The Wadge hierarchy on Zariski topologies

Joint work with C. Massaza

Basic definition

Let X, Y be topological spaces, $A \subseteq X, B \subseteq Y$.
Then A *continuosly reduces* — or *Wadge reduces* — to B iff

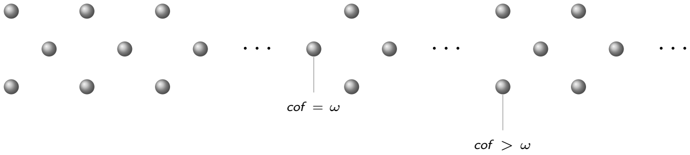
$$\exists f : X \to Y \text{ continuous s.t. } A = f^{-1}(B)$$

This is denoted $A \leq_W B$.
If $A \leq_W B \leq_W A$, write $A \equiv_W B$.

The restriction of $\leq_W$ to the subsets of a fixed topological space X is a preorder on $\mathcal{P}(X)$, the powerset of X. This restriction is sometimes denoted $\leq_W^X$, to point out the ambient space.
The goal is to understand the structure of this preorder for various topological spaces.

Examples

- (Wadge) The Wadge hierarchy on the Baire space $(\mathbb{N}^{\mathbb{N}}, \mathcal{T})$. On the Borel sets it looks like



- Similar behaviours for Polish zero-dimensional spaces.
- (Schlicht) For positive-dimensional Polish spaces, there are antichains of size the continuum among the Borel sets.
- (Damiani, C.) Let τ be the compact complement topology on $\mathbb{N}^{\mathbb{N}}$. Then the longest antichains among sets in $\Sigma^0_2(\mathcal{T}) \cup \Pi^0_2(\mathcal{T})$ have size 4.

Affine varieties

Let an infinite commutative field k be fixed.

Definition

- An *affine variety* in k^n is the set $V(I)$ of the common zeros of a collection of polynomials $I \subseteq k[X_1, \ldots, X_n]$.
  Equivalently, it is the set $V(I)$ of the common zeros of an ideal of polynomials $I \subseteq k[X_1, \ldots, X_n]$.
  Equivalently, it is the set $V(f_1, \ldots, f_r)$ of the common zeros of a finite list of polynomials $f_1, \ldots, f_r \in k[X_1, \ldots X_r]$.
- If $\mathcal{V}$ is an affine variety, the Zariski topology on $\mathcal{V}$ is the topology whose closed sets are the subsets of $\mathcal{V}$ that are affine varieties themselves.

Problem: Understand the Wadge hierarchy on an affine variety $\mathcal{V}$ endowed with the Zariski topology.

Dimension

Definition

Let $\mathcal{V}$ be an affine variety.

- $\mathcal{V}$ is *irreducible* if it is not the union of two proper subvarieties; otherwise it is *reducible*.
- The *irreducible components* of $\mathcal{V}$ are the maximal irreducible subvarieties of $\mathcal{V}$.
- The *dimension* of an affine vatiery is the biggest d such that there exists a strictly increasing chain

$$C_0 \subset C_1 \subset \ldots \subset C_d$$

of non-empty irreducible subvarieties of $\mathcal{V}$.

It turns out that every affine variety has finitely many irreducible components. Therefore

$$\mathcal{V} = \mathcal{V}_1 \cup \ldots \mathcal{V}_s$$

where $\mathcal{V}_1, \ldots, \mathcal{V}_s$ are the irreducible components of $\mathcal{V}$. This is referred to as the (unique) *decomposition* of an affine variety in its irreducible components.

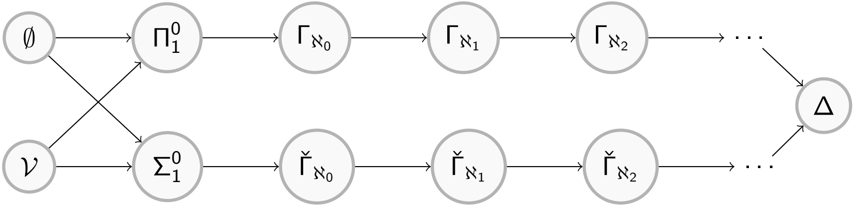
Baby case: curves

An affine variety of dimension 1 is called a *curve*.

An irreducible curve $\mathcal{V}$ is just an infinite set endowed with the cofinite topology. Therefore, $f : \mathcal{V} \to \mathcal{V}$ is continuous if and only if either it is constant or it is finite-to-1.
Consequently, given $A, B \in \mathcal{P}(\mathcal{V}) \setminus \{\emptyset, \mathcal{V}\}$, one has $A \leq_W B$ if and only if:

- either A, B are both finite or both cofinite; or
- $\mathrm{card}(A) \leq \mathrm{card}(B)$ and $\mathrm{card}(\mathcal{V} \setminus A) \leq \mathrm{card}(\mathcal{V} \setminus B)$

The Wadge hierarchy on an irreducible curve



where, for $\kappa < \mathrm{card}(\mathcal{V})$:

- $\Gamma_\kappa = \{A \subseteq \mathcal{V} \mid \mathrm{card}(A) = \kappa\}$
- $\check{\Gamma}_\kappa = \{A \subseteq \mathcal{V} \mid \mathrm{card}(\mathcal{V} \setminus A) = \kappa\}$
- $\Delta = \{A \subseteq \mathcal{V} \mid \mathrm{card}(A) = \mathrm{card}(\mathcal{V} \setminus A) = \mathrm{card}(\mathcal{V})\}$

If instead $\mathcal{V}$ consists of an irreducible curves plus some isolated points, it is enough to add to the picture the degree $\Delta^0_1 \setminus \{\emptyset, \mathcal{V}\}$.

More complex curves

For more complicated curves, the situation is less easy.
On the one hand:

Theorem

If $\mathcal{V}$ is any curve, then $\leq_W$ is a wqo, actually a bqo, on $\mathcal{P}(\mathcal{V})$.

However:

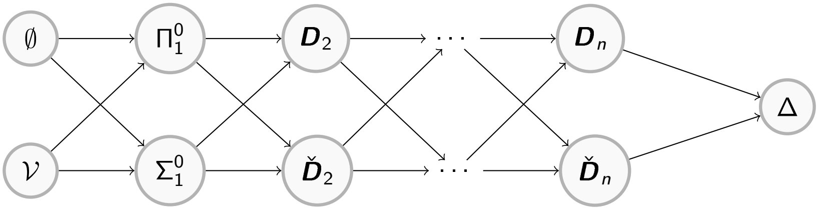
Theorem

- For every m there exists a curve $\mathcal{V}$ such that $\leq_W$ has antichains of size m
- If the curve $\mathcal{V}$ has at least two irreducible components of cardinality $\geq \aleph_\omega$, then $\leq_W$ has antichains of arbitrarily high finite cardinalities

Problem, for later: How many points does a curve have? Certainly $\leq \mathrm{card}(k)$.

Higher dimensions: the countable irreducible case

The Wadge hierarchy on a countable, irreducible, *n*-dimensional, affine variety is:



where

- $\boldsymbol{D}_i$ = true *i*-th differences of closed sets (sets in $\bigcup_{i\in\mathbb{N}} \boldsymbol{D}_i$ are called *constructible* sets in topology)
- $\Delta = \{A \subseteq \mathcal{V} \mid$ for some subvariety $\mathcal{W}, A \cap \mathcal{W}$ is dense and condense in $\mathcal{W}\}$

A corollary

Definition

- (Weihrauch) If X is a second countable, T_0 space, an *admissible representation* is a continuous $\rho : Y \subseteq \mathbb{N}^{\mathbb{N}} \to X$ such that for every continuous $\sigma : Z \subseteq \mathbb{N}^{\mathbb{N}} \to X$ there exists a continuous $h : Z \to Y$ such that $\sigma = \rho h$.
- (Tang, Pequignot) If $A, B \subseteq X$, let

$$A \preceq^X_{TP} B \Leftrightarrow \rho^{-1}(A) \leq^Y_W \rho^{-1}(B)$$

for some/any admissible representation ρ.

Fact. For any second countable, T_0 space X, it holds that $\leq^X_W \subseteq \preceq^X_{TP}$.

Corollary

If $\mathcal{V}$ is a countable, irreducible, affine variety, then $\leq^{\mathcal{V}}_W = \preceq^{\mathcal{V}}_{TP}$.

The uncountable case: how many points are there?

Question

Assume that k is uncountable and let $\mathcal{V}$ be an infinite affine variety over k. What is $\mathrm{card}(\mathcal{V})$?

Example. If k is algebraically closed then $\mathrm{card}(\mathcal{V}) = \mathrm{card}(k)$.

Definition

Say that k is reasonable if every affine variety over k has the same cardinality as k.

Non-reasonable fields

Theorem

There are non-reasonable field.

In fact:

Theorem

For every infinite cardinals $\lambda < \kappa$ there exist a field K of cardinality κ and a curve over K of cardinality λ.

Proof.

Transversal sets

Transversal sets are a useful tool to construct continuous functions between affine varieties.

Definition

Let $\mathcal{V}$ be an infinite affine variety. A *transversal set* in $\mathcal{V}$ is a subset $T \subseteq \mathcal{V}$ such that:

- $\mathrm{card}(T) = \mathrm{card}(\mathcal{V})$
- $T \cap \mathcal{W}$ is finite for every proper subvariety $\mathcal{W} \subsetneq \mathcal{V}$

Suppose that

- T is a transversal set in some irreducible subvariety of $\mathcal{V}$
- $\mathcal{W}$ is an affine variety, and $f : \mathcal{W} \to \mathcal{V}$ is such that
  - the range of f is contained in T, and
  - f is finite-to-1

then f is continuous.

Adequate varieties and fields

Definition

- An infinite affine variety $\mathcal{V}$ is *adequate* if all infinite irreducible subvarieties of $\mathcal{V}$ have the same cardinality as $\mathcal{V}$ and admit a transversal set
- The field k is *adequate* if every infinite affine variety over k is adequate.

Therefore every adequate field is reasonable.

Question

Is every reasonable field adequate?

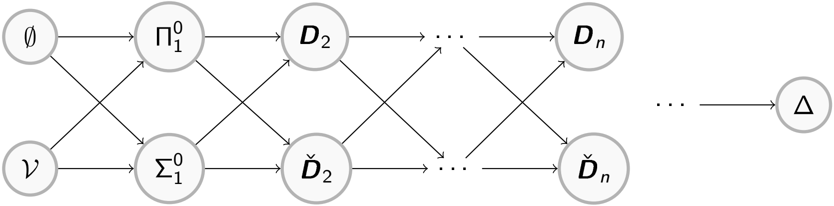
Conjecture. Yes.

Examples of adequate fields are:

- countable fields: using reasonability+diagonalisation
- algebraically closed fields: using reasonability+diagonalisation
- $\mathbb{R}$: using some mild tools from differential topology

Adequate, irreducible, affine varieties

If $\mathcal{V}$ is an adequate, irreducible, *n*-dimensional affine variety, then the Wadge hierarchy is:



where $\Delta = \{A \subseteq \mathcal{V} \mid \exists W$ irreducible subvariety of $\mathcal{V}$ s.t. both $\mathcal{W} \cap A$ and $\mathcal{W} \setminus A$ contain a transversal set of $\mathcal{W}\}$.

The $\cdots$ zone

The Wadge hierarchy in the dotted zone of the previous picture can be wild. For instance:

Theorem

- If $\mathcal{V}$ contains irreducible curve $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \ldots$ such that
  - each $\mathcal{C}_i$ is uncountable, and
  - $\mathcal{C}_i \cap \mathcal{C}_j \Leftrightarrow |i - j| \leq 1$

  then $\leq_W$ has antichains of arbitrarily high finite cardinalities
- If $\mathcal{V}$ contains irreducible curve $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \ldots$ such that
  - all $\mathcal{C}_i$ have the same cardinality $\geq \aleph_\omega$, and
  - $\mathcal{C}_i \cap \mathcal{C}_j \Leftrightarrow |i - j| \leq 1$

  then $\leq_W$ has antichains of cardinality the continuum

Some questions

1. The term *Zariski topology* also appears in a wider context.
Let R be a commutative ring, and denote

$$SpecR = \{\mathfrak{p} \mid \mathfrak{p} \text{ is a prime ideal in } R\}$$

The *Zariski topology* of $SpecR$ is the topology generated by the closed sets

$$V(I) = \{\mathfrak{p} \in SpecR \mid I \subseteq \mathfrak{p}\}$$

where I ranges over all ideals of R.

The set of closed points in $SpecR$ is $Max(SpecR)$, the set of the maximal ideals of R.

Given an affine variety $\mathcal{V}$, there is a homeomorphism
$\mathcal{V} \to Max(Spec\mathcal{O}(\mathcal{V}))$. So, $\mathcal{V}$ sits inside $Spec\mathcal{O}(\mathcal{V})$.

Some questions

Question

How is the Wadge hierarchy on $Spec\mathcal{O}(\mathcal{V})$?
More generally, how is the Wadge hierarchy on $SpecR$?

Comment. Some insight to these questions, at least in the countable case, might come from the study of $\preceq_{TP}$ and some work in progress on the relationship between $\leq_W$ and $\preceq_{TP}$ on Alexandrov spaces.

Some questions

2. The Zariski topology on affine varieties appears to be more a synthetic way to express things than a real interesting object of investigation in algebraic geometry. In other words, the category of affine varieties of real interest in algebraic geometry does not have the continuous functions $\mathcal{V} \to \mathcal{W}$ as morphisms, but the polynomial ones.

Definition. If $A, B \subseteq \mathcal{V}$, let

$$A \leq_{pol} B \Leftrightarrow \exists f : \mathcal{V} \to \mathcal{V} \text{ polynomial s.t. } A = f^{-1}(B)$$

Notice that $\leq_{pol} \subseteq \leq_W$.

Question

How is the structure of $\leq_{pol}$?

Polynomial reducibility on k

$\leq_{pol}$ is already non-trivial in the case $\mathcal{V} = k$.

$\leq_{pol}$ is much more sensible than $\leq_W$ to the algebraic properties of k.

An example. If k is an ordered field, then $\leq_{pol}, \leq_W$ coincide on $\Pi^0_1(k) \setminus \{\emptyset, k\}$: given $A, B \subseteq k$ finite, non-empty, it holds that $A \equiv_p B$:

- $A \leq_{pol} \{0\}$: witnessed by $\prod_{a \in A}(X - a)$ (holds for any k)
- $\{0\} \leq_{pol} A$: witnessed by $X^2 + \max A$

Question. Is this true for any non-algebraically closed field?

Polynomial reducibility on k algebraically closed

Hence assume that k is algebraically closed.

Proposition. If $A, B \in \mathcal{P}(k) \setminus \{\emptyset, k\}$ and $A \leq_{pol} B$, then either A, B are both finite and $\mathrm{card}(B) \leq \mathrm{card}(A)$, or $\mathrm{card}(A) = \mathrm{card}(B)$.
Pf. Every non-constant polynomial is surjective, and every elements has finitely many preimages.

For $1 \leq \kappa < \mathrm{card}(k)$ let

$$\mathcal{P}_\kappa = \{A \subseteq k \mid \mathrm{card}(A) = \kappa\}, \quad \check{\mathcal{P}}_\kappa = \{k \setminus A\}_{A \in \mathcal{P}_\kappa}$$

Also, let $\mathcal{P}_{fin} = \bigcup_{n \in \mathbb{N} \setminus \{0\}} \mathcal{P}_n$.

It appears (unsurprisingly) that different tools are needed for the study of $\leq_{pol}$ on $\mathcal{P}_{fin}$, and on $\mathcal{P}_\kappa$ for infinite κ.

The case of finite sets

Proposition

- $\mathcal{P}_1$ and $\mathcal{P}_2$ each consist of a single class, say $[A_1], [A_2]$, respectively, and $[A_2] \leq_{pol} [A_1]$.
- Let $n \geq 3$ and $A, B \in \mathcal{P}_n$. If $A \leq_{pol} B$ and f is a polynomial such that $A = f^{-1}(B)$, then f is linear. Consequently $A \equiv_{pol} B$.

Therefore, denote $\mathcal{Q}_n = \mathcal{P}_n / \equiv_{pol}$.

Geometric representations

Under some technical conditions, the space of orbits $\mathcal{V}/G$ of the action by isomorphisms of an algebraic group on an affine variety

$$G \curvearrowright \mathcal{V}$$

can be endowed with the structure of affine variety.

This is always possible when G is finite.

Geometric representations

Sym_n acts on k^n by permutation. An orbit is a n-element sets, with some elements counted possibly several times.
The set of points with distinct coordinates is an open invariant set of k^n: it is the complement of the union of the hyperplanes $x_j - x_{j'} = 0$. Its quotient $\mathcal{P}_n$ can be given the structure of a n-dimensional affine variety.

Each $\equiv_{pol}$-class is a 2-dimensional subvariety of $\mathcal{P}_n$.

Geometric representations

The quotient $\mathcal{S}_n = \mathcal{P}_n / \equiv_{pol}$ is a $(n-2)$-dimensional affine variety.
The elements of $\mathcal{S}_n$ are the $\equiv_{pol}$-classes.

In other words, the restriction of $\leq_{pol}$ to the $\equiv_{pol}$-classes of n-element subsets of k^n is a preorder defined on an affine variety.
This extra structure gives a framework to measure quantitatively the behaviour of $\leq_{pol}$, especially as many sets naturally defined using $\leq_{pol}$ turn out to be subvarieties of some $\mathcal{S}_n$.

Geometric representations

Example. Given $m > n$:

1. How many classes of $\mathcal{S}_m$ reduce to a fixed class of $\mathcal{S}_n$?
2. How many classes of $\mathcal{S}_n$ a fixed class of $\mathcal{S}_m$ reduces to?

The set of classes in (1) is a subvariety of $\mathcal{S}_m$.
The set of classes in (2) is a subvariety of $\mathcal{S}_n$.
Therefore questions (1) and (2) can be made quantitatively more precise by asking what is the dimension of such sets.

Example (cont.)

- If there is no integer g such that $\frac{m}{n} \leq g \leq \frac{m-1}{n-1}$, then there are no $[A] \in \mathcal{S}_m, [B] \in \mathcal{S}_n$ such that $[A] \leq_{pol} [B]$. In particular, given any n, the least m such that there exists $[B] \in \mathcal{S}_m$ with $[B] \leq_{pol} A$ is $m = 2n - 1$.
- For any $[A] \in \mathcal{S}_m$, it holds that $\{[B] \mid [A] \leq_{pol} [B]\}$ is finite.
- For any $[B] \in \mathcal{S}_{n-1}$, there exists a unique $[A] \in \mathcal{S}_{2n-1}$ such that $[A] \leq_{pol} B$.
- The set of $[B] \in \mathcal{S}_4$ that reduces to the unique element of $\mathcal{S}_2$ is a subvariety of dimension 1.

The case of infinite and coinfinite sets

- There exist $2^{\mathrm{card}(k)}$ maximal elements
- If $k \subseteq \mathbb{C}$ has chains of order type ζ