Realizations of equivalence relations and subshifts

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Borel equivalence relations

A **Borel equivalence relation** on a Polish space X is an equivalence relation E on X such that $E \subseteq X^2$ is Borel (consider it as a set of pairs). Many natural classification problems in mathematics arise as Borel equivalence relations:

- Classification of finitely generated groups up to isomorphism.
- Classification of (open) Riemann surfaces up to conformal equivalence.
- Classification of finitely generated groups up to quasi-isometry.

Some of these do not have "reasonable" invariants which classify them. One aim of the program of Borel equivalence relations is to make these kinds of statements precise.

Borel equivalence relations

Let E and F be Borel equivalence relations on X and Y respectively.
We say that E is **Borel reducible** to F (denoted $E \leq_B F$) if there is a Borel map $X \to Y$ such that

$$x \; E \; x' \iff f(x) \; F \; f(x').$$

This defines a preorder on Borel equivalence relations.
We say that E is **smooth** if it is Borel reducible to $=_{\mathbb{R}}$, the equality relation on $\mathbb{R}$.
This corresponds to those classification problems which have concrete invariants.
For instance, the classification of 5×5 unitary matrices up to similarity (aka conjugacy) is smooth, where the concrete invariants are the 5 eigenvalues.

Countable Borel equivalence relations

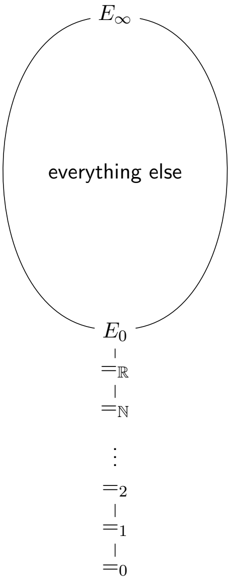
Today we'll work in the context of **countable Borel equivalence relations (CBER)**, which are Borel equivalence relations with every class countable.
Canonical example:
Let Γ be a countable group, let X be a Polish space, and fix a continuous action $\Gamma \curvearrowright X$.
The **orbit equivalence relation** E_Γ^X is defined as follows:

$$x \; E_\Gamma^X \; x' \iff \exists\gamma\, [\gamma \cdot x = x']$$

This is a CBER. (In fact, all CBERs arise in this manner!)

- Irrational rotation $\mathbb{Z} \curvearrowright S^1$.
- Bernoulli shift $\mathbb{Z} \curvearrowright 2^{\mathbb{Z}}$.
- Bernoulli shift $\Gamma \curvearrowright 2^{\Gamma}$.

CBERs under $\leq_B$



Topological realizations

Henceforth, assume all CBERs are **aperiodic**, i.e. every class is infinite.
Topology is useful to study CBERs.
The prototypical example:

Proposition

Let $\Gamma \curvearrowright X$ be a Borel action of a countable group on a Polish space X, with no finite orbits. If E_Γ^X is minimal, then it is not smooth.

A CBER E on a Polish space is **minimal** if every class is dense.
A CBER E has a **minimal action realization** if there is some countable group Γ, some Polish space X, and a continuous action $\Gamma \curvearrowright X$ such that E_Γ^X is Borel isomorphic to E.
If E has a minimal action realization, then E is not smooth. Converse?

Minimal realizations

A CBER E has a **minimal realization** if it is Borel isomorphic to a minimal CBER on a Polish space.
We show that every CBER has a minimal realization:

Theorem ([FKSV21])

Let E be an aperiodic CBER and let X be a perfect Polish space. Then E has a minimal realization on X.

Even smooth ones! Define the function $f : \prod_{n<\omega} 2^n \to 2^\omega$ by

$$f((x_n)_n) = \begin{cases} \lim_n x_n & \text{if the limit exists} \\ x_0{}^\frown x_1{}^\frown x_2{}^\frown \cdots & \text{otherwise} \end{cases}$$

Then let E be the smooth CBER induced by f: define E on $\prod_{n<\omega} 2^n$ by

$$(x_n)_n \; E \; (y_n)_n \iff f((x_n)_n) = f((y_n)_n)$$

Example application: stronger marker lemma

Here's a stronger version of the classical marker lemma:

Theorem ([FKSV21])

Let E be an aperiodic CBER. Then there is a family $(A_s)_{s \in 2^{<\mathbb{N}}}$ of Borel sets such that

1. $A_\varnothing = X$.
2. *If* $s \preceq t$, *then* $A_s \supseteq A_t$.
3. A_{s0} *and* A_{s1} *are disjoint.*
4. *Each* A_s *is a complete section for* E.
5. *For every* $x \in 2^{\mathbb{N}}$, *we have* $\bigcap_{s \preceq x} A_s = \varnothing$.

CBERs under inclusion

Define $E \subseteq_B F$.
Insert manual tikz:

Minimal action realizations

Back to minimal action realizations!

We can realize every hyperfinite CBER.

Question

Does every non-smooth aperiodic CBER have a minimal action realization?

Compact actions

There is an analogous statement for compact spaces:

Proposition

Let $\Gamma \curvearrowright X$ be a continuous action of a countable group on a compact Polish space X, with no finite orbits. Then E_Γ^X is not smooth.

A CBER E has a **compact action realization** if there is some countable group Γ, some compact Polish space X, and a continuous action $\Gamma \curvearrowright X$ such that E_Γ^X is Borel isomorphic to E.
We know for instance...

- Hyperfinite CBERs.
- Free parts of the shift $(2^{\mathbb{N}})^\Gamma$.
- The universal compressible CBER.

A gluing construction

There is a general gluing construction.
Suppose $\Gamma \curvearrowright X$ and $\Delta \curvearrowright Y$. Fix an infinite orbit X_0 in X.
Glue a copy of Y to every $x \in X_0$, resulting in a compact space Z with an action of $\Gamma \times \Delta$.

Theorem

If E has a locally compact action realization, then $E \times I_{\mathbb{N}}$ has a compact action realization.
So if E is furthermore compressible, then E has a compact action realization.

Corollary

If $(E_n)_n$ are compressible CBERs with compact action realizations, then so is $\bigoplus E_n$.

K_σ realizations

A subset of a Polish space is K_σ if it is the countable union of compact sets.
If $\Gamma \curvearrowright X$ with X compact, then E_Γ^X is K_σ:

$$x \, E_\Gamma^X \, x' \iff \exists \gamma \, [\gamma \cdot x = x']$$

Clinton Conley asked the following question:
Does every E have a K_σ realization?
Yes!

Theorem ([FKSV21])

Every aperiodic CBER E has a K_σ realization.
That is, E is Borel isomorphic to a K_σ CBER.

We do not know if we can get minimal K_σ realizations, but we know that smooth ones are not possible (due to Solecki).

Sketch of K_σ realization

Let E be an aperiodic CBER.
Let C be Cantor space.
Write $C = N \sqcup Q$, where N is isomorphic to Baire space.

Proposition ([FKSV21])

Let E be an aperiodic CBER. Then there is a continuous action $F_\infty \curvearrowright \mathcal{N}$ such that $E_{\mathcal{N}}^{F_\infty} \cong_B E$.

For each $\gamma \in \Gamma$, let R_γ be the graph of γ.
Then take the union of the $\overline{R_\gamma}$, plus I_Q (which is countable).

Realizations as subshifts

A natural question is to consider compact realizations not just on an arbitrary compact Polish space, but as a subshift.
Let X be a Polish space.
A **subshift** of X^Γ is a closed Γ-invariant subset $K \subseteq X^\Gamma$.
One can realize a universal CBER as a minimal subshift:

Theorem ([FKSV21])

There is a minimal subshift K of 2^{F_3} such that E_K is a universal CBER.

In general, we know many groups Γ for which 2^Γ has a minimal subshift with universal CBER (certain wreath products).

Side remark on amenability

A countable group Γ is **amenable** if every continuous action $\Gamma \curvearrowright X$ on a compact space has an invariant measure.
We show that it suffices to check subshifts of 2^Γ.

Theorem ([FKSV21])

A group Γ is amenable iff every subshift of 2^Γ has an invariant measure.

The proof is fairly explicit.
Andy Zucker has noted that there is an abstract proof using ideas from topological dynamics, in particular, using strongly proximal actions.

The space of subshifts

A natural object to consider when studying subshifts is the space of subshifts.
For a Polish space X, let $\mathrm{Sh}(X)$ be the standard Borel space of subshifts of X^{F_∞}.
Every compact Polish space is a closed subspace of $[0,1]^{\mathbb{N}}$ (the Hilbert cube).
$\mathrm{Sh}([0,1]^{\mathbb{N}})$ is a **universal space for compact actions**.
Similarly, $\mathrm{Sh}(\mathbb{R}^{\mathbb{N}})$ is a **universal space for arbitrary actions**, since every Polish space is a closed subspace of $\mathbb{R}^{\mathbb{N}}$.

Topological and descriptive complexity

Theorem

The set

$$\{K \in \mathrm{Sh}([0,1]^{\mathbb{N}}) : K \text{ is smooth}\}$$

is meager and $\mathbf{\Pi}^1_1$*-complete (not Borel).*

Question

The set

$$\{K \in \mathrm{Sh}([0,1]^{\mathbb{N}}) : K \text{ is hyperfinite}\}$$

is $\mathbf{\Sigma}^1_2$*.*
Is this upper bound exact?
Is this set comeager?

Topological and descriptive complexity

A CBER E on X is **measure-hyperfinite** if for every Borel probability measure μ on X, there is a μ-conull subset $Y \subseteq X$ such that $E \restriction Y$ is hyperfinite.

Theorem

The set

$$\{K \in \mathrm{Sh}([0,1]^{\mathbb{N}}) : E_K \text{ is measure-hyperfinite}\}$$

is comeager and $\mathbf{\Pi}^1_1$-complete.

Theorem

The set

$$\{K \in \mathrm{Sh}([0,1]^{\mathbb{N}}) : K \text{ is free and measure-hyperfinite}\}$$

is dense G_δ.

G_δ is very surprising!

Measure-amenability

This follows from:

Theorem

The set

$$\{K \in \mathrm{Sh}([0,1]^{\mathbb{N}}) : K \textit{ is measure-amenable}\}$$

is dense G_δ.

A Borel action $\Gamma \curvearrowright X$ is **(Borel) amenable** if there is a sequence $p_n : X \to \mathrm{Prob}(\Gamma)$ such that

$$\|p_n^{\gamma\cdot x} - \gamma \cdot p_n^x\|_1 \to 0$$

for every $\gamma \in \Gamma$ and $x \in X$.
Define μ-**amenable**, then **measure-amenable**.

Topological amenability

A continuous action $\Gamma \curvearrowright X$ on a Polish space is **topologically amenable** if for every finite $S \subseteq \Gamma$, every compact $K \subseteq X$, and every $\varepsilon > 0$, there is some continuous $p\colon X \to \mathrm{Prob}(\Gamma)$ such that

$$\max_{\substack{\gamma \in S \\ x \in K}} \|p^{\gamma \cdot x} - \gamma \cdot p^x\|_1 < \varepsilon.$$

This is actually equivalent to measure-amenability (for σ-compact spaces). From this characterization, we obtain that the set of measure-amenable shifts is $\mathbf{\Sigma}^1_1$.

Going from $\mathbf{\Sigma}^1_1$ to G_δ

Theorem

The set

$$\{K \in \mathrm{Sh}([0,1]^{\mathbb{N}}) : K \text{ is measure-amenable}\}$$

is dense G_δ.

So far, we have $\mathbf{\Sigma}^1_1$.
To get G_δ, we use an argument of Kechris-Louveau-Woodin:

Proposition

Let I be a σ-ideal in $\mathrm{Sh}(\Gamma, [0,1]^{\mathbb{N}})$. If I is F_σ-hard, then I is $\mathbf{\Pi}^1_1$-hard.

A no-go theorem

We've seen that the class of smooth subshifts is not Borel.
This implies that even if every CBER has a compact realization, there is no **effective** way to obtain this realization.
Precisely:

Theorem ([FKSV21])

There is a non-smooth aperiodic subshift $F \in \mathrm{Sh}(\mathbb{R}^{\mathbb{N}})$, such that for every $K \in \mathrm{Sh}([0,1]^{\mathbb{N}})$, there is no $\Delta^1_1(F)$ isomorphism of E_F with E_K.

Thank you!

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