Measurable perfect matchings

Matt Bowen, Gábor Kun, Marcin Sabok

McGill University, Rényi Institute+Eötvös University

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Graphings and equidecompositions

A probability measure preserving (pmp) Borel graph is called a graphing. Graphings are locally finite today.

Γ a set of isometries of $\mathbb{R}^n$. $A, B \subseteq \mathbb{R}^n$ are Γ-equidecomposable if there are finite partitions $A = \cup_{n=1}^k A_n$, $B = \cup_{n=1}^k B_n$ and $\gamma_1, \ldots, \gamma_k \in \Gamma$ such that $A_i = \gamma_i B_i$.

Given a finite Γ, A and B admit a Γ-equidecomposition iff the measurably bipartite graphing $V(G) = A \cup^* B$, $E(G) = \{(a, b) : \exists \gamma \in \Gamma \ \gamma a = b\}$ admits a perfect matching Γ.

Banach, Tarski (1924): Any two bounded sets of nonempty interior in $\mathbb{R}^3$ are equidecomposable.

Amenable group actions

A group is amenable if it admits no paradoxical decomposition.
E.g., isometries of the plane.

Tarski (1925): Are the unit square and the disc of unit area equidecomposable by isometries?

Laczkovich (1990): Yes!

Grabowski, Máthé, Pikhurko (2017): Measurable equidecomposition

Marks, Unger (2017): Borel equidecomposition

Máthé, Noel, Pikhurko (2021): Jordan-measurable pieces

These proofs use translations only. They give equidecomposition of bounded sets with equal measure and box dimension $< n$ in $\mathbb{R}^n$.

Gardner's conjecture

$\langle\Gamma\rangle$ is the subgroup generated by Γ.

Gardner (1991): $A, B \subseteq \mathbb{R}^n$ bounded measurable, Γ set of isometries. $\langle\Gamma\rangle$ is amenable and A and B are Γ-equidecomposable. Are A and B measurably equidecomposable by isometries?

Cieśla, Sabok (2019): Holds if Γ is commutative and A and B are Γ-uniform. Including free pmp group actions.

Bowen, K, Sabok (2021): Holds if Γ is amenable, $\mathbb{Z}_2 * \mathbb{Z}_2$ is not a quotient by a finite normal subgroup and A and B are Γ-uniform. Including free pmp group actions.

Laczkovich (1988): Γ is not sufficient.

K (2021): $\langle\Gamma\rangle$ is not sufficient.

Expansion (in bipartite graphings)

We assume that for every measurable set $\mu(N(S)) > (1+\varepsilon)\mu(S)$. Includes pmp ergodic actions of Kazhdan Property (T) groups.

Banach-Ruziewicz problem: For $n > 1$ is the only $SO(n)$-invariant finitely additive probability measure on S^n the Lebesgue measure?

Margulis (1980), Sullivan (1981): n ≥ 4

Drinfeld (1984): n ≥ 2

Lyons, Nazarov (2011): Every bipartite Cayley graph of a non-amenable group admits a factor of iid perfect matching.

Grabowski, Máthé, Pikhurko (2017): $n \geq 3, A, B \subseteq \mathbb{R}^n$ bounded measurable of nonempty interior, $\lambda(A) = \lambda(B)$. Then A and B are measurably equidecomposable.

Graphings without measurable matchings

Laczkovich (1988): 2-regular acyclic graphing without measurable perfect matching.

Conley, Kechris (2013): Modified it to d-regular for even d.

An (essentially) acyclic graphing is called a treeing.

Marks (2013): d-regular treeing without Borel perfect matching for $d > 2$.

Kechris, Marks (2018): Does every 3-regular graphing admit a measurable perfect matching?

K (2021): Measurably bipartite d-regular treeing without measurable perfect matching for $d > 2$. Moreover, every bounded measurable circulation in L_1 is a.e. zero.

Ends of hyperfinite graphings

A graphing is hyperfinite if the induced equivalence relation is an increasing union of equivalence relations with finite classes. Includes pmp Schreier graphs of amenable groups.

The number of ends in a graph is the supremum of infinite components after the removal of a finite cutset. A hyperfinite graphing is a.e. zero-ended iff a.e. component is finite, a.e. one-ended if it has a.e. superlinear growth, and a.e. two-ended iff it has a.e. linear growth.

Timár (2018): A one-ended Cayley graph admits a factor of iid a.s. one-ended spanning subforest.

Conley, Gaboriau, Marks, Tucker-Drob (2021): An a.e. one-ended graphing has a measurable a.e. one-ended spanning subforest.

Main results

BKS: Assume that a bipartite graphing G admits a measurable fractional perfect matching τ and the support of τ is hyperfinite and one-ended. Then G admits a measurable perfect matching.

Same holds for rounding flows with integral capacities. Applications to equidecompositions: Gardner, measurable circle squaring.

Regular hyperfinite one-ended bipartite graphings admit measurable perfect matchings.

BKS: A bipartite Cayley graph admits a factor of iid perfect matching iff the group is not isomorphic to $\mathbb{Z} \ltimes H$ for an odd H.

Timár (2021): Factor matching of optimal tail between Poisson processes on the plane

Measurable circle squaring using independent translations

Laczkovich (1990): Circle squaring using random translations

Consider the measurable bipartite graphing. Take two set of random translations (black+white) and the two equidecompositions. This gives a black (white) measurable fractional perfect matching.

We suffice to show that the union of the two supports is a.e. one-ended or finite. If a black+white component does not contain an infinite black (white) then OK. If it contains infinitely many infinite black (white) then it has superlinear growth.

The union of the set of components containing finitely many but > 0 infinite black and white components is a nullset: we can choose a "Vitali type set", e.g., the set of vertices in infinite black components closest to infinite white ones.

A close construction

Lemma: $\forall d \geq 3, \varepsilon > 0$ there exists a treeing T s.t. $\Delta(T) = d$, the set of vertices with degree less than d has measure less than ε and every circulation in $L_1^{as}(E(T))$ is zero a. e.

We construct a sequence of finite graphs $\{G_n\}_{n=1}^{\infty}$ recursively.
$G_1 = K_{d,d}$. $A_n \subset \{-1, 0, +1\}^{E(G_n)}$ antisymmetric functions.
Choose $N(n, \varepsilon) = \frac{2^n 3^{|E(G_n)|}}{\varepsilon}$ and set
$V_{n+1} = \Pi_{f \in A(G_n)} \{1, \ldots, N(\varepsilon, n)\}$,
$V(G_{n+1}) = V(G_n) \times V_{n+1}$ and $E(G_{n+1}) =$
{ ((x,v), (x',v')): (x,x') $\in E(G_n), \forall f \in A_n \; v'_f - v_f = f(v, v')\}$.

Set $V(T) = \Pi_{n=1}^{\infty} V_n$. This is a treeing. $E(T) = \cap_{n=1}^{\infty} \pi_n^{-1} E(G_n)$.

Why are circulations trivial?

For every $n, g \in A_n$, circulation $f \in L_1^{as}(E(T))$, we have
$\int_{e \in E(T)} f(e) g(\pi_n(e)) = 0$,
since there exists $p \in \mathbb{R}^{V(G)}$ potential s.t. $g(x, y) = p(y) - p(x)$.

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Let $f \in L_1^{as}(E(T))$ circulation. There exist an n and $g \in A_n$ approximating $sign(f)$ such that $\|fg(\pi_n) - |f|\|_1 \leq \|f\|_1/2$.

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By the triangle inequality
$0 = \int_{E(T)} fg(\pi_n) = \int_{E(T)} |f| + \int_{E(T)} fg(\pi_n) - |f| \geq \|f\|_1/2$, hence f is zero a. e.

Sketch of the rest

We suffice to find a treeing without non-zero bounded circulation s.t. the degrees are divisible by d: the vertices of such a treeing can be split into vertices of degree d.

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We start with a treeing T with large maximum degree $(2d^2)$ and close average degree $(2d^2 - 1/d)$. As a preparation we find a subtreeing where the minimum degree is at least d and every vertex has at least three neighbors with degree divisible by d.

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Choose a d-tuple of vertices at even, close distance all with the same degree modulo d and rewire the extra edges by replacing the with paths. Iterate. The total length of these paths is $<< \int_{[0,1]} log(x)dx < \infty$.

Proof sketch for matchings: extreme points

Consider the set of measurable fractional perfect matchings. By Krein-Milman this is the convex hull of the extreme points.

Lemma: The fractional part of an extreme point τ is a.e. zero or half. The set of edges where τ is half is essentially a vertex-disjoint union of bi-infinite paths.

The set of edges where the fractional value is integral is essentially acyclic: else we could add or subtract a circuit on a set of positive measure. By hyperfiniteness this is a disjoint union of paths. It should be half on every path, else we could round it.

Proof sketch for matchings: using cycles from tiling

Given τ that is integral everywhere, but on a disjoint union of infinite paths where its fractional part is half. Consider a family of cycles $\mathcal{C}$ that covers at least half of the edges of these paths at least K times for a large K, and every other edge at most once.

Consider $\tau' = (1-\lambda)\tau + \lambda\tau_0$ and add or subtract for every cycle in $\mathcal{C}$ the $+/-\varepsilon$-circuit randomly and for disjoint cycles independently.

The total distance from integers on τ' is smaler than on τ: the gain will have a factor $O(K^{1/2})$ to outweigh the loss...

Thank you!