New examples of bounded degree acyclic graphs with large Borel chromatic number

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Caltech Logic Seminar

Preliminaries

Let G be a graph on the vertex set $V(G)$. Define

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The *chromatic number* of G is the minimal n for which G has an n-coloring. Notation: $\chi(G)$.

On Borel combinatorics

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$\mathcal{F}$-measurable chromatic numbers, if $\mathcal{F} \subset \mathcal{P}(V(G))$ is a σ-algebra. Notation: $\chi_{\mathcal{F}}(G)$.

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Theorem. (Conley–Marks–Tucker-Drob) Let G be a Borel graph with $V(G)$ Polish, and μ be a measure on $V(G)$. Let $d \geq 3$. If $\Delta(G) \leq d$ then $\chi_{BM}(G), \chi_\mu(G) \leq d$ unless G contains a K_{d+1}.

Marks' method

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Consider $d = 3$. Let $\Gamma = \langle \alpha, \beta, \gamma | \alpha^2 = \beta^2 = \gamma^2 = 1 \rangle$. Γ acts on the space n^Γ by the left-shift action, i.e., for $\delta, \delta' \in \Gamma$ let

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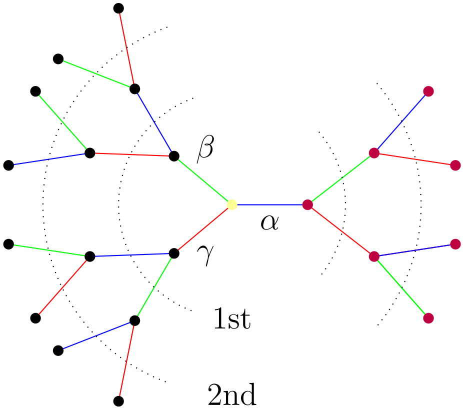
The Schreier-graph of this action on n^Γ is defined by making x, x' adjacent if for some $\delta \in S = \{\alpha, \beta, \gamma\}$ we have $\delta \cdot x = x'$. Let G be the restriction of this graph to
$Free(n^\Gamma) = \{x : \Gamma \text{ acts freely on the component of } x\}$.

Marks' method



New examples

Let H be a Borel graph. Γ acts on the space $V(H)^\Gamma$ by the left-shift action, and define the Schreier-graph as before.

Let $Hom(\Gamma, S; H)$ be the restriction of the Schreier-graph to the set

$$\{h \in V(H)^\Gamma : h \text{ is a homomorphism from } Cay(\Gamma; S) \text{ to } H\}.$$

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- $\chi_B(H) \leq 3$ implies $\chi_B(Hom(\Gamma, S; H)) \leq 3$.
- $\chi_{\Delta^1_2-abs}(H) > 3$ implies $\chi_B(Hom(\Gamma, S; H)) > 3$.

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Open questions

- Is the collection of compact free subshifts of 2^Γ with Borel chromatic number ≤ 3 also Σ^1_2-complete?

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- Is toastability Σ^1_2-complete on bounded degree acyclic Borel graphs?

Thank you for your attention!