A dichotomy for countable unions of smooth Borel equivalence relations

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- We say that E continuously embeds into F, denoted by $E \sqsubseteq_c F$, if there is a continuous embedding from E to F.

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We have the following initial segment of the hierarchy of Borel equivalence relations:

$$\Delta_1 <_B \Delta_2 <_B \ldots <_B \Delta_{\mathbb{N}} <_B \Delta_{\mathbb{R}} <_B \mathbb{E}_0,$$

which is exhaustive in the sense that every Borel equivalence relation is either bireducible with one of the elements of this initial segment, or is strictly greater than $\mathbb{E}_0$.

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Proposition (Folklore)

The relation $\mathbb{E}_1$ is not essentially countable.

Kechris–Louveau's dichotomy

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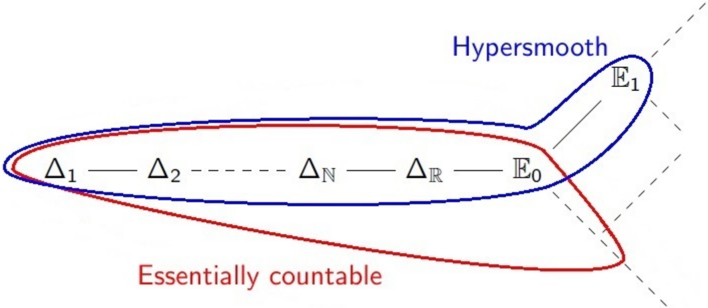

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There are other examples, for instance the disjoint union of $\mathbb{E}_1$ and of a non-hypersmooth countable Borel equivalence relation.

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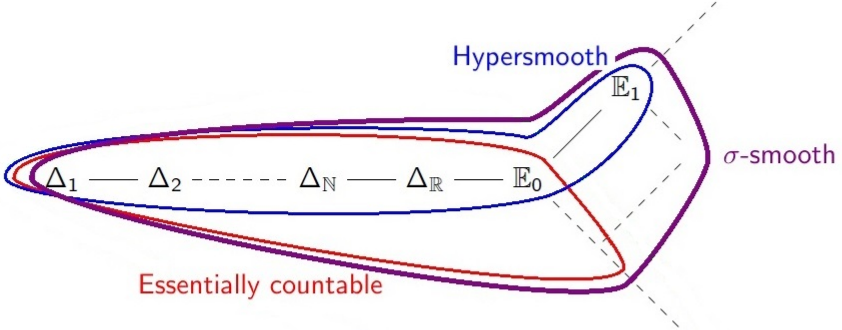

Idealisticity

A Borel equivalence relation E on a Polish space X is said to be idealistic (resp. strongly idealistic) if there is an E-invariant assignment $x \mapsto \mathcal{I}_x$ sending each point in X to a σ-ideal on X in such a way that:

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The equivalence relation E is said to be ccc idealistic (resp. strongly ccc idealistic) if for every $x \in X$ and every uncountable family $(B_i)_{i \in I}$ of pairwise disjoint Borel subsets of X, one of the B_i's is in $\mathcal{I}_x$.

Group actions

Given a Borel action $G \curvearrowright X$ of a Polish group on a Polish space, we can consider the orbit equivalence relation associated to this action, i.e. the analytic equivalence relation E_G^X on X defined by:

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Theorem (Feldman–Moore)

Let E be a countable Borel equivalence relation on a Polish space X. Then there is a Borel action $\Gamma \curvearrowright X$ of a countable discrete group such that $E = E_\Gamma^X$.

Idealisticity and $\mathbb{E}_1$

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Let E be a Borel equivalence relation on a Polish space. Then exactly one of the following two conditions holds:

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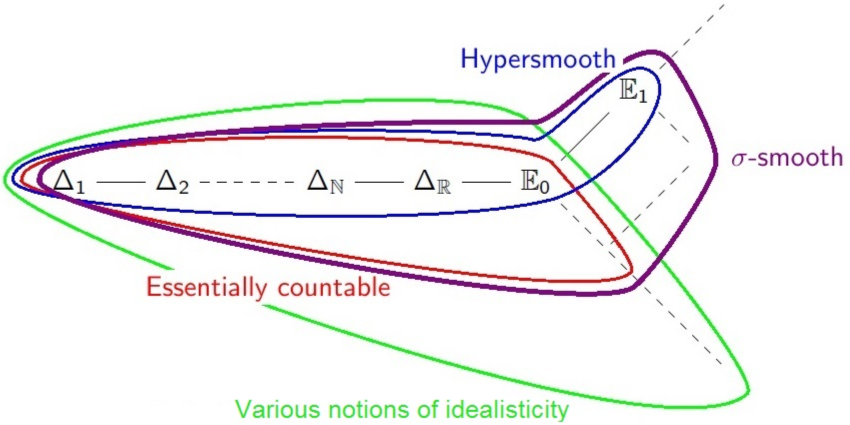
Kechris–Louveau's dichotomy solves this conjecture in the special case of hypersmooth Borel equivalence relations. Our dichotomy solves it in the special case of σ-smooth Borel equivalence relations.

A picture



Hypersmooth
$\mathbb{E}_1$
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Δ_1 — Δ_2 - - - $\Delta_\mathbb{N}$ — $\Delta_\mathbb{R}$ — $\mathbb{E}_0$
Essentially countable
Various notions of idealisticity

Stability under countable unions

Theorem (Rephrasing of the main theorem)

Let E be a Borel equivalence relation on a Polish space. Suppose that $\mathbb{E}_1 \not\leq_B E$ (this holds, for instance, if E is ccc idealistic).

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An equivalence relation E on a Polish space X is said to be potentially F_σ if it is Borel reducible to an F_σ equivalence relation on a Polish space.

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For every $n \in \mathbb{N}$, let E_n be an equivalence relation on a set X_n. The disjoint union of the E_n's is the equivalence relation E on $X := \bigsqcup_{n \in \mathbb{N}} X_n$ defined by $x \mathrel{E} x' \Leftrightarrow (\exists n \in \mathbb{N})(x, x' \in X_n \text{ and } x \mathrel{E_n} x')$.

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If $\mathcal{F}$ is a family of Borel equivalence relations on Polish spaces, denote by $\mathcal{F}^{\leqslant_B}$ the family of all equivalence relations on Polish spaces that are Borel reducible to an element of $\mathcal{F}$, and by $\sigma(\mathcal{F})$ the class of all equivalence relations on Polish spaces that can be expressed as countable unions of subequivalence relations belonging to $\mathcal{F}$.

Our most general result

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Our dichotomy for σ-smooth equivalence relations is the special case when $\mathcal{F}$ is the class of all countable Borel equivalence relations.

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This proves Kechris–Louveau's conjecture for the class of equivalence relations that can be expressed as countable unions of subequivalence relations that are Borel reducible to strongly ccc idealistic potentially F_σ equivalence relations on Polish spaces.

F^+ and $F^\cap$

Definition

Let F be an equivalence relation on a Polish space X.

- For $A \subseteq X$, denote by $[A]_F$ the F-saturation of A, that is, the set $\{x \in X \mid (\exists x' \in A)(x \mathrel{F} x')\}$.

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- For $A \subseteq X$, denote by $[A]_F$ the F-saturation of A, that is, the set $\{x \in X \mid (\exists x' \in A)(x \, F \, x')\}$.
- The Friedman–Stanley jump of F is the equivalence relation F^+ on $X^{\mathbb{N}}$ defined by $x \, F^+ \, x'$ iff $[x(\mathbb{N})]_F = [x'(\mathbb{N})]_F$.

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Definition

- A homomorphism from a binary relation R on a set X to a binary relation S on a set Y is a mapping $f \colon X \to Y$ such that $(f \times f)[R] \subseteq S$.
- A reduction from R to S is a mapping $f \colon X \to Y$ which is both a homomorphism from R to S and from $\sim R$ to $\sim S$.

F^+ and $F^\cap$

Proposition

Let E be an equivalence relation on a Polish space X, and F be a strongly idealistic Borel equivalence relation on a Polish space Y.

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Let E and F be Borel equivalence relations on Polish spaces X and Y, respectively. The following are equivalent:

- *E Borel reduces to $(F \times \Delta_{\mathbb{N}})^\cap$;*

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Proposition

Let E and F be Borel equivalence relations on Polish spaces X and Y, respectively. The following are equivalent:

- *E Borel reduces to $(F \times \Delta_{\mathbb{N}})^\cap$;*
- *E is a countable union of subequivalence relations that are Borel reducible to $F \times \Delta_{\mathbb{N}}$.*

Another consequence

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Let E be an equivalence relation on a Polish space which is Borel reducible to a ccc idealistic Borel equivalence relation.

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Let E be an equivalence relation on a Polish space which is Borel reducible to a ccc idealistic Borel equivalence relation. Let $\tilde{F}$ be a strongly idealistic potentially F_σ equivalence relation on a Polish space and let $F = \tilde{F} \times \Delta_{\mathbb{N}}$.

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Moreover, if these conditions are satisfied, then $E \leqslant_B F^+$.

Thank you for your attention!