Jumps in the Borel complexity hierarchy

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Classification and Borel complexity

In Borel complexity theory we study equivalence relations E on a standard Borel space X. We sometimes think of them as classification problems.

Example

We may regard the class of countable linear orders as a subset $X \subset 2^{\mathbb{N}\times\mathbb{N}}$, and identify the classification problem for countable linear orders with the isomorphism equivalence relation $\cong_X$.

Definition

The complexity of a classification problem E on X is measured by its position in the Borel reducibility hierarchy:
$E \leq_B F$ iff there is a Borel function $f\colon X \to Y$ such that

$$x \mathrel{E} x' \iff f(x) \mathrel{F} f(x')$$

Jump operators

Definition

A proper jump operator on Borel equivalence relations is a mapping $E \mapsto J(E)$ which is:

- (monotone) $E \leq_B F$ implies $J(E) \leq_B J(F)$, and;
- (proper) $E <_B J(E)$ whenever E has at least two equivalence classes.

Remark

One may wish to impose a definability condition; our examples will all be suitably definable.

Example 1

Example

If E is a Borel equivalence relation on X, the Friedman–Stanley jump of E is defined on X^ω by:

$$x\ E^+\ y \iff \{[x(n)]_E : n \in \omega\} = \{[y(n)]_E : n \in \omega\}$$

Theorem (Friedman–Stanley)

The FS jump is proper.

Remark

The tower $F_\alpha = \Delta(2)^{+\alpha}$ of iterates of the Friedman–Stanley jump is unbounded in complexity among Borel equivalence relations, and is often used as a yardstick.

Example 2

Example

If E is a Borel equivalence relation on X, the Louveau jump of E with respect to the filter $\mathcal{F}$ is defined on X^ω by:

$$x \; E^{\mathcal{F}} \; y \iff \{n \in \omega : x(n) \; E \; y(n)\} \in \mathcal{F}$$

Theorem (Louveau, later Hjorth–Kechris–Louveau)

If $\mathcal{F}$ is a free filter, the Louveau jump with respect to $\mathcal{F}$ is proper.

Bernoulli jumps

Definition

Let E be an equivalence relation on X, and let Γ be a countable group. The Γ-jump of E is the equivalence relation $E^{[\Gamma]}$ defined on X^Γ by

$$x \, E^{[\Gamma]} \, y \iff (\exists \gamma \in \Gamma)\,(\forall \alpha \in \Gamma)\; x(\gamma^{-1}\alpha) \; E \; y(\alpha)$$

Remark

- In words $E^{[\Gamma]}$ consists of Γ-many factors of E, modulo translation by Γ.
- $\Delta(2)^{[\Gamma]}$ is the orbit equivalence relation of the "Bernoulli" shift action of Γ on 2^Γ.
- The Γ-jump may be iterated through countable ordinals. We write $E^{[\Gamma]^\alpha}$ for the α-iterated Γ-jump.

Properties of Bernoulli jumps

Proposition

For any countable group Γ we have:

- *If E is Borel then $E^{[\Gamma]}$ is Borel*
- *(monotone) If $E \leq_B F$ then $E^{[\Gamma]} \leq_B F^{[\Gamma]}$*
- *(pre-proper?) $E \leq_B E^{[\Gamma]}$*
- *$E^\omega \leq_B E^{[\Gamma]}$ (for Γ infinite)*
- *If Λ is a subgroup or quotient of Γ, then $E^{[\Lambda]} \leq_B E^{[\Gamma]}$*

Gentleness of Bernoulli jumps

Proposition

For any countable group Γ we have:

- *If E is pinned then $E^{[\Gamma]}$ is pinned*
- *If E_Λ is the orbit equivalence relation of $\Lambda \curvearrowright X$, and Γ is any countable group, then $(E_\Lambda)^{[\Gamma]}$ is the orbit equivalence relation of $(\Lambda \wr \Gamma) \curvearrowright X^\Gamma$*
- *If E is induced by a Polish (resp. solvable, cli, closed in S_∞) group, then $E^{[\Gamma]}$ is too.*

Remark

The Friedman–Stanley jump quickly becomes non-pinned, and the Louveau jump quickly becomes non-Polish-induced. Thus the Bernoulli jumps are "kindler, gentler".

Comparison of Bernoulli jumps and FS jumps

Theorem

- $\Delta(2)^{[\mathbb{Z}]^\alpha} \leq_B F_\alpha$
- $\Delta(2)^{[\Gamma]^\alpha} \leq_B F_{1+\alpha}$
- $F_2 \not\leq_B \Delta(2)^{[\Gamma]^\alpha}$

Theorem

If E has perfectly many classes and $E \times E \leq_B E$, then $E^{[\mathbb{Z}]} \leq E^+$.

Theorem (Allison–Shani)

$(E_0)^{[\mathbb{Z}^2]}$ is not potentially $\mathbf{\Pi}^0_3$. In particular, $(E_0)^{[\mathbb{Z}^2]}$ and $(E_0)^+$ are Borel incomparable.

Question

When is $E^{[\Gamma]} \leq_B E^+$?

Properness of Bernoulli jumps

So far we have postponed the question of whether $E \mapsto E^{[\Gamma]}$ is really a proper jump operator.

For an obvious example, if Γ is finite then $E \mapsto E^{[\Gamma]}$ is easily seen to be improper: $E_0 \sim_B (E_0)^{[\Gamma]}$.

As we will see on the next two frames, even for infinite groups Γ, the answer depends on Γ.

Not all Bernoulli jumps are proper

Definition

A countable group satisfies the d.c.c. if it has no infinite properly descending sequence of subgroups.

Example

The Prüfer group $\mathbb{Z}(p^\infty)$ satisfies the d.c.c.

Theorem

If Γ satisfies the d.c.c. then the Γ-jump is not proper.

Proof idea.

It is straightforward to exhibit a reduction function:

$$\Delta(2)^{[\Gamma]^{\omega+1}} \leq_B \Delta(2)^{[\Gamma]^{\omega}}$$

in this case. □

Most Bernoulli jumps are proper

Theorem

Let Γ be a countable group such that $\mathbb{Z}$ or $\mathbb{Z}_p^{<\omega}$ for p prime is a quotient of a subgroup of Γ. Then the Γ-jump is proper.

The proof consists of two pieces.

- A theorem of Solecki, which implies there are Γ^ω actions of arbitrarily high complexity;
- An adaptation of the Hjorth–Kechris–Louveau proof of Friedman–Stanley's theorem, which implies that Γ^ω actions are Borel reducible to iterates of the Γ-jump.

Solecki's theorem

Definition

A family $\mathcal{F}$ of Borel equivalence relations has cofinal essential complexity if for every α there exists $E \in \mathcal{F}$ such that E is not Borel reducible to any equivalence relation in Π^0_α.

Theorem (Solecki)

If Γ is one of the groups $\mathbb{Z}$ or $\mathbb{Z}_p^{<\omega}$ for p a prime, then the family of Γ^ω-actions has cofinal essential complexity.

Remark

Solecki's proof involves constructing structures called *group trees* of unbounded rank.

Hjorth–Kechris–Louveau argument

Definition

Let Γ be a countable group. The infinite Γ-tree T_Γ consists of the tree $\Gamma^{<\omega}$ together with the structure of Γ on every set of siblings.

Theorem

Let Γ be a countable group, and E an equivalence relation induced by a continuous action of a closed subgroup of $\mathrm{Aut}(T_\Gamma)$. If E is $\mathbf{\Pi}^0_\alpha$ then $E \leq_B \Delta(2)^{[\Gamma]^{\omega\cdot\alpha}}$.

Proof idea.

Hjorth–Kechris–Louveau show that $[x]_E$ is determined by the orbit closure of x in a topology $\tau_{x,\beta}$ (β is roughly $\omega \cdot \alpha$).
For a closed subgroup of S_∞, the topology and orbit-closure can be coded in a tree of rank roughly β.
We show that for a closed subgroup of $\mathrm{Aut}(T_\Gamma)$, the topology and orbit-closure can be coded in the β'th iterate of the Γ-jump. □

Conclusion of proof of properness

Theorem

Let Γ be a countable group such that $\mathbb{Z}$ or $\mathbb{Z}_p^{<\omega}$ for p prime is a quotient of a subgroup of Γ. Then the Γ-jump is proper.

Proof sketch.

- By the Hjorth–Kechris–Louveau machinery we proved: any orbit equivalence relation induced by an action of Γ^ω is Borel reducible to some iterate $\Delta(2)^{[\Gamma]^\alpha}$.
- Solecki's theorem: The family of Γ^ω-actions has cofinal essential complexity. In particular the family of iterates $\Delta(2)^{[\Gamma]^\alpha}$ has cofinal essential complexity.
- Now if $E^{[\Gamma]} \sim_B E$, then all iterates $\Delta(2)^{[\Gamma]^\alpha}$ are Borel reducible to E. Since the iterates have cofinal essential complexity, E is not Borel, a contradiction. □

What is left for properness

Question

Which groups Γ give rise to a proper jump?

Example

A "test" group that (1) fails the d.c.c. and (2) fails to have $\mathbb{Z}$ or $\mathbb{Z}_p^{<\omega}$ as a quotient of a subgroup is:

$$\Gamma = \bigoplus_{p \text{ prime}} \mathbb{Z}_p$$

Scattered orders definition

The definition of the Beroulli jumps was initially motivated by the classification of countable scattered orders.

Definition

A linear order L is said to be scattered if there does not exist an embedding from $\mathbb{Q}$ to L.

Example

- α, for α an ordinal
- α^* (reverse), for α an ordinal
- $\mathbb{Z}$
- $\mathbb{Z}^k$, the lexicographic power
- $\mathbb{Z} \cdot (\mathbb{Z} + 2 + \mathbb{Z}) + \mathbb{Z} \cdot (\mathbb{Z} + 3 + \mathbb{Z}) + \mathbb{N}$, combinations using sums and products

Derivatives

Definition

The derivative of L is the quotient $L/\sim$ where:

$$x \sim y \iff \text{the interval between } x, y \text{ is finite}$$

Definition

The α-derivative of L is the quotient $L/\sim_\alpha$ where:

$$\begin{aligned} x \sim_{\beta+1} y &\iff [x]_\beta \sim [y]_\beta \\ x \sim_\lambda y &\iff (\exists \beta < \lambda)\, x \sim_\beta y \end{aligned}$$

Proposition

L is scattered if and only if there exists α such that $L/\sim_\alpha$ is trivial (has just one equivalence class).

Scattered orders derivative

Example

$$L = \mathbb{Z} \cdot (\mathbb{Z} + 2 + \mathbb{Z}) + \mathbb{Z} \cdot (\mathbb{Z} + 3 + \mathbb{Z}) + \mathbb{N}$$

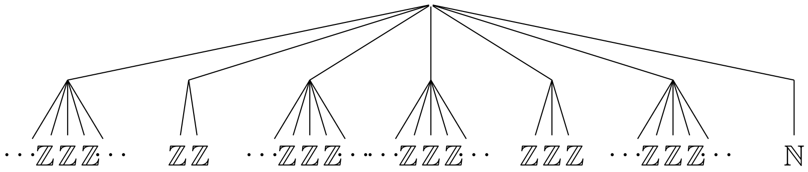
$$L/{\sim} = \mathbb{Z} + 2 + \mathbb{Z} + \mathbb{Z} + 3 + \mathbb{Z} + 1$$

$$L/{\sim_2} = 7$$

$$L/{\sim_3} = 1$$

Remark

We can view L as a well-founded $\mathbb{Z}$-tree: each set of siblings carries a suborder of $\mathbb{Z}$.



Scattered orders rank

Definition

The rank of L is the least α such that $L/{\sim_\alpha} = 1$. (Or, the rank of the corresponding tree is $1 + \alpha$.)

Observation

Incrementing the rank allows up to $\mathbb{Z}$-many structures of the previous rank. This suggests that incrementing the rank results in a $\mathbb{Z}$-jump in complexity.

Scattered orders and the $\mathbb{Z}$-jump

Theorem

The isomorphism relation $\cong_{1+\alpha}$ on countable scattered linear orders of rank $1+\alpha$ is Borel bireducible with the αth iterated jump of the identity $\Delta(\mathbb{N})$ (that is, with $\Delta(\mathbb{N})^{[\mathbb{Z}]^\alpha}$).

Proof sketch.

- First we can confirm that $\cong_{1+\alpha}$ is Borel bireducible with isomorphism of $\mathbb{Z}$-trees of rank $2+\alpha$.
- Second we show that incrementing the rank of the $\mathbb{Z}$-trees corresponds with taking a $\mathbb{Z}$-jump. □

Corollary

Since we know the $\mathbb{Z}$-jump is proper, we conclude that the classification of countable scattered linear orders increases properly in complexity with the rank.

Generic ergodicity

Here we present a selection recent results comparing jumps against one another, and against standard complexities. Many of these comparisons derive from generic ergodicity results.

Definition

E is generically F-ergodic if whenever $x\,E\,x' \implies f(x)\,F\,f(x')$ then f maps a comeager set into a single F-class.

Theorem

For any infinite Γ, we have $(E_0)^{[\Gamma]}$ is generically $(E_\infty)^\omega$-ergodic.

Theorem (Allison–Panagiotopoulos)

$(E_0)^{[\mathbb{Z}]}$ is generically F-ergodic for any F induced by a TSI polish group.

Applications below F_2

The $\mathbb{Z}$-jump provides new examples of complexity points in the Borel reducibility hierarchy.

Theorem

- $(E_0)^\omega <_B (E_0)^{[\mathbb{Z}]} <_B F_2$
- $(E_\infty)^\omega <_B (E_\infty)^{[\mathbb{Z}]} <_B F_2$
- $(E_0)^{[\mathbb{Z}]}$ *and* $(E_\infty)^\omega$ *are Borel incomparable*

Question

We do not know whether there are complexities properly between $(E_0)^\omega$ and $(E_0)^{[\mathbb{Z}]}$.

Varying the group

Theorem (Shani)

Suppose E is generically $\Delta(2)$-ergodic. If Γ is not a quotient of a subgroup of Γ', then $E^{[\Gamma]}$ is generically $(E_\infty)^{[\Gamma']}$-ergodic.

Corollary

- $(E_0)^{[\mathbb{Z}]} <_B (E_0)^{[\mathbb{Z}^2]} <_B \cdots <_B (E_0)^{[\mathbb{Z}^{<\omega}]} <_B (E_0)^{[F_2]}$.
- $(E_0)^{[\mathbb{Z}]}$ *and* $(E_0)^{[\mathbb{Z}_2^{<\omega}]}$ *are Borel incomparable.*

Remark

Shani recently extended his result using tools of Larson–Zapletal: If every homomorphism $\Gamma \to \Gamma'$ has finite image and kernel isomorphic to Γ, then $\Delta(2)^{[\Gamma]^2}$ is generically $\Delta(2)^{[\Gamma']^\alpha}$-ergodic.

Thank you!