Subgroups of $\mathcal{L}_I$ which do not embed into $F$
This is joint work with James Hyde (Cornell)

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PLoI: The group of all orientation preserving homeomorphisms of $\mathbb{R}^2$ which are piecewise linear.

"Subgroups of PLoI ... have been a source of groups with interesting properties in which calculations are practical." — Matt Brin

Thompson's Group $F$: All $f \in \text{PLoI}$ which have breakpoints at dyadic rationals and whose slopes are powers of $2$. 
How to think about $F$:

**Brin’s Ubiquity Theorem**

If $G \leq \text{Homeo} I$, $J$ is an orbital of $J$ and some element of $G$ reaches one end of $J$ but not the other, then $F \rightarrow G$.

**Example**

Suppose that $f, g \in \text{Homeo} I$

$supt(f) = (a, b)$  \hspace{1cm} $supt(g) = (c, d)$

$a < c < b < d$ \hspace{1cm} and \hspace{1cm} $f(c) \geq g(b)$.

Then $\langle f, g \rangle \equiv F$. 
General Program Understand the quasi-order of finitely generated subgroups of $\text{Pl}_0 \text{I}$ ordered by embeddability (homo-morphic).

The Thesis This order should be highly structured (but complex!) below Thompson's group $F$, whose structure should completely break down above $F$.

Conjecture (Brin) Any finitely generated subgroup of $\text{Pl}_0 \text{I}$ either is embeddable into $F$ or else contains $F$.

Conjecture (Brin, Sapir) If $G \leq \text{Pl}_0 \text{I}$ does not contain $F$, $G$ is an elementary (amenable) group
Theorem (Bleak, Brin, M.) There are elementary groups \( G_\xi \) \( (\xi \leq \xi_0) \) such that:

1. \( G_0 \) is the trivial group and \( G_{\xi+1} \cong G_\xi + \mathbb{Z} \).
2. For all \( \xi, \eta \leq \xi_0 \), \( G_\xi \cong G_\eta \) iff \( \xi = \eta \).
3. Each \( G_\xi \) is elementary with class \( \alpha_\xi < \xi_0 \) where \( \sup_{\xi \leq \xi_0} \alpha_\xi = \xi_0 \).
This talk: Establish a criteria for when a subgroup of $\text{PL}_0I$ does not embed into $F$.

**Theorem (Lodha)** The Stein groups $F_{p,q}$ do not embed into $F$ if $p,q$ are relatively prime.

**Reason:** The groups of germs have rank $>1$ and this is not possible in $F$.

**Theorem (Hyde, M.)** If $f, g \in \text{PLO}I$ are an $F$-obstruction and $\varphi : <f, g> \overset{1:1}{\rightarrow} \text{PLO}I$ is a monomorphism, then $(\varphi(f), \varphi(g))$ is an $F$-obstruction.
Prop If \( f \in F \leq Pz_0 I \), then \( f_\gamma \) is not an \( F \)-obstruction.

Remark: \( F \) contains an \( F \)-obstruction for each \( p, q \) with \( \gcd(p, q) = 1 \).

Cleary's "Golden ratio \( F \)" contains an \( F \)-obstruction.
Poincaré's Rotation Number

Suppose that \( \gamma \) is a homeomorphism of \( \mathbb{R}/\mathbb{Z} \) and \( \gamma \) is a lift. The rotation \( \Omega \) of \( \gamma \) is the limit

\[
\theta = \lim_{n \to \infty} \frac{\gamma^n(x) - x}{n}
\]

modulo 1. Does not depend on \( x \) or \( \gamma \).

**Theorem (Poincaré)** The rotation \( \Omega \) being irrational implies that \( \gamma \) is topologically semiconjugate to a rotation by \( \theta \).

**Theorem (Herman)** If \( \gamma \) is PL, e.g. Then \( \gamma \) is in fact topologically conjugate to a rotation by its rotation \( \Omega \).
What is an F-obstruction?

Suppose $f, g \in \text{PL}_0 I$ and $s \in I$ and $s \leq f(s) \leq g(s) \leq f(g(s))$.

Define $\gamma : [s, g(s)) \to [s, g(s))$ by $\gamma(t) = \gamma^m(f(t))$ where $m$ is unique such that $\gamma^m(f(t)) \in [s, g(s))$. (m is unique and either 0 or -1).

This $\gamma$ is a homeomorphism of the circle.
The rotation of $f$ is the rotation number of $f$ modulo $g$ at $s$.

$fg$ is an $F$-obstruction if for some $s$, the rotation number of $f$ modulo $g$ at $s$ is irrational.

(Also, symmetric to this so that if $fg$ is an $F$-obstruction, so is $f^{-1}$, $g$, $f$, $g^{-1}$, etc.)
F = PL, I doesn't contain F-obstructions

Theorem (Ghys-Sergiescu) Thompson's group T does not contain elements with irrational rotation #.
Analysis of 1-orbital F-obstructions

The first step is to show that if \( f \) is an F-obstruction and \( J \) is the orbital of \( \phi_g \) witnessing \( \text{th} \), then there is an abuse \( A, B \subseteq J \) such that

if \( a < b \) and \( a \in A, \ b \in B \), then \( \text{th}(A, B) \).

\[ \text{Supp}(A) \cap J = (a, b). \]
A dichotomy theorem for subgroups of Pl"oI

Suppose that $G \leq \text{Pl}_0I$ and $J, K_0, \ldots, K_n$ are orbitals of $G$ and $G$ is resolvable on $J$. Then either:

1. There is some $g \in G$ such that $\text{supp}(g) \supset J + 1$ and yet $\text{supp}(g)$ is disjoint from $K_i$'s.

2. There is a $G$-equivariant monotone $\gamma : K_i \rightarrow J$ for some $i = c, \ldots, n$.

$\gamma(g(x)) = g(\gamma(x))$
Open Problems

1. If $G \leq \text{PL}_0 I$ does not contain an $F$-obstruction and is finitely generated, must $G \subset F$?

2. Suppose $G \leq \text{PL}_0 I$ is finitely generated and does not embedd into $F$. Must $G$ have an orbit which is somewhere dense?

3. Does $F + \mathbb{Q} = \{ f^{t + t + q} : q \in \mathbb{Q} \} \leq \text{Homeo} \mathbb{R}$ fail to embed into $F$?