

Schreier decorations of unimodular random graphs

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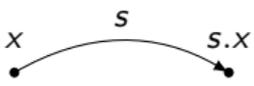
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$\Gamma = \langle S \rangle$ finitely generated group, X set, $\Gamma \curvearrowright X$.

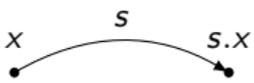
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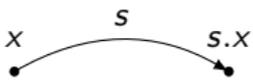
$\Gamma = \langle S \rangle$ finitely generated group, X set, $\Gamma \curvearrowright X$.

Vertices: $x \in X$. Edges: A diagram showing a directed edge from a vertex labeled x to a vertex labeled $s.x$. The edge is a curved arrow pointing from x to $s.x$ and is labeled with the letter s above it.

Notation: $\text{Sch}(\Gamma, S, X)$. Oriented, labeled graph.

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Schreier graphs of Γ



transitive

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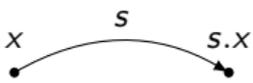
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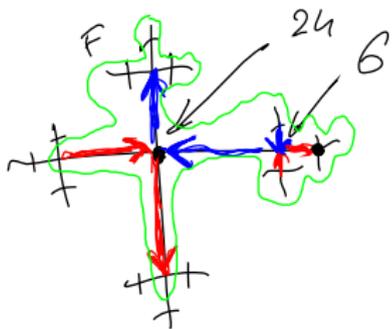
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Folklore combinatorics: every $2d$ -regular finite graph is a Schreier graph of the free group F_d .

I.e. the edges have an orientation and d -coloring such that at each vertex there is exactly one incoming and outgoing edge of each color. (We call this a Schreier decoration.)

An invariant random Schreier decoration

Task: Find an $\text{Aut}(T_4)$ -invariant random Schreier decoration of T_4 .



Start from root, go
radially outwards.

Claim: does not depend on
the choice of root in distr.

Proof:

$$\begin{aligned} \mathbb{P}[\text{seeing a fixed dec in } F] \\ = \frac{1}{24} \cdot \binom{|F|-1}{1/6} \end{aligned}$$

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- Invariant random rooted Schreier graph = coset Schreier graph of an IRS.
- Grebik has a **quasi-p.m.p.** version now.

Graphings and unimodular random graphs

Definition (graphing)

Let (X, μ) be a standard Borel probability space. A **graphing** is a locally finite graph \mathcal{G} on $V(\mathcal{G}) = X$ with Borel edge set $E(\mathcal{G}) \subset X \times X$ satisfying

$$\int_A \deg_B(x) d\mu(x) = \int_B \deg_A(x) d\mu(x)$$

for all measurable sets $A, B \subseteq X$, where $\deg_S(x)$ is the number of edges from $x \in X$ to $S \subseteq X$.

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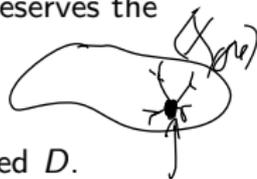
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Intrinsic formulation of the p.m.p. condition: **involution invariance**, or **Mass Transport Principle**, or **reversibility of random walk**.

Examples

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Less boring example: Galton-Watson tree.

Every connected component is (P, o) , the rooted bi-infinite path.

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Invariant random (rooted) Schreier graphs

Random rooted Schreier graph: $(\tilde{G}, \tilde{\sigma}, \text{dec})$ random, where dec is the combinatorial data. I.e. $\tilde{\nu} \in P(\mathfrak{G}_{\bullet}^{\text{Sch}})$, where $\mathfrak{G}_{\bullet}^{\text{Sch}}$ is the space of rooted Schreier graphs.

Invariance: F_d acts on rooted Schreier graphs by moving the root:
 $s.(\tilde{G}, \tilde{\sigma}, \text{dec}) = (\tilde{G}, s.\tilde{\sigma}, \text{dec})$. Want $F_d \curvearrowright (\mathfrak{G}_{\bullet}^{\text{Sch}}, \tilde{\nu})$ to be p.m.p.

Two viewpoints

Random rooted graphs

Unimodular random rooted graph
 $(G, o) \sim \nu$ where $\nu \in P(\mathcal{G}_{\bullet}^D)$ unimod.

Measurable combinatorics

Graphing
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$\Phi : \mathfrak{G}_{\bullet}^{\text{Sch}} \rightarrow \mathfrak{G}_{\bullet}^D$ forgets the decoration,

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NOT Thm: Every $2d$ -reg graphing
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IS Thm: Every $2d$ -reg \mathcal{G} is a *local isomorphic image* of some \mathcal{G}' that is a Schreier graphing of F_d .

Back to the example

Factors of iid and Bernoulli graphings

$d = 1$

Invariant random language

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Subquestion: What about Bernoulli graphings? On what fixed graphs are there FIID Schreier decorations?

Theorem (Bencs, Hrušková, T., 2021)

For any $d \geq 1$ there are $2d$ -reg graphs that have no factor of iid Schreier decoration.

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Builds on toasts, could have Borel version.

Factor of iid Schreier decorations II

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*Non-amenable quasi-transitive unimodular graphs (e.g. regular infinite trees) of even degree have a factor of iid **balanced orientation**.*

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- Standard representation theoretic proof for Bernoulli shifts of non-amenable groups;

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- Is measurably decorating the Bernoulli graphing the most difficult among graphings with the same local statistics?

THANK YOU FOR YOUR ATTENTION

Unimodular random rooted graphs

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