Algorithmic information theory, effective descriptive set theory and geometric measure theory

Theodore A. Slaman

University of California Berkeley

Abstract

We will describe how the perspectives of Recursion Theory and Set Theory suggest lines of investigation into Geometric Measure Theory. We will discuss the extent of capacitability for Hausdorff dimension and the question of existence of sets of strong gauge dimension, which is a property generalizing that of strong measure zero.

Abstract

We will describe how the perspectives of Recursion Theory and Set Theory suggest lines of investigation into Geometric Measure Theory. We will discuss the extent of capacitability for Hausdorff dimension and the question of existence of sets of strong gauge dimension, which is a property generalizing that of strong measure zero.

Caveat: This project is in an early stage, both in terms of the results obtained and in terms of my understanding of the well-developed machinery already available.

Hausdorff Dimension

Define a family of outer measures, parameterized by $s \in [0,1]$. For $A \subseteq 2^\omega$,

$$\mathcal{H}^s(A) = \lim_{r \to 0} \inf \left\{ \sum_i \frac{1}{2^{|\sigma_i|\, s}} : \begin{array}{l} \text{there is a cover of } A \text{ by balls} \\ B(\sigma_i) \text{ with } 1/2^{|\sigma_i|} < r \end{array} \right\}.$$

Definition

The *Hausdorff dimension* of A is as follows.

$$\begin{aligned} \dim_{\mathrm{H}}(A) &= \inf\{s \geq 0 : \mathcal{H}^s(A) = 0\} \\ &= \sup\left(\{s \geq 0 : \mathcal{H}^d(A) = \infty\} \cup \{0\}\right) \end{aligned}$$

Hausdorff Dimension

Example

- The Cantor Middle Third Set has Hausdorff dimension $\frac{\log 2}{\log 3}$.
- A line segment in the plane has Hausdorff dimension 1.

Definition

We write $\dim_{\mathrm{H}}(A)$ for the Hausdorff dimension of A.

Capacitability

Theorem (Besicovitch and Davies 1952 (independently))

If A is an analytic subset of 2^ω and $\dim_{\mathrm{H}}(A) = s$, then for every $t < s$ there is a closed set $C \subseteq A$ such that $s \leq \dim_{\mathrm{H}}(C) \leq t$.

Remark

- *In other words, the Hausdorff dimension of analytic A is carried by the dimensions of its closed subsets. We say that A is capacitable.*
- *Capacitability is a generalization of the familiar regularity of Lebesgue measure.*

Effective Hausdorff Dimension

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Definition

- For $A \subseteq 2^\omega$, define A has *effective s-dimension Hausdorff measure* 0 iff there is a uniformly computably enumerable sequence of open sets $O_i = \bigcup_j B(\sigma_{i,j})$ such that for each i, $A \subseteq O_i$ and $\sum_j (1/2^{|\sigma_{i,j}|})^s < 1/2^i$.
- The *effective Hausdorff dimension* $\dim_{\mathrm{H}}^{\mathit{eff}}(A)$ of A is the infimum of those s such that A has effective s-dimension Hausdorff measure 0.

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- The *effective Hausdorff dimension* $\dim_{\mathrm{H}}^{\mathit{eff}}(A)$ of A is the infimum of those s such that A has effective s-dimension Hausdorff measure 0.

Remark

- *For all* A, $\dim_{\mathrm{H}}(A) \leq \dim_{\mathrm{H}}^{\mathit{eff}}(A)$
- *If* x *is Martin-Löf random then* $\dim_{\mathrm{H}}^{\mathit{eff}}(\{x\}) = 1$.
  - *Here* x *is ML-random iff there is a constant* C *such that for all* ℓ *the prefix-free Kolmogorov complexity of* $x \upharpoonright \ell$ *is greater than* $\ell - C$.

Effective Hausdorff Dimension

formulated by compressibility

Theorem (Mayordomo 2002)

For any $x \in 2^\omega$,

$$\dim_{\mathrm{H}}^{\mathit{eff}}(\{x\}) = \liminf_{\ell \to \infty} \frac{K(x \upharpoonright \ell)}{\ell}.$$

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- We will abbreviate and write $\dim_{\mathrm{H}}^{\mathit{eff}}(x)$ for $\dim_{\mathrm{H}}^{\mathit{eff}}(\{x\})$.
- We can relativize to a real B and write $\dim_{\mathrm{H}}^{\mathit{eff}(B)}(x)$.

Point-to-Set for Hausdorff Dimension

Theorem (J. Lutz and N. Lutz 2017)

For $A \subseteq 2^\omega$, the Hausdorff dimension of a set A is equal to
the infimum over all $B \subseteq \mathbb{N}$
of the supremum over all $x \in A$
of $\dim_{\mathrm{H}}^{\mathit{eff}(B)}(x)$, the effective-relative-to-B H-dimension of x.

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One direction of the above is obtained by considering the reals that can compute appropriate open covers of A. The other direction can be proven by using the relativized version of the fact that the set of reals with effective Hausdorff dimension s has Hausdorff dimension s.

Co-analytic Sets

Consistency results

We will look at capacitability in Gödel's universe of constructible sets L and also in models of determinacy.

Co-analytic Sets

Working in $V = L$

Definition

Define P by

$$P = \left\{ x : \begin{array}{l} x \text{ can compute a representation of the ordinal at} \\ \text{which } x \text{ is constructed} \end{array} \right\}$$

Theorem (Gaspari, Kechris, Sacks)

P is the maximal thin Π^1_1 set:

- *P is co-analytic.*
- *P has no perfect subset.*
- *If $V = L$ then P is not countable. In fact, the set of Turing degrees of elements of P is cofinal in the Turing degrees.*

Co-analytic Sets

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Theorem

If $V = L$ then $\dim_H(P) = 1$.

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Working in $V = L$

Theorem

If $V = L$ *then* $\dim_H(P) = 1$.

Consequently, it is consistent with *ZFC* that the Hausdorff dimensions of co-analytic sets are not carried by their closed subsets.

Applying Point-to-Set Reasoning

Working in $V = L$

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Step 1. There is an infinite computable set $S \subseteq \mathbb{N}$ such that for all B and for all x, if x is Martin-Löf random relative to B and y is equal to x at all places not in S then $\dim_{\mathrm{H}}^{\mathit{eff}(B)}(y) = 1$.

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In fact, S could be the iterated powers of 2. To verify the claim, use Mayordomo's theorem and estimate the compressibility of y relative to B.

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In fact, S could be the iterated powers of 2. To verify the claim, use Mayordomo's theorem and estimate the compressibility of y relative to B.

Step 2. By the Lutz and Lutz theorem, it is sufficient to show that for every z there is a y in P such that $\dim_{\mathrm{H}}^{\mathit{eff}(z)}(y) = 1$.

Applying Point-to-Set Reasoning

Working in $V = L$

Step 3. Suppose that $B \in 2^\omega$ is given.

- Let x be Martin-Löf random relative to B.
- Let $m \in P$ be such that m can compute x and B.
- Let y be the result of replacing the bit values of x on the elements of S by the bit values of m.

Then, m can compute the ordinal at which y is constructed and y can compute m. Thus, $y \in P$.

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Step 4. Conclude, $\dim_H(P) = 1$, as required.

AD

At the opposite end of the set theoretic spectrum, AD/Large Cardinals show that capacitability extends far beyond that class of analytic sets.

Theorem (Crone, Fishman and Jackson 2020)

Assume AD. Let A be a subset of $\mathbb{R}^d$ and $0 \leq \delta \leq d$. Either A has a compact subset C such that $\dim_{\mathrm{H}}(C) \geq \delta$ or $\dim_{\mathrm{H}}(A) \leq \delta$.

The proof artfully combines ingredients from Descriptive Set Theory and Geometric Measure Theory, in particular a geometric formulation of $\dim_{\mathrm{H}}(A)$ in terms of the measure of A within neighborhoods of elements of A.

Gauge Functions and General Hausdorff Dimension

The following is joint work with Jan Reimann.

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Definition

A *gauge function* is a function $f : \mathbb{R} \to \mathbb{R}$ which has the following properties:

- continuous
- increasing and positive valued on $(0, \infty)$
- $\lim_{t \to 0^+} f(t) = 0$

Example

$f(t) = t^s$, for $s > 0$.

Gauge Functions and General Hausdorff Dimension

Definition

Let f be a gauge function. For a set $E \subseteq 2^\omega$ (or ω^ω, $\mathbb{R}^d$ etc.), define

$$H^f(E) = \lim_{\delta \to 0} \inf_{\substack{E \subseteq \cup F_i \\ \max d(F_i) < \delta}} \sum_{i=1}^{\infty} f(d(F_i))$$

where $\{F_i\}$ is a sequence of closed (open) sets covering E and $d(F_i)$ is the diameter of F_i.

A general gauge function may vary the relative weight it gives intervals depending upon the scale of those intervals.

Gauge Functions and General Hausdorff Dimension

Definition

Write $h \prec g$ to indicate that $\lim_{t\to 0^+} \frac{g(t)}{h(t)} = 0$.

Note, $h \prec g$ means that is it easier for a set to be H^g-null than it is to be H^h-null.

Example

- $\sqrt{t} \prec t$, since $\lim_{t\to 0^+} \frac{t}{\sqrt{t}} = \lim_{t\to 0^+} \sqrt{t} = 0$.
- Sets of Hausdorff dimension $1/2$ are null with respect to linear measure.

Sets of non-σ-finite measure

Definition

A set A is σ-finite for H^f iff A is a countable union of sets A_i, for which $H^f(A_i)$ is finite.

Theorem (Davies 1956 for x^s, Sion and Sjerve 1962)

If E is analytic and is non-σ-finite for H^h, then there is a compact subset of E that is non-σ-finite for H^h.

Sets of Strong Dimension h

Definition

A set E has *strong dimension h* iff

$$\forall f [f \prec h \Rightarrow H^f(E) = \infty]$$

$$\forall g [h \prec g \Rightarrow H^g(E) = 0]$$

As a limiting case, E has strong dimension 0 iff for all g, $H^g(E) = 0$.

Sets of Strong Dimension h

Theorem (Besicovitch 1956, generalized Rogers 1962)

If E is compact and is non-σ-finite for H^h, then there is a g such that $h \prec g$ and E is non-σ-finite for H^g.

- Thus, a compact set E cannot have strong dimension h and be non-σ-finite for H^h.
- By the previous capacitability theorem, the same is true for which E is analytic.

Sets of Strong Dimension h

Theorem (Besicovitch 1963)

If CH then there is a set $E \subset \mathbb{R}^2$ such that E has strong linear dimension and is not σ-finite for linear measure.

Theorem (Combining Besicovitch 1963 with Erdős, Kunen and Mauldin 1981)

If $V = L$ there there is a Π^1_1 set E such that E has strong linear dimension and is not σ-finite for linear measure.

Borel Conjecture

Definition

A set $E \subseteq \mathbb{R}$ has *strong measure 0* iff for any sequence of positive real numbers $\{\epsilon_i\}$ there is a sequence of open intervals $\{O_i\}$ such that for each i, O_i has length ϵ_i, and $E \subseteq \cup_{i=1}^{\infty} O_i$.

Borel Conjectured that strong measure 0 implies countable (BC).

Theorem

- *(Sierpiński 1928) CH implies that there is an uncountable set of strong measure 0.*
- *(Laver 1976) $Con(ZFC)$ implies $Con(ZFC + BC)$.*

Borel Conjecture

Theorem (Besicovitch 1955)

A set E has strong dimension 0 iff it has strong measure 0.

Theorem (Another variation on Besicovitch 1963)

$\neg BC$ implies that there is a subset of $\mathbb{R}^2$ which has strong linear dimension and which is not σ-finite for linear measure.

It is open whether BC implies that there do not exist f and E such that E has strong dimension f and E is not σ-finite for H^f.

The End