Borel factor maps and embeddings between $\mathbb{Z}^d$ actions

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Ergodic theory

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Definition

Let a and b be Borel actions of a group Γ on spaces X and Y respectively. A function $f : X \to Y$ is equivariant if it commutes with the actions a and b, that is, for all $x \in X$ and $\gamma \in \Gamma$, $f(\gamma \cdot x) = \gamma \cdot f(x)$.

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Note that in the ergodic theory context if a and b are measure preserving actions on (X, μ) and (Y, ν) respectively, then saying that f is measurable means that it only needs to be defined on some μ-conull set.

Shift spaces

Recall:

Definition

The k-shift action is the space k^Γ *of functions* $x : \Gamma \to \{0, \dots k-1\}$ *with the action given by* $(\gamma \cdot x)(\delta) = x(\delta \cdot \gamma^{-1})$.

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A shift space is a closed subset of k^Γ for some k and Γ.

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Theorem (Hochman)

The shift action on $2^{\mathbb{Z}}$ is universal.

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This is a nice Borel graph which captures many of the properties of the action.
For instance, saying that the Borel chromatic number of some Cayley graph G_a is at most k is equivalent to the existence of an equivariant Borel map from X into a natural space of k-colorings of the Cayley graph of $\mathbb{Z}^d$.

Some nice shift spaces

Definition

A rectangle in $\mathbb{Z}^d$ is a product of intervals. For a finite set of rectangles $\mathcal{T}$, we define a space $X_{\mathcal{T}}$ to be the set of functions $x : \mathbb{Z}^d \to \mathcal{T}$ such that $f^{-1}R$ is a disjoint union of translates of R.

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The space of graph homorphisms generalizes vertex colorings.

Main theorems

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Then there is an equivariant Borel map from X to Y where the range consists of aperiodic points.

Some similar theorems are referenced as "to appear in a forthcoming paper" in various papers of Gao, Jackson, Krohne and Seward.

Main theorems continued

Theorem (Chandgotia-U)

Suppose that X is a closed subset of a shift space $k^{\mathbb{Z}^d}$ consisting of aperiodic points and Y is either of the following spaces:

1. *The space of homomorphisms of the Cayley graph of $\mathbb{Z}^d$ in to a finite graph H of size at least 3 which is not bipartite.*
2. *The space of domino tilings of $\mathbb{Z}^d$.*

if $h_{top}(X) < h_{top}(Y)$ then there exists an equivariant Borel embedding $\phi : X \to Y$.

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A larger project

Question (Weiss)

Suppose that a is a free Borel action of a finitely generated amenable group. Is a hyperfinite?

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The current best known result is due to Conley, Jackson, Marks, Seward and Tucker-Drob who extract a combinatorial condition (finite Borel asymptotic dimension) that implies hyperfiniteness.

Restrictions on hyperfiniteness

The following is a theorem of Gao, Jackson, Krohne and Seward.

Theorem

Let a be a free minimal action of a countable group Γ *on a compact Polish space* X *by homeomorphisms. Let* $B_n \subseteq X$ *be a sequence of Borel sets such that for all finite* $F \subseteq \Gamma$ *and for all sufficiently large* n*, the set* $\{x \in X \mid \gamma \cdot x \in B_n \text{ for all } \gamma \in F\}$ *is a complete section of a comeager set. Then the set* $\{x \in X \mid x \text{ belongs to } B_n \text{ for infinitely many } n\}$ *is comeager.*

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3. By our assumption and the first item $B_{n,F}$ is nonmeager for all large enough n.
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6. It follows that for all large enough n, there is $\gamma \in F$ such that $\gamma \cdot U \cap B_{n,F}$ is nonmeager.

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6. It follows that for all large enough n, there is $\gamma \in F$ such that $\gamma \cdot U \cap B_{n,F}$ is nonmeager.
7. Now $\gamma^{-1} \cdot (\gamma \cdot U \cap B_{n,F}) \subseteq U \cap B_n$ is nonmeager for all large enough n, so we are done.

A consequence for $\mathbb{Z}^d$ actions

We can derive another theorem of Gao, Jackson, Krohne and Seward from this.

Theorem

Let $d \geq 2$ and a be a free minimal action of $\mathbb{Z}^d$ such that subaction with respect to the $\mathbb{Z} \times \{0\}^{d-1}$ is also minimal. Given a sequence of Borel sets $B_n \subseteq X$ with the following properties:

1. *B_n is a complete section.*
2. *The connected components of B_n are finite rectangles such that if v_n is the minimum side length of a rectangle in B_n, then $\lim_{n\to\infty} v_n = \infty$.*

Then the set

$$\{x \in X \ : \ x \text{ belongs to } \partial B_n \text{ for infinitely many } n\}$$

is comeager.

Almost squares

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We can extend this notion to finite subsets F of X where we have an action of $\mathbb{Z}^d$ and F is contained in a single orbit.

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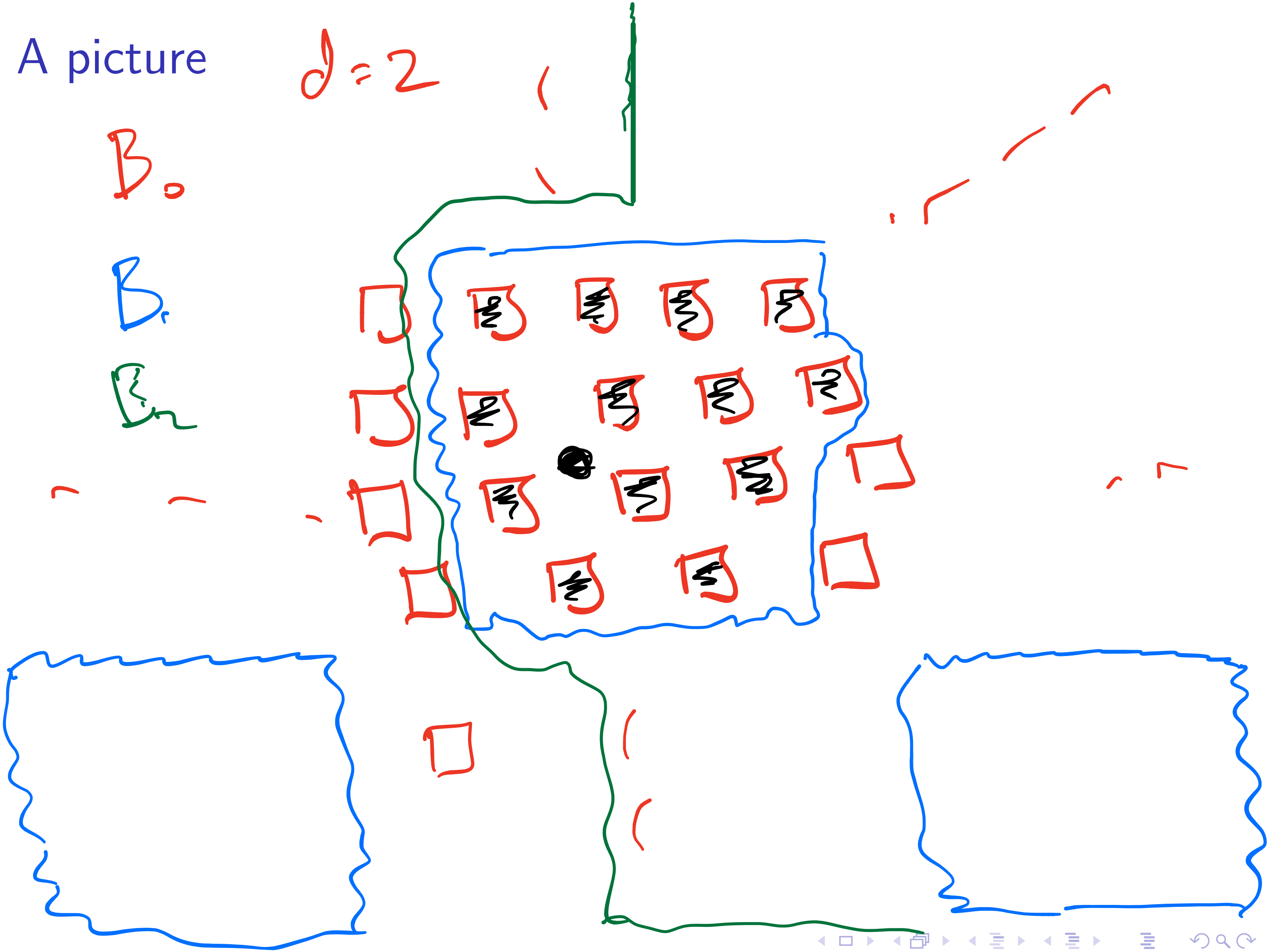
Theorem

Let a be a free action of $\mathbb{Z}^d$ on X with $d > 1$ and $\delta > 0$. If $r_1 < r_2 \dots$ is a sequence of natural numbers satisfying $12 \sum_{j<k} r_j < \delta r_k$, then there is a sequence of Borel sets $B_1, B_2, \dots$ such that

1. *the connected components C of B_j are $(1+\delta)$-almost squares of side length r_j whose complement is connected.*
2. *for all $x \in X$, there is $k \in \mathbb{N}$ such that $x \in B_k$ and*
3. *if C, D are connected components of B_l and B_m respectively with $l \leq m$, then $d(\partial C, \partial D) > r_l$.*

A picture



How to define a map from this?

We make use of the connection with measurable combinatorics. Given a, X, $B_1, B_2, \ldots$ and a target space $Y \subseteq k^{\mathbb{Z}^d}$ we define maps $f_n : B_n \to k$ such that setting $f = \bigcup_{n \geq 1} f_n$ we have that $\hat{f}$ defined by

$$\hat{f}(x) = (\gamma \mapsto f(\gamma \cdot x))$$

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This means for example that if Y is a space of tilings, then the functions f_n define tilings of the Cayley graph of a restricted to B_n which has finite connected components.
There is an interplay here between the shape of the connected components of the B_i and our ability to extend patterns on components of B_i for $i < n$ to B_n.

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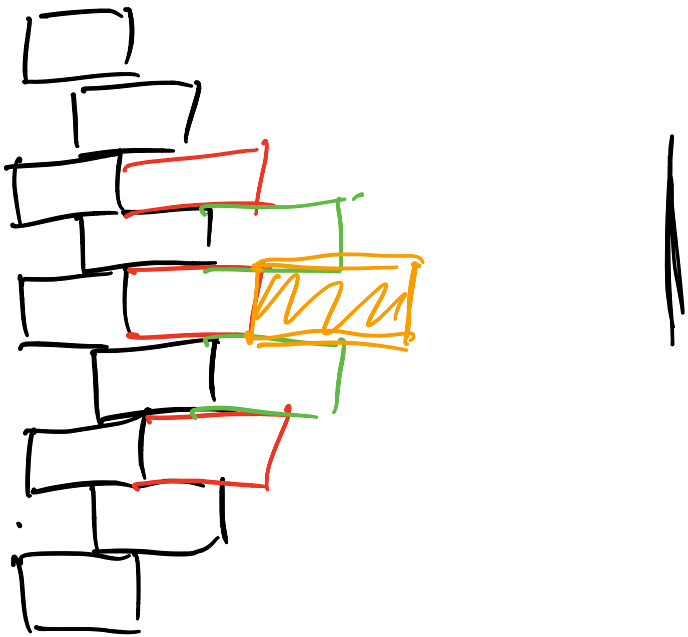
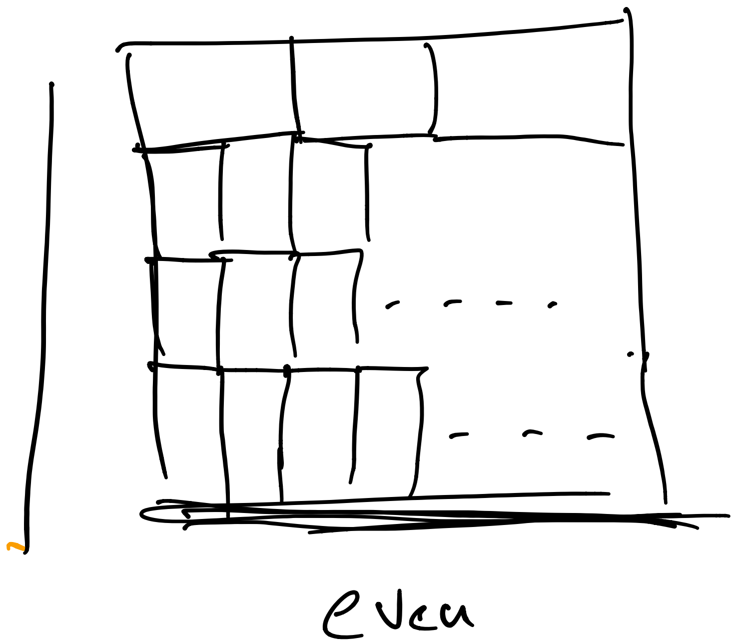
Not all patterns extend

Domino Tilings, $d = 2$



Some patterns do extend



Open problems

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2. Consider tilings by a coprime set of boxes. Is there a collection of *extendible* finite patterns whose entropy is the same as the entropy of the space of all tilings? By our work, this would give embeddings of shift spaces of smaller entropy into spaces of coprime tilings.