq \mathbb{R} : A \text{ is nonempty} \}$. 

Example: if $Q$ is a quantifier, then its dual $\tilde{Q}$ is also a quantifier. Here, $A \in \tilde{Q}$ if and only if $\mathbb{R} \setminus A \not\in Q$. 

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$$\exists^R = \{A \subset \mathbb{R} : \text{Player I has a w.s. on the game of length } \omega \text{ on } \mathbb{R} \text{ with payoff } A\}.$$ 

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Example: if $Q$ is a quantifier, then its dual $\check{Q}$ is also a quantifier. Here, $A \in \check{Q}$ if and only if $\mathbb{R} \setminus A \notin Q$. 

Note: $Q$ is closed under supersets, so $Qx \ A(x)$ is equivalent to $\exists Y \in Q \ \forall x \in Y \ A(x)$. Using this triviality, we can make sense of expressions such as

$$Qx_1 \ Qx_2 \ldots \ \phi(x_1, x_2, \ldots).$$
Note: $Q$ is closed under superset, so $Qx A(x)$ is equivalent to $\exists Y \in Q \forall x \in Y A(x)$. Using this triviality, we can make sense of expressions such as

$$Qx_1 Qx_2 \ldots \phi(x_1, x_2, \ldots).$$

Namely, this formula holds if and only if Player I has a winning strategy in the following game:

- Player I begins by playing $Y_1 \in Q$,
- Player II responds by playing $x_1 \in Y_1$,
- Player I responds with $Y_2 \in Q$, etc.
- After infinitely many rounds, Player I wins if and only if $\phi(x_1, x_2, \ldots)$ holds.
We consider operators $\phi : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$. 
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We say that an operator $\phi$ is definable by a formula $\psi(x, X)$ if for every set $A$

$$\phi(A) = \{ x \in \mathbb{R} : \psi(x, A) \}.$$
Inductive definability

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  We say that $\psi(x, X)$ is positive if $X$ appears only positive in it.
- We are concerned with operators $\phi$ definable by positive second-order formulas $\psi(x, X)$ in the language of second-order arithmetic with constants for every real number and expanded by the quantifiers $Q$ and $\bar{Q}$.
Such positive operators can be iterated:

\[
\begin{align*}
\phi^0 &= \emptyset \\
\phi^\alpha &= \phi\left( \bigcup_{\beta < \alpha} \phi^\beta \right) \\
\phi^\infty &= \bigcup_{\alpha} \phi^\alpha
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**Definition**

A set \( A \subset \mathbb{R} \) is \( Q \)-**inductive** if \( A = \{ x : (x, a) \in \phi^\infty \} \) for some \( \phi \) as above.
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Definition

A set \( A \subseteq \mathbb{R} \) is \( Q \)-inductive if \( A = \{ x : (x, a) \in \phi^\infty \} \) for some \( \phi \) as above. We say that \( A \) is \( Q \)-hyperprojective if both \( A \) and \( \mathbb{R} \setminus A \) are \( Q \)-inductive.
Spector classes

- We will need some more notions from generalized recursion theory.

**Definition**

A *Spector class* on $\mathbb{R}$ is an $\mathbb{R}$-parametrized collection of subsets of $\mathbb{R}$ closed under finite conjunctions and disjunctions, $\exists^\mathbb{R}$ and $\forall^\mathbb{R}$, containing all projective sets, and having the prewellordering property.
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**Theorem (Aczel)**

The $Q$-inductive sets form the smallest Spector class on $\mathbb{R}$ closed under $Q$ and $\bar{Q}$. 
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**Theorem (Aczel)**

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**Theorem (Aczel)**

The $Q$-inductive sets are precisely those sets defined by a formula of the form

$$Qx_1 \check{Q}x_2 \exists x_3 \forall x_4 Qx_5 \check{Q}x_6 \ldots A(x_1, x_2, x_3, \ldots),$$

where $A$ is projective.
Finally, we will need one of the companion theorems of Moschovakis:

**Theorem (Moschovakis)**

Let $\Gamma$ be a Spector class on $\mathbb{R}$. Then, there is an admissible set $M$ with $\mathbb{R} \in M$ and such that $\mathcal{P}(\mathbb{R}) \cap M = \Gamma \cap \check{\Gamma}$. 
The following are equivalent over ZFC:

1. There is a transitive model of $\text{KP} + \text{DC} + \text{AD}_R$ containing $\mathbb{R}$;
2. All open games of length $\omega^3$ with moves in $\mathbb{N}$ are determined.
Back to the theorem

- Proof sketch for the first half. Suppose that all open games of length $\omega^3$ are determined.
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Consider the quantifier $\mathcal{D}_{\omega^2}^\mathbb{R}$ consisting of all sets of reals $A$ such that Player I has a winning strategy for the game on $A$ with moves in $\mathbb{R}$ and length $\omega^2$. 
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- Consider the quantifier $\mathcal{D}_\omega^\mathbb{R}_{\omega^2}$ consisting of all sets of reals $A$ such that Player I has a winning strategy for the game on $A$ with moves in $\mathbb{R}$ and length $\omega^2$.
- Given $A \subseteq \mathbb{R}^2$, we write $\mathcal{D}_\omega^\mathbb{R}_A$ for $\{ y \in \mathbb{R} : \{ x : (x, y) \in A \} \in \mathcal{D}_\omega^\mathbb{R} \}$. 
Proof sketch for the first half. Suppose that all open games of length $\omega^3$ are determined.

Consider the quantifier $\mathcal{D}^{\mathbb{R}}_{\omega^2}$ consisting of all sets of reals $A$ such that Player I has a winning strategy for the game on $A$ with moves in $\mathbb{R}$ and length $\omega^2$.

Given $A \subset \mathbb{R}^2$, we write $\mathcal{D}^{\mathbb{R}}_{\omega^2} A$ for $\{y \in \mathbb{R} : \{x : (x, y) \in A\} \in \mathcal{D}^{\mathbb{R}}_{\omega^2}\}$.

Write $\mathcal{D}^{\mathbb{R}}_{\omega^2} \Sigma^0_1$ for the pointclass of all sets of the form $\mathcal{D}^{\mathbb{R}}_{\omega^2} A$, with $A$ open.
Proof sketch

- **Lemma 1.** The pointclass of all $\mathcal{D}_R^R$-inductive sets is contained in $\mathcal{D}_\omega^\omega \Sigma^0_1$.

Proof idea: Naively, sets in $\mathcal{D}_\omega^\omega \Sigma^0_1$ are those defined by a formula of the form $\exists x_1 \forall x_2 \ldots \phi(x_1, x_2, \ldots)$, where the string of quantifiers has length $\omega^2$ and $\phi$ is $\Sigma^0_1$ with parameters. By Aczel's characterization, $\mathcal{D}_R^R$-inductive sets can be defined by a formula of the form $\mathcal{D}_R x_1 \mathcal{D}_R x_2 \ldots \psi(x_1, x_2, \ldots)$, where $\psi$ is projective. This formula has a specific semantics, but naively, we should be allowed to replace each game quantifier by an infinite string of real quantifiers. Thus, we obtain a definition of a given $\mathcal{D}_R^R$-inductive set by a formula of the form $\exists x_1 \forall x_2 \ldots \psi^*(x_1, x_2, \ldots)$, where $\psi^*$ is projective. With some extra work, we can replace $\psi^*$ by a $\Sigma^0_1$ formula, thus obtaining the result.

Indeed, the converse of the lemma is true. We shall not prove that, but it will be used as well in the future.
Proof sketch

Lemma 1. The pointclass of all $\mathcal{D}^R$-inductive sets is contained in $\mathcal{D}^\omega_2 \Sigma^0_1$.

Proof idea: Naively, sets in $\mathcal{D}^\omega_2 \Sigma^0_1$ are those defined by a formula of the form $\exists x_1 \forall x_2 \ldots \phi(x_1, x_2, \ldots)$, where the string of quantifiers has length $\omega^2$ and $\phi$ is $\Sigma^0_1$ with parameters.
Proof sketch

- **Lemma 1.** The pointclass of all $\mathcal{D}^R$-inductive sets is contained in $\mathcal{D}^R\omega^2\Sigma^0_1$.
  - Proof idea: Naively, sets in $\mathcal{D}^R\omega^2\Sigma^0_1$ are those defined by a formula of the form $\exists x_1 \forall x_2 \ldots \phi(x_1, x_2, \ldots)$, where the string of quantifiers has length $\omega^2$ and $\phi$ is $\Sigma^0_1$ with parameters.
  - By Aczel’s characterization, $\mathcal{D}^R$-inductive sets can be defined by a formula of the form $\mathcal{D}^R x_1 \mathcal{D}^R x_2 \ldots \psi(x_1, x_2, \ldots)$, where $\psi$ is projective.
Proof sketch

- **Lemma 1.** The pointclass of all $\mathcal{D} R$-inductive sets is contained in $\mathcal{D} R_{\omega^2} \Sigma^0_1$.

- Proof idea: Naively, sets in $\mathcal{D} R_{\omega^2} \Sigma^0_1$ are those defined by a formula of the form $\exists x_1 \forall x_2 \ldots \phi(x_1, x_2, \ldots)$, where the string of quantifiers has length $\omega^2$ and $\phi$ is $\Sigma^0_1$ with parameters.

- By Aczel's characterization, $\mathcal{D} R$-inductive sets can be defined by a formula of the form $\mathcal{D} R x_1 \mathcal{D} R x_2 \ldots \psi(x_1, x_2, \ldots)$, where $\psi$ is projective.

- This formula has a specific semantics, but naively, we should be allowed to replace each game quantifier by an infinite string of real quantifiers. Thus, we obtain a definition of a given $\mathcal{D} R$-inductive set by a formula of the form

$$\exists x_1 \forall x_2 \ldots \psi^*(x_1, x_2, \ldots),$$

where $\psi^*$ is projective. With some extra work, we can replace $\psi^*$ by a $\Sigma^0_1$ formula, thus obtaining the result.
Lemma 1. The pointclass of all $\mathcal{D}^R$-inductive sets is contained in $\mathcal{D}_{\omega^2}^R \Sigma^0_1$.

Proof idea: Naively, sets in $\mathcal{D}_{\omega^2}^R \Sigma^0_1$ are those defined by a formula of the form $\exists x_1 \forall x_2 \ldots \phi(x_1, x_2, \ldots)$, where the string of quantifiers has length $\omega^2$ and $\phi$ is $\Sigma^0_1$ with parameters.

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where $\psi^*$ is projective. With some extra work, we can replace $\psi^*$ by a $\Sigma^0_1$ formula, thus obtaining the result.

Indeed, the converse of the lemma is true. We shall not prove that, but it will be used as well in the future.
Lemma 2. Suppose that all open games on of length $\omega^3$ on $\mathbb{N}$ are determined. Then, all $\mathcal{O}^\mathbb{R}_{\omega^2}\Sigma^0_1$-games of length $\omega^2$ on $\mathbb{N}$ are determined.
Proof sketch

- Lemma 2. Suppose that all open games on of length $\omega^3$ on $\mathbb{N}$ are determined. Then, all $\mathcal{D}_{\omega^2}^R \Sigma^0_1$-games of length $\omega^2$ on $\mathbb{N}$ are determined.

  - Proof idea: as before, we can naively define each set in $\mathcal{D}_{\omega^2}^R \Sigma^0_1$ by a formula of the form $\exists x_1 \forall x_2 \ldots \phi(x_1, x_2, \ldots)$, where the string of quantifiers has length $\omega^2$ and $\phi$ is $\Sigma^0_1$.

Lemma 2. Suppose that all open games on of length $\omega^3$ on $\mathbb{N}$ are determined. Then, all $\mathcal{D}R^\omega_2 \Sigma^0_1$-games of length $\omega^2$ on $\mathbb{N}$ are determined.

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Thus, we can consider a game in which players play $\omega^2$ many turns and then they are required to play the game given by the formula

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Lemma 2. Suppose that all open games on of length $\omega^3$ on $\mathbb{N}$ are determined. Then, all $\mathcal{D}_{\omega^2}^R \Sigma_1^0$-games of length $\omega^2$ on $\mathbb{N}$ are determined.

Proof idea: as before, we can naively define each set in $\mathcal{D}_{\omega^2}^R \Sigma_1^0$ by a formula of the form $\exists x_1 \forall x_2 \ldots \phi(x_1, x_2, \ldots)$, where the string of quantifiers has length $\omega^2$ and $\phi$ is $\Sigma_1^0$.

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which takes $\omega^3$ turns and has a $\Sigma_1^0$ winning condition.

Combining the two lemmata: if all open games of length $\omega^3$ are determined, then all games of length $\omega^2$ on $\mathbb{N}$ with $\mathcal{D}_R$-inductive payoff are also determined.
Proof sketch

Lemma 3. Suppose that $\mathcal{D}^R$-hyperprojective games of length $\omega^2$ on $\mathbb{N}$ are determined. Then, every $\mathcal{D}^R$-hyperprojective game of length $\omega$ on $\mathbb{R}$ has a $\mathcal{D}^R$-hyperprojective winning strategy.
Lemma 3. Suppose that $\mathcal{D}^R$-hyperprojective games of length $\omega^2$ on $\mathbb{N}$ are determined. Then, every $\mathcal{D}^R$-hyperprojective game of length $\omega$ on $\mathbb{R}$ has a $\mathcal{D}^R$-hyperprojective winning strategy.

Proof idea: First, one adapts Moschovakis’ argument for showing that inductive sets have inductive scales in order to show that, under the hypotheses of the lemma, $\mathcal{D}^R$-hyperprojective sets have $\mathcal{D}^R$-hyperprojective scales. This requires (the proof of) Martin’s theorem on the propagation of scales under the real-game quantifier.
Proof sketch

Lemma 3. Suppose that $\mathcal{D}^R$-hyperprojective games of length $\omega^2$ on $\mathbb{N}$ are determined. Then, every $\mathcal{D}^R$-hyperprojective game of length $\omega$ on $\mathbb{R}$ has a $\mathcal{D}^R$-hyperprojective winning strategy.

Proof idea: First, one adapts Moschovakis’ argument for showing that inductive sets have inductive scales in order to show that, under the hypotheses of the lemma, $\mathcal{D}^R$-hyperprojective sets have $\mathcal{D}^R$-hyperprojective scales. This requires (the proof of) Martin’s theorem on the propagation of scales under the real-game quantifier.

Then, one adapts the proof of Moschovakis’ Third Periodicity Theorem to prove the lemma. This requires the scale property, as well as the fact that $\mathcal{D}^R$-hyperprojective relations can be uniformized by $\mathcal{D}^R$-hyperprojective functions (this follows from the existence of scales).
Proof sketch

- From the lemmata, one half of the theorem is straightforward.
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- Suppose that open games of length $\omega^3$ are determined. By the first two lemmata, all games of length $\omega^2$ on $\mathbb{N}$ with $\mathcal{D}^\mathbb{R}$-inductive payoff are also determined.
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- From the lemmata, one half of the theorem is straightforward.

- Suppose that open games of length $\omega^3$ are determined. By the first two lemmata, all games of length $\omega^2$ on $\mathbb{N}$ with $\mathcal{D}^\mathbb{R}$-inductive payoff are also determined.

- By the third lemma, every $\mathcal{D}^\mathbb{R}$-hyperprojective game of length $\omega$ on $\mathbb{R}$ has a $\mathcal{D}^\mathbb{R}$-hyperprojective winning strategy.
Proof sketch

- From the lemmata, one half of the theorem is straightforward.
- Suppose that open games of length $\omega^3$ are determined. By the first two lemmata, all games of length $\omega^2$ on $\mathbb{N}$ with $\mathcal{D}^{\mathbb{R}}$-inductive payoff are also determined.
- By the third lemma, every $\mathcal{D}^{\mathbb{R}}$-hyperprojective game of length $\omega$ on $\mathbb{R}$ has a $\mathcal{D}^{\mathbb{R}}$-hyperprojective winning strategy.
- Let $M$ be the companion model of the $\mathcal{D}^{\mathbb{R}}$-hyperprojective sets obtained from Moschovakis’ theorem. Then, the sets of reals in $M$ are precisely the $\mathcal{D}^{\mathbb{R}}$-hyperprojective sets. Thus, for each game in $M$ of length $\omega$ on $\mathbb{R}$, there is a strategy in $M$. Therefore, $M \models AD$. 
Let us finish by sketching the argument for the converse. Let $M$ be a transitive model of $\text{KP} + \text{DC} + \text{AD}_R$ such that $R \in M$. We claim that all open games of length $\omega^3$ are determined.
Proof sketch

- Let us finish by sketching the argument for the converse. Let $M$ be a transitive model of $\text{KP} + \text{DC} + \text{AD}_R$ such that $R \in M$. We claim that all open games of length $\omega^3$ are determined.

- First, we need a stronger determinacy hypothesis in $M$; namely that all games of length $\omega^2$ with moves in $\mathbb{N}$ are determined in $M$. This is proved using the uniformization property for sets in $M$. 
Proof sketch

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- First, we need a stronger determinacy hypothesis in $M$; namely that all games of length $\omega^2$ with moves in $\mathbb{N}$ are determined in $M$. This is proved using the uniformization property for sets in $M$.

- Thus, every $\mathcal{D}^R$-hyperprojective game of length $\omega^2$ on $\mathbb{N}$ is determined.
Proof sketch

- We now need the following determinacy transfer theorem:

**Theorem**

Let $\alpha$ be a countable limit ordinal with $\omega^2 \leq \alpha$. Let $\Gamma$ be an $\omega$-parametrized pointclass containing all recursive sets and satisfying the prewellordering property. Suppose that $\Gamma$ is closed under recursive substitution, finite unions and intersections, and the quantifier $\exists^N \alpha$ for games of length $\alpha$ on $\mathbb{N}$. Suppose moreover that all games of length $\alpha$ with moves in $\mathbb{N}$ and payoff in $\Gamma \cap \check{\Gamma}$ are determined. Then, all games of length $\alpha$ with moves in $\mathbb{N}$ and payoff in $\Gamma$ are determined.
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**Theorem**

Let $\alpha$ be a countable limit ordinal with $\omega^2 \leq \alpha$. Let $\Gamma$ be an $\omega$-parametrized pointclass containing all recursive sets and satisfying the prewellordering property. Suppose that $\Gamma$ is closed under recursive substitution, finite unions and intersections, and the quantifier $\forall^N_\alpha$ for games of length $\alpha$ on $\mathbb{N}$. Suppose moreover that all games of length $\alpha$ with moves in $\mathbb{N}$ and payoff in $\Gamma \cap \check{\Gamma}$ are determined.

Then, all games of length $\alpha$ with moves in $\mathbb{N}$ and payoff in $\Gamma$ are determined.

- The theorem is an extension of a determinacy transfer theorem due to Kechris and Solovay, and its proof is a very simple modification of Kechris and Solovay’s proof.
Proof sketch

Its consequence of relevance to us is that from the determinacy of all $\mathcal{D}^R$-hyperprojective games of length $\omega^2$ on $\mathbb{N}$, we can conclude the determinacy of all $\mathcal{D}^R$-inductive games of length $\omega^2$ on $\mathbb{N}$. 
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Hence, we can conclude the determinacy of all games of length $\omega^2$ with moves in $\mathbb{N}$ and payoff in $\mathcal{D}_{\omega^2}^R \Sigma^0_1$. 

To finish, we need to show that this implies open determinacy for games of length $\omega^3$. 

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Hence, we can conclude the determinacy of all games of length $\omega^2$ with moves in $\mathbb{N}$ and payoff in $\mathcal{D}^R_{\omega^2} \Sigma^0_1$.

To finish, we need to show that this implies open determinacy for games of length $\omega^3$. 

Proof sketch
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- The idea is as follows: Let $G$ be an open game of length $\omega^3$ for which Player I does not have a winning strategy. Divide $G$ into infinitely many blocks $G_1, G_2, \ldots$, of length $\omega^2$ and consider each of them a separate game.
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- We consider an auxiliary game $H_1$ where the players play $\omega^2$ moves $x_1 \in \mathbb{N}^{\omega^2}$, after which Player I wins if and only if she has a winning strategy for $A$ with which $x_1$ is consistent.
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This can be regarded as a game of length $\omega^2$ with payoff in $\mathcal{D}^{\mathbb{R}}_{\omega^2} \Sigma^0_1$, so it is determined.
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- This can be regarded as a game of length $\omega^2$ with payoff in $\mathcal{D}_{\omega^2} \Sigma^0_1$, so it is determined.

- Observe that Player I does not have a winning strategy for $H_1$, because this would induce a winning strategy for $G$. 
Proof sketch

- Suppose that the auxiliary game is determined in favor of Player II. Then, by playing $G$ according to the strategy of $H_1$, after $\omega^2$ turns, a real $x$ is produced from which Player I does not have a winning strategy for $G$. 

We then consider an auxiliary game $H_x$ where the players play $\omega^2$ moves $x_2 \in \mathbb{N}_{\omega^2}$, after which Player I wins if and only if she has a winning strategy for $A$ with which $x \downarrow x_2$ is consistent. This game is determined, but Player I cannot have a winning strategy for it, as before. Continuing this way, we produce a sequence $x_i$ with each $x_i$ an element of $\mathbb{N}_{\omega^2}$. The point is that this sequence is a winning play for Player II in $G$. This is because the game is open, so if Player I were to win, she would do so at some bounded stage, but we argued that this was impossible. We have just described a winning strategy for Player II in $G$. 

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Proof sketch

- Suppose that the auxiliary game is determined in favor of Player II. Then, by playing $G$ according to the strategy of $H_1$, after $\omega^2$ turns, a real $x$ is produced from which Player I does not have a winning strategy for $G$.

- We then consider an auxiliary game $H_2^x$ where the players play $\omega^2$ moves $x_2 \in \mathbb{N}^{\omega^2}$, after which Player I wins if and only if she has a winning strategy for $A$ with which $x_1 \dashv x_2$ is consistent.
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- We then consider an auxiliary game $H^x_2$ where the players play $\omega^2$ moves $x_2 \in \mathbb{N}^{\omega^2}$, after which Player I wins if and only if she has a winning strategy for $A$ with which $x_1 \dashv x_2$ is consistent.

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- We then consider an auxiliary game $H_2^x$ where the players play $\omega^2$ moves $x_2 \in \mathbb{N}^{\omega^2}$, after which Player I wins if and only if she has a winning strategy for $A$ with which $x_1 \wedge x_2$ is consistent.

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- We have just described a winning strategy for Player II in $G$. 
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- Thank you!