An algebraic approach to Borel CSPs

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Introduction

For a finite relational structure $\mathcal{D}$, CSP($\mathcal{D}$) is the set of finite structures which admit a homomorphism to $\mathcal{D}$. And, $\mathrm{CSP}_B(\mathcal{D})$ is the set of Borel structures which admit a Borel homomorphism to $\mathcal{D}$. Note that CSP($\mathcal{D}$) is NP, and $\mathrm{CSP}_B(\mathcal{D})$ is Σ^1_2. Examples:

1. CSP(K_n) is the n-coloring problem for graphs
2. If $\mathcal{D} = (\{0, 1\}; P(\{0, 1\}^3))$, then CSP($\mathcal{D}$) is 3SAT
3. Let $3\mathrm{LIN}_p$ be the finite field $\mathbb{F}_p$ equipped with all affine subsets of $\mathbb{F}_p^3$, then CSP($\mathcal{D}$) is the problem of solving a system of 3-variable linear equations

We call $\mathcal{D}$ the **template** for CSP($\mathcal{D}$), and structures we test for homomorphisms **instances** of the CSP.

Introduction

We can ask **complexity questions** about CSPs. For example: when is CSP($\mathcal{D}$)

1. polynomial time solvable?
2. solvable by constraint propogation?
3. solvable by linear relaxation?

These questions (and many others like them) have all been solved by algebraic methods

Introduction

We can ask similar questions about Borel CSPs. For example:

1. When is $\mathrm{CSP}_B(\mathcal{D})$ Π^1_1?
2. When is a solution in ZFC enough to guarantee a Borel solution? (we'll call these templates **classical**)
3. When is a Borel solution enough to guarantee a Δ^1_1 solution? (we'll call these templates **effectivizable**)

I conjecture that these questions also have algebraic solutions

Introduction

One application is to questions about **bases**: a family F of structures is a basis for $\mathrm{CSP}(\mathcal{D})$ if

$$\mathcal{X} \notin \mathrm{CSP}(\mathcal{D}) \Leftrightarrow (\exists \mathcal{Y} \in F, f)\ f : \mathcal{Y} \to \mathcal{X} \text{ is a homomorphism.}$$

Bases for $\mathrm{CSP}_B(\mathcal{D})$ are defined similarly.

Theorem

$\mathrm{CSP}(\mathcal{D})$ *has a finite basis if and only if it is finitely axiomatizable*

Theorem (Carroy, Miller, Schrittesser, Vidnyanszky)

$\mathrm{CSP}_B(K_2)$ *has a 1-element basis*

Introduction

In the finite setting, many questions about bases have algebraic answers. In general, complexity lower bounds rule out certain kinds of bases:

Proposition

$\mathrm{CSP}_B(\mathcal{D})$ *is classical if and only if it admits a basis of finite structures.*

Theorem (Todorcevic, Vidnyanszky)

$\mathrm{CSP}_B(K_3)$ *is* $\utilde{\Sigma}^1_2$*-complete, so does not admit a* $\utilde{\Sigma}^1_2$ *basis.*

Algebraic Tools

Definition

For a structure $\mathcal{D}$ on D, a **polymorphism** is an n-ary operation on D which is homomorphism from $\mathcal{D}^n$ to $\mathcal{D}$.

Polymorphisms combine solutions to instances of CSP($\mathcal{D}$). They always include projections and are closed under compositions. Such algebras are called clones.

Definition

For a structure $\mathcal{D}$, Pol($\mathcal{D}$) is its clone of polymorphisms.

Algebraic Tools

Examples:

1. The only polymorphisms of 3SAT are projections
2. The polymorphisms of 2SAT are generated by the majority function:

$$\text{maj}(x, y, z) = \text{ the repeated value among } x, y, z$$

3. The polymorphisms of $3LIN_2$ are generated by the minority function

$$\text{min}(x, y, z) = x + y + z$$

Algebraic Tools

The function Pol is one part of a Galois correspondence. On one side we have the lattice of algebras on a set ordered by containment; on the other we have sets of relations ordered by a notion of simulation:

Definition

For a structure $\mathcal{D}$ on D, a relation $R \subseteq D^n$ is **pp-definable** in $\mathcal{D}$ if there are atomic formulae in $\mathcal{D}$ (including equality!) $\alpha_i(\bar{x}, \bar{z})$ so that

$$R(\bar{x}) :\Leftrightarrow (\exists \bar{z}) \bigwedge_i \alpha_i(\bar{x}, \bar{z}).$$

Theorem (Geiger, Bodnartchuk, Kaluznin, Kotov, Romov)

$\mathrm{Pol}(\mathcal{D}) \subseteq \mathrm{Pol}(\mathcal{E})$ *if and only if* $\mathcal{E}$ *is pp-definable in* $\mathcal{D}$.

Algebraic Tools

It follows that $\text{Pol}(\mathcal{D})$ controls the complexity of $\mathcal{D}$:

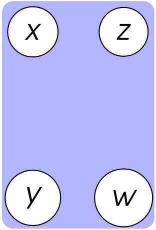
Theorem

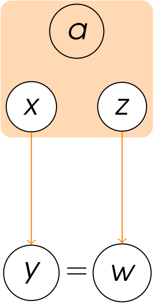
If $\mathcal{D}$ pp-defines $\mathcal{E}$, then $\text{CSP}(\mathcal{E})$ is polynomial time (in fact logspace) reducible to $\text{CSP}(\mathcal{D})$

For example, suppose relations R, S, T satisfy the following:

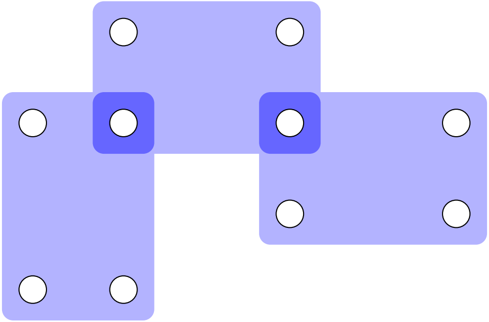
$$R(w,x,y,z) \Leftrightarrow S(x,y) \wedge S(z,w) \wedge w = y \wedge (\exists a)\ T(x,y,a)$$

Then R is pp-definable in $\{S, T\}$.

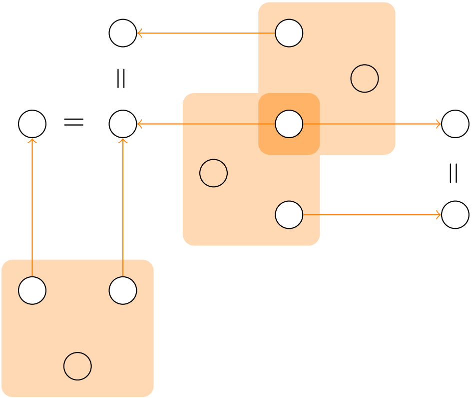




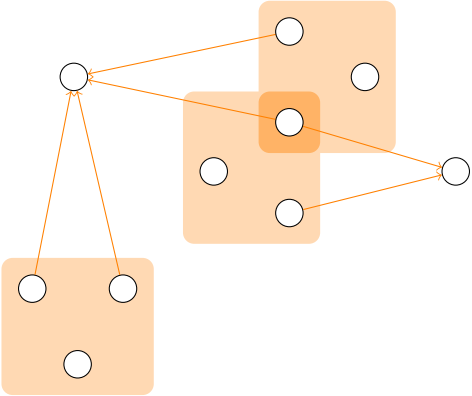
Algebraic Tools



Algebraic Tools



Algebraic Tools



Algebraic Tools

Definition

A structure $\mathcal{E}$ is **pp-interpretable** in $\mathcal{D}$ if there is an onto function $f : A \to \mathcal{E}$ so that the relations $A \subseteq D^n$ and $f^{-1}(R)$ for every relation $R \in \mathcal{E}$ (including equality!) are pp-definable in $\mathcal{D}$.

Theorem (Ess. Bulatov and Jeavons)

$\mathcal{E}$ is pp-interpretable in $\mathcal{D}$ if and only if (a reduct of) $\mathrm{Pol}(\mathcal{E})$ is in the variety generated by $\mathrm{Pol}(\mathcal{D})$ (in the sense of universal algebra).

So, $\mathcal{E}$ is pp-interpretable in $\mathcal{D}$ if every identity satisfied by operations $\mathrm{Pol}(\mathcal{D})$ is satisfied by operations in $\mathrm{Pol}(\mathcal{E})$.

Algebraic Tools

Definition

Two structures are **equivalent** if there are homomorphisms between them. A structure is a **core** if it is not equivalent to any of its proper substructure.
A structure $\mathcal{E}$ is **pp-constructible** in $\mathcal{D}$ if there is sequence of structures

$$\mathcal{E}_0 = \mathcal{E}, \mathcal{E}_1, ..., \mathcal{E}_n = \mathcal{D}$$

so that each $\mathcal{E}_i$ is either pp-interpretable in $\mathcal{E}_{i+1}$, is equivalent to $\mathcal{E}_{i+1}$, or is a core and $\mathcal{E}_{i+1}$ is an expansion of $\mathcal{E}_i$ by a singleton unary relation.

Note that for any $\mathcal{D}$ there is an $\mathcal{D}'$ so that $\mathcal{D}$ and $\mathcal{D}'$ pp-construct each other and $\mathcal{D}'$ includes all singletons as unary relations. One can also characterize pp-constructibility by so-called "height-1 identities"

Algebraic Tools

A theorem of Taylor from the 70s characterizes when an idempotent algebra does not have any projection algebra in its variety

Theorem (Taylor, Siggers)

For a structure $\mathcal{D}$, the following are equivalent:

1. $\mathrm{Pol}(\mathcal{D})$ *does not contain an operation f satisfying*

$$f(a, r, e, a) = f(r, a, r, e)$$

2. $\mathcal{D}$ *pp-constructs 3SAT*
3. $\mathcal{D}$ *pp-constructs every structure*

Definition

A structure $\mathcal{D}$ is **intractable** if it satisfies any of the above properties. It is **tractable** otherwise.

Algebraic Tools

Assuming P≠NP, if $\mathcal{D}$ is intractable, it is not polynomial time solvable. Remarkably, the converse is true.

Theorem (Bulatov, Zhuk)

If $\mathcal{D}$ is tractable it is polynomial time solvable.

Corollary

$\{\mathcal{D} : \mathrm{CSP}(\mathcal{D}) \in P\} \in NP$

Many other classes of structures admit similar characterizations.

Borel CSPs

The polynomial time reductions given by pp-constructions adapt almost word for word to the descriptive setting, except when we need to enforce equality

Definition

$\mathcal{E}$ is **simply definable** (or interpretable or constructible) in $\mathcal{D}$ if it is pp-definable (or interpretable or constructible) using predicates which don't include $=$.

Borel CSPs

Proposition

If $\mathcal{E}$ is simply constructible in $\mathcal{D}$, then $\mathrm{CSP}_B(\mathcal{E})$ is Borel reducible to $\mathrm{CSP}_B(\mathcal{D})$. In fact, there are maps F, G, H which are Δ^1_1 in the codes so that,

1. *whenever $\mathcal{X}$ is an instance of $\mathcal{E}$, $F(\mathcal{X})$ is an instance of $\mathcal{D}$*
2. *if h is a solution to $\mathcal{X}$, then $G(h)$ is a solution to $F(\mathcal{X})$*
3. *if g is a solution to $F(\mathcal{X})$, then $H(g)$ is a solution to $\mathcal{X}$.*

Corollary

If $\mathcal{D}$ and $\mathcal{E}$ have equality in their signature and $\mathrm{Pol}(\mathcal{D})$ and $\mathrm{Pol}(\mathcal{E})$ satisfy the same identities, then $\mathrm{CSP}_B(\mathcal{D}) \equiv_B \mathrm{CSP}_B(\mathcal{E})$.

Some technical lemmas let us remove assumptions about equality:

Lemma

1. *If $\mathcal{D}$ has a transitive automorphism group and is a core, then $\mathcal{D}$ simply defines equality.*
2. *If R is a relation so that $\pi_{ij}R \not\subseteq (=)$, and R is invariant under a quotient of a subalgebra of $\mathrm{Pol}(\mathcal{D})$, then R is simply interpretable in $\mathcal{D}$.*

Borel CSPs

Tools from tame congruence theory give us a further refinement of intractability:

Definition

Let N be the relation on $\{0, 1\}$ given by

$$N(x, y, z) :\Leftrightarrow x, y, z \text{ are not all equal}$$

Theorem (Bulatov and Jeavons)

If $\mathcal{D}$ is intractible and includes every singleton unary relation, then N is invariant under a quotient of a subalgebra of $\mathrm{Pol}(\mathcal{D})$

Borel CSPs

Theorem

If $\mathcal{D}$ is intractable, then $\mathcal{D}$ simply constructs every structure. In particular $\mathrm{CSP}_B(\mathcal{D})$ *is* $\underset{\sim}{\Sigma}^1_2$*-complete.*

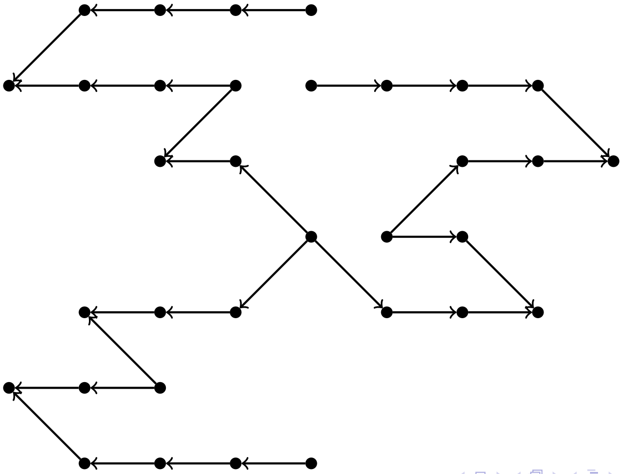
Proof sketch.

We can replace $\mathcal{D}$ with a structure $\mathcal{D}'$ that defines all of its singletons. Then N is invariant under a quotient of a subalgebra of $\mathrm{Pol}(\mathcal{D}')$. Since N does not imply any equations, $\mathcal{D}'$ simply interprets N. Since N has a transitive automorphism group and is intractable, it simply constructs every structure. □

Borel CSPs

Corollary

The directed graph below has a $\underset{\sim}{\Sigma}^1_2$-complete Borel CSP



Borel CSPs

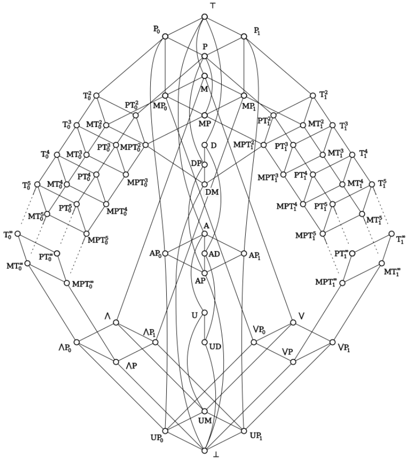
Corollary

For a simple undirected graph G, the following are equivalent:

1. *G is tractable*
2. *G is bipartite*
3. *G is effectivizable*
4. $\mathrm{CSP}_B(G)$ *is* Π^1_1
5. $\mathrm{CSP}_B(G)$ *is not* $\underset{\sim}{\Sigma}^1_2$*-complete.*

A partial Schaefer type theorem

In the 40s, Post classified the clones on 2 elements:



A partial Schaefer type theorem

Examining the bottom part of this picture gives Schaefer's theorem:

Theorem (Schaefer)

If $\mathcal{D}$ is a structure on $\{0, 1\}$, then one of the following must hold:

1. *$\mathcal{D}$ is pp-constructible in HornSAT*
2. *$\mathcal{D}$ is pp-definable in 2SAT*
3. *$\mathcal{D}$ is pp-definable in $3LIN_2$*
4. *$\mathcal{D}$ pp-defines $(\{0, 1\}, N)$*

In the first 3 cases, $\mathcal{D}$ is tractable, in the last case it is not.

A partial Schaefer type theorem

The first case of Schaefer's theorem has a simple algebraic characterization:

Definition

An n-ary operation f is **totally symmetric** if

$$f(x_1, \dots, x_n) = f(y_1, \dots, y_n)$$

whenever $\{x_1, \dots, x_n\} = \{y_1, \dots, y_n\}$.

For example, constant functions, $\sup$ and $\inf$ are totally symmetric. A structure is pp-constructible in HornSAT if and only if it has a totally symmetric polymorphism.

A partial Schaefer type theorem

Theorem

If $\mathcal{D}$ has a totally symmetric polymorphism g of arity at least $|\mathcal{D}|$, then $\mathrm{CSP}_B(\mathcal{D})$ is classical

Proof.

If an instance $\mathcal{X}$ of $\mathcal{D}$ has a solution, then there is a function $f : \mathcal{X} \to \mathcal{P}(\mathcal{D})$ such that, whenever $a \in f(x)$ and $x = x_i$ for some $(x_1, ..., x_n) \in R^{\mathcal{X}}$, there is $(a_1, ..., a_n) \in R$ with $a_i \in f(x_i)$ and $a_i = a$. Using a reflection argument, we can find a Borel function f with the same property. Then $g \circ f$ is a Borel solution to $\mathcal{X}$. □

The converse of the above theorem is also true

A partial Schaefer type theorem

$\mathcal{D}$ is pp-definable in 2SAT if and only if $\mathtt{maj}$ is a polymorphism of $\mathcal{D}$. We can prove effectivization for a slightly more general class of problems.

Definition

The **dual descriminator operation** on a domain D is the function

$$d(x,y,z) = \begin{cases} x & y \neq z \\ y & \text{otherwise} \end{cases}$$

A partial Schaefer type theorem

Proposition (Folklore)

A relation on D is invariant under the dual discriminator iff it is pp-definable in $\mathcal{D} = (D; \tau)$, where τ is the set of the following relations:

1. $R_{a,b}(x, y) :\Leftrightarrow x = a \vee y = b$ *for* $a, b \in D$
2. $R_{\pi}(x, y) :\Leftrightarrow y = \pi(x)$ *for some* $\pi \in S_D$
3. *any unary predicate*

A partial Schaefer type theorem

Theorem

The structure $\mathcal{D}$ from the previous slide admits effectivization (and therefore so does any structure with a dual discriminator polymorphism)

Proof.

An instance $\mathcal{X}$ of $\mathcal{D}$ has a solution if and only if there is a countable sequence of partial functions $\langle f_i : i \in \omega \rangle$ with $f_i : X \to D$ so that,

1. If $R_{a,b}(x, y)$, $x \in \mathrm{dom}(f_i)$, and $f_i(x) \neq a$, then $f_i(y) = b$
2. If $R_\pi(x, y)$ and $x \in \mathrm{dom}(f_i)$, then $f_i(y) = \pi(f_i(x))$
3. If $U(x)$, then $f_i(x) \neq a$ for any $a \notin U$
4. $X = \bigcup_i \mathrm{dom}(f_i)$

Conditions $(1 - 3)$ are closure properties and independence properties, so a general effectivization theorem applies. □

A partial Schaefer type theorem

Putting this all together we get:

Theorem

For $\mathcal{D}$ any structure on $\{0, 1\}$, one of the following is true:

- $\mathrm{Pol}(\mathcal{D})$ *contains a totally symmetric term, and* $\mathcal{D}$ *is classical*
- $\mathrm{Pol}(\mathcal{D})$ *contains* $\mathtt{maj}$*, and* $\mathcal{D}$ *is effectivizable*
- $\mathcal{D}$ *is intractable and* $\mathrm{CSP}_B(\mathcal{D})$ *is* $\utilde{\Sigma}^1_2$*-complete*
- $\mathrm{CSP}_B(\mathcal{D}) \equiv_B \mathrm{CSP}_B(3LIN_2)$.

A partial Schaefer type theorem

It is unclear how hard $\mathrm{CSP}_B(3\mathrm{LIN}_2)$ is. But, we have the following:

Theorem (Barto, Kozik)

For any $\mathcal{D}$, either $\mathrm{Pol}(\mathcal{D})$ has some affine algebra as a quotient of a subalgebra, or $\mathcal{D}$ is bounded width.

Bounded width structures are solvable by greedy algorithms. Arguments similar to the previous theorem gives effectivization in many special cases.

Questions

Problem

Is every $\mathrm{CSP}_B(\mathcal{R})$ either $\undertilde{\Pi}^1_1$ or $\undertilde{\Sigma}^1_2$-complete? (True under $\undertilde{\Sigma}^1_2$ Determinacy)

Problem

Is $\mathrm{CSP}_B(3\mathrm{LIN}_2)$ $\undertilde{\Sigma}^1_2$-complete?

Problem

If $\mathcal{E}$ is pp-interpretable in $\mathcal{D}$, and $\mathcal{D}$ is effectivizable (or $\undertilde{\Pi}^1_1$) is the same true of $\mathcal{E}$?