An unpublished theorem of Solovay on OD partitions of the reals into two non-OD parts, revisited

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Let $a \in 2^\omega$ be Sacks generic over $L$. Then it is true in $L[a]$ that

1) there is a partition $2^\omega \setminus L = A \cup B$, of the $\Pi^1_2$ set $2^\omega \setminus L$ of all nonconstructible reals, such that

2) the associated equivalence relation on $2^\omega \setminus L$ is lightface $\Pi^1_2$, hence the partition is OD as an unordered pair

3) $A, B$ are non-OD, equivalently, $A, B$ are OD-indiscernible.
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The theorem also holds for Miller forcing (superperfect sets in $\omega^\omega$) and $E_0$-large forcing (Borel sets $X \subseteq 2^\omega$ s. t. $E_0 \upharpoonright X$ is nonsmooth).
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The theorem also holds for **Miller forcing** (superperfect sets in $\omega^\omega$) and **$\mathcal{E}_0$-large forcing** (Borel sets $X \subseteq 2^\omega$ s. t. $\mathcal{E}_0 \upharpoonright X$ is nonsmooth).

**Problem**

Figure out the cases of **Cohen, random, Silver etc.** forcing notions.
Example (early years of forcing)

Let $\langle a, b \rangle$ be a Sacks $\times$ Sacks generic pair of reals over $L$. Then it is true in $L[a]$ that the $L$-degrees $[a]_L$ and $[b]_L$ are indiscernible non-OD sets but their unordered pair \{ $[a]_L$, $[b]_L$ \} is OD.

Example (GKL)

There is a generic extension of $L$ in which it holds that there exist disjoint countable indiscernible non-OD sets $X, Y \subseteq 2^{\omega}$ such that their union $X \cup Y$ and the associated equivalence relation are $\Pi^1_2$.

It is a key novelty of the Solovay partition theorem that, unlike these and similar examples, the indiscernible partition in the Sacks extension of $L$ is not related to any sort of mutually generic reals.

See e.g. FGH, GH on some modern research related to indiscernible sets.
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Silver’s canonization theorem

**Theorem (Silver)**

Let $E$ be a Borel equivalence relation on a Borel uncountable set $X$ in a Polish space. There is a perfect set $Y \subseteq X$ such that:

- either $E \upharpoonright Y$ is the equality
- all elements of $Y$ are $E$-equivalent.

If $E$ is ctble then we have only the either case.

This is a sine qua non of the proof of the Solovay partition theorem for the Sacks extensions.

Suitable more complex canonization results known from KSZ are used for the cases of Miller and $E_0$-large forcing.
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The proof is a bit too involved to type in using a web-interface like yahoo. (Shades of Fermat’s margin!) […]

– Bob
A double-bubble pair, DBP, is a pair of countable Borel equivalence relations \( \langle E, D \rangle \) on \( 2^\omega \), such that each D-class is the union of exactly two distinct E-classes (in particular \( E \subsetneq D \)).
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Thus a DBP \( \langle E, D \rangle \) can be seen as a Borel partition of \( 2^\omega \) into countable parts by D, plus a finer Borel partition by E that splits each D-class in exactly two non-empty half-classes.
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A double-bubble pair, DBP, is a pair of *countable* Borel equivalence relations \( \langle E, D \rangle \) on \( 2^\omega \), such that each D-class is the union of exactly two distinct E-classes (in particular \( E \not\subseteq D \)).

Thus a DBP \( \langle E, D \rangle \) can be seen as a *Borel* partition of \( 2^\omega \) into *countable* parts by D, plus a *finer* Borel partition by E that splits each D-class in exactly two non-empty *half-classes*.

Example

Define \( E_0^{\text{even}} \) on \( 2^\omega \) so that \( x \ E_0^{\text{even}} y \) iff \( \{ n : x(n) \neq y(n) \} \) has a finite *even* number of elements. Then \( \langle E_0^{\text{even}}, E_0 \rangle \) is a DBP.
Extension of DBPs

Definition

A DBP $\langle E', D' \rangle$ extends $\langle E, D \rangle$, in symbol $\langle E, D \rangle \lessapprox \langle E', D' \rangle$, if $D \subseteq D'$ and $E \subseteq E'$, and $D \setminus E \subseteq D' \setminus E'$, so that, for any $x, y \in 2^\omega$, if $x \in [y]_{D \setminus E}$, then we still have $x \in [y]_{D' \setminus E'}$.

Thus extension of a DBP $\langle E, D \rangle$ means coarsening (that is merging classes into bigger classes) of the D-partition and E-subpartition, that honors the original splitting of D-classes into E-halfclasses.
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- \( D \setminus E \subseteq D' \setminus E' \), so that, for any \( x, y \in 2^\omega \), if \( x \in [y]_D \setminus [y]_E \) then we still have \( x \in [y]_{D'} \setminus [y]_{E'} \).
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Thus extension of a DBP \( \langle E, D \rangle \) means coarsening (that is merging classes into bigger classes) of the D-partition and E-subpartition, that honors the original splitting of D-classes into E-halfclasses.
It follows that if \( \lambda \in \text{Ord} \) is limit and \( \langle E_{\alpha}, D_{\alpha} \rangle_{\alpha \lessdot \lambda} \) is a \( \preceq \)-increasing sequence then the limit pair \( \lim \limits_{\alpha \to \lambda} \langle E_{\alpha}, D_{\alpha} \rangle = \langle \bigcup_{\alpha < \lambda} E_{\alpha}, \bigcup_{\alpha < \lambda} D_{\alpha} \rangle \) is a DBP extending each \( \langle E_{\alpha}, D_{\alpha} \rangle \).
It follows that if $\lambda \in \text{Ord}$ is limit and $\langle E_\alpha, D_\alpha \rangle_{\alpha<\lambda}$ is a $\preceq$-increasing sequence then the limit pair $\lim_{\alpha \to \lambda} \langle E_\alpha, D_\alpha \rangle = \langle \bigcup_{\alpha<\lambda} E_\alpha, \bigcup_{\alpha<\lambda} D_\alpha \rangle$ is a DBP extending each $\langle E_\alpha, D_\alpha \rangle$.

This will allow us to define, in $\mathbf{L}$, an increasing transfinite sequence $\langle E_\alpha, D_\alpha \rangle_{\alpha<\omega_1}$ of DBPs such that $\bigcup_{\alpha<\omega_1} D_\alpha$ will be essentially the total equivalence while accordingly the union $E = \bigcup_{\alpha<\lambda} E_\alpha$ will lead to the proof of the Solovay theorem.
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But we have to specify passages from $\langle E_\alpha, D_\alpha \rangle$ to $\langle E_{\alpha+1}, D_{\alpha+1} \rangle$. 
Definition

Given a set $X \subseteq 2^\omega$ and a map $f : X \to 2^\omega$, a DBP $\langle E, D \rangle$:

- contains $f$ if $f \subseteq D$, that is, $f(x) \in \{x\} \cap D$ for all $x \in X$;
- negatively contains $f$ if $f(x) \in \{x\} \setminus \{y\} \cap E$ for all $x \in X$.
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**Lemma 1 (Containment Lemma 1)**

Assume that $\langle E, D \rangle$ is a DBP, $X \subseteq 2^\omega$ is a perfect set, and $f : X \rightarrow 2^\omega$ is Borel and 1-1. Then there exist:

- a perfect set $Y \subseteq X$, and
- a DBP $\langle E', D' \rangle$ which extends $\langle E, D \rangle$ and contains $f \upharpoonright Y$. 
Proof of Lemma 1

WLOG assume that \( f(x) \neq x \) for all \( x \in X \).
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WLOG assume that $f(x) \neq x$ for all $x \in X$.
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- **a** $x \not\in y$ — hence $x \not\in E y$ as well,

- **b** $f(x) \not\in f(y)$ — hence $f(x) \not\in E f(y)$ (also use that $f$ is 1-1),
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Let \( \Delta = [Y \cup f[Y]]_D \), **critical domain**.
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Now cook up \( \langle E', D' \rangle \).

- If \( x \not\in \Delta \) then no extension: \( [x]_{D'} = [x]_D \) and \( [x]_{E'} = [x]_E \).
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- If \( x \in Y \) then \([x]_{D'} = [x]_D \cup [f(x)]_D \), \([x]_{E'} = [x]_E \cup [f(x)]_E \), and let the other \( E' \)-class within \([x]_{D'} \) be \([x]_{D'} \setminus [x]_{E'} \).
Lemma 2

Let $\langle D, E \rangle$ be a DBP, and $X \subseteq 2^\omega$ a perfect set. Then there exist:

- A perfect set $Y \subseteq X$.
- A Borel $1 - 1$ map $f : Y \to Y$.
- A DBP $\langle D', E' \rangle$ that extends $\langle D, E \rangle$ and negatively contains $f \upharpoonright Y$, so that $f(y) \in [y]_{D'} \setminus [y]_{E'}$ for all $y \in Y$. 
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  2) $[y]_{E'} = [y]_E \cup ([f(y)]_D \setminus [f(y)]_E)$,
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Using the two containment lemmas, we define, in $L$, an $\leq$-increasing sequence $\langle E_\alpha, D_\alpha \rangle_{\alpha < \omega_1}$ of DBPs such that

A: if $X \subseteq 2^{\omega}$ is a perfect set and $f: X \to 2^{\omega}$ Borel and 1-1, then there exist $Y$ a perfect set $\subseteq X$ and an ordinal $\alpha < \omega_1$ such that $\langle E_\alpha, D_\alpha \rangle$ contains $f|_Y$.

B: if $X \subseteq 2^{\omega}$ is a perfect set then there exist a perfect set $Y \subseteq X$, an ordinal $\alpha < \omega_1$, and a Borel 1-1 map $f: Y \to Y$, such that $\langle E_\alpha, D_\alpha \rangle$ contains $f$ negatively.

C: the sequence of pairs $\langle E_\alpha, D_\alpha \rangle$ is $\Delta^1_2$, in the sense that there exists a $\Delta^1_2$ sequence of codes for Borel sets $E_\alpha$ and $D_\alpha$.

This item is not really easy.
Using the two containment lemmas, we define, in $L$, an $\preceq$-increasing sequence $\langle E_\alpha, D_\alpha \rangle_{\alpha<\omega_1}$ of DBPs such that

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Definition (Solovay’s equivalence relations)

Let $E = \bigcup_{\alpha < \omega_1} E_{\alpha}$ and $D = \bigcup_{\alpha < \omega_1} D_{\alpha}$. This makes sense in any $\omega_1$-preserving extension of $L$. 

Theorem (implies Solovay’s partition theorem)

Let $a_0 \in 2^{\omega}$ be Sacks generic over $L$. It is true in $L[a_0]$ that

1. $E$ and $D$ are equivalence relations and $E$ is a subrelation of $D$;
2. all reals $x, y \in 2^{\omega} \setminus L$ are $D$-equivalent;
3. there are at most two $E$-classes intersecting $2^{\omega} \setminus L$—say $A, B$;
4. the sets $A, B$ are not $\text{OD}$, and we have $A \cup B = 2^{\omega} \setminus L$;
5. $E|_{2^{\omega} \setminus L}$ is lighface $\Pi^1_2$. 

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5. \( E \upharpoonright (2^\omega \setminus L) \) is lighface \( \Pi^1_2 \).
Proof By Shoenfield, because $E_\alpha$, $D_\alpha$ are Borel equiv. relations in $L$.

Let $x \in 2^{\omega} \setminus L$ in $L[a_0]$. There is a perfect set $X \subseteq 2^{\omega}$ coded in $L$ and a continuous 1-1 $f : X \to 2^{\omega}$ coded in $L$ such that $a_0 \in X$ and $x = f(a_0)$.

By $A$ and since $a_0$ is Sacks, there exist $Y \subseteq X$ coded in $L$ and some $\alpha < \omega_1$ such that still $a_0 \in Y$ and $\langle E_\alpha, D_\alpha \rangle$ contains $f \upharpoonright Y$, meaning that $a_0 D_\alpha f(a_0)$.

Thus $a_0 D_\alpha x$, as required.

Remark The following is true in $L[a_0]$ as well: if $x \in 2^{\omega} \cap L$ and $y \in 2^{\omega} \setminus L$ then $x \not\sim D y$.

The construction can be modified to ensure that all reals in $2^{\omega} \cap L$ are $D$-equivalent and $2^{\omega} \cap L$ has exactly two $E$-classes (similar to $2^{\omega} \setminus L$).
1 By Shoenfield, because $E_\alpha, D_\alpha$ are Borel equiv. relations in $L$. 

2 Let $x \in 2^{\omega} \setminus L \in L[a]$. There is a perfect set $X \subseteq 2^{\omega}$ coded in $L$ and a continuous 1-1 $f: X \to 2^{\omega}$ coded in $L$ such that $a_0 \in X$ and $x = f(a_0)$.

By $A$ and since $a_0$ is Sacks, there exist $Y \subseteq X$ coded in $L$ and some $\alpha < \omega_1$ such that still $a_0 \in Y$ and $\langle E_\alpha, D_\alpha \rangle$ contains $f|Y$, meaning that $a_0 D_\alpha f(a_0)$.

Thus $a_0 D_x$, as required.

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There is a perfect set $X \subseteq 2^\omega$ coded in $\mathcal{L}$ and a continuous 1-1 $f : X \to 2^\omega$ coded in $\mathcal{L}$ such that $a_0 \in X$ and $x = f(a_0)$. 

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1 By Shoenfield, because $E_\alpha, D_\alpha$ are Borel equiv. relations in $L$.

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By A and since $a_0$ is Sacks, there exist: a perfect $Y \subseteq X$ coded in $L$ and some $\alpha < \omega_1$ such that still $a_0 \in Y$ and $\langle E_\alpha, D_\alpha \rangle$ contains $f \upharpoonright Y$, meaning that $a_0 \ D_\alpha \ f(a_0)$.
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Thus $a_0 D x$, as required.
1. By Shoenfield, because $E_\alpha$, $D_\alpha$ are Borel equiv. relations in $L$.

2. Let $x \in 2^\omega \setminus L$ in $L[a_0]$. There is a perfect set $X \subseteq 2^\omega$ coded in $L$ and a continuous 1-1 $f : X \rightarrow 2^\omega$ coded in $L$ such that $a_0 \in X$ and $x = f(a_0)$. By $\text{A}$ and since $a_0$ is Sacks, there exist: a perfect $Y \subseteq X$ coded in $L$ and some $\alpha < \omega_1$ such that still $a_0 \in Y$ and $\langle E_\alpha, D_\alpha \rangle$ contains $f \upharpoonright Y$, meaning that $a_0 D_\alpha f(a_0)$. Thus $a_0 D x$, as required.

**Remark**

The following is true in $L[a_0]$ as well: if $x \in 2^\omega \cap L$ and $y \in 2^\omega \setminus L$ then $x \not\sqsubseteq y$. 
1 By Shoenfield, because $E_\alpha$, $D_\alpha$ are Borel equiv. relations in $L$.

2 Let $x \in 2^\omega \setminus L$ in $L[a_0]$. There is a perfect set $X \subseteq 2^\omega$ coded in $L$ and a continuous 1-1 $f : X \to 2^\omega$ coded in $L$ such that $a_0 \in X$ and $x = f(a_0)$. By $A$ and since $a_0$ is Sacks, there exist: a perfect $Y \subseteq X$ coded in $L$ and some $\alpha < \omega_1$ such that still $a_0 \in Y$ and $\langle E_\alpha, D_\alpha \rangle$ contains $f \upharpoonright Y$, meaning that $a_0 D_\alpha f(a_0)$. Thus $a_0 D x$, as required.

Remark

The following is true in $L[a_0]$ as well: if $x \in 2^\omega \cap L$ and $y \in 2^\omega \setminus L$ then $x \not\in y$. The construction can be modified to ensure that all reals in $2^\omega \cap L$ are D-equivalent and $2^\omega \cap L$ has exactly two E-classes (similar to $2^\omega \setminus L$).
3 Let $x, y, z \in 2^\omega \setminus L$ in $L[a_0]$. 
Let $x, y, z \in 2^\omega \setminus L$ in $L[a_0]$. There is a perfect set $X \subseteq 2^\omega$ coded in $L$ and continuous 1-1 maps $f, g, h : X \to 2^\omega$ coded in $L$ such that $a_0 \in X$ and $x = f(a_0)$, $y = g(a_0)$, $z = h(a_0)$. 
Let $x, y, z \in 2^\omega \setminus L$ in $L[a_0]$. There is a perfect set $X \subseteq 2^\omega$ coded in $L$ and continuous 1-1 maps $f, g, h : X \to 2^\omega$ coded in $L$ such that $a_0 \in X$ and $x = f(a_0), y = g(a_0), z = h(a_0)$. By $A$, there exist: a perfect $Y \subseteq X$ coded in $L$ and some $\alpha < \omega_1$ such that $a_0 \in Y$ and $\langle E_\alpha, D_\alpha \rangle$ contains $f \restriction Y, g \restriction Y, h \restriction Y$.
Let $x, y, z \in 2^\omega \setminus L$ in $L[a_0]$. There is a perfect set $X \subseteq 2^\omega$ coded in $L$ and continuous 1-1 maps $f, g, h : X \to 2^\omega$ coded in $L$ such that $a_0 \in X$ and $x = f(a_0)$, $y = g(a_0)$, $z = h(a_0)$.

By **A**, there exist: a perfect $Y \subseteq X$ coded in $L$ and some $\alpha < \omega_1$ such that $a_0 \in Y$ and $\langle E_\alpha, D_\alpha \rangle$ contains $f \upharpoonright Y$, $g \upharpoonright Y$, $h \upharpoonright Y$. Thus

$$\forall a \in X \left( a D_\alpha f(a) D_\alpha g(a) D_\alpha h(a) \right)$$

holds in $L$,
Let $x, y, z \in 2^\omega \setminus L$ in $L[a_0]$. There is a perfect set $X \subseteq 2^\omega$ coded in $L$ and continuous 1-1 maps $f, g, h : X \to 2^\omega$ coded in $L$ such that $a_0 \in X$ and $x = f(a_0), y = g(a_0), z = h(a_0)$. By $A$, there exist: a perfect $Y \subseteq X$ coded in $L$ and some $\alpha < \omega_1$ such that $a_0 \in Y$ and $\langle E_\alpha, D_\alpha \rangle$ contains $f \upharpoonright Y, g \upharpoonright Y, h \upharpoonright Y$. Thus
\[
\forall a \in X \left( a \ D_\alpha \ f(a) \ D_\alpha \ g(a) \ D_\alpha \ h(a) \right)
\]
holds in $L$, hence, as $\langle E_\alpha, D_\alpha \rangle$ is a DBP,
\[
\forall a \in X \left( f(a) \ E_\alpha \ g(a) \lor f(a) \ E_\alpha \ h(a) \lor g(a) \ E_\alpha \ h(a) \right)
\]
in $L$. 
3 Let \( x, y, z \in 2^\omega \setminus \mathbb{L} \) in \( \mathbb{L}[a_0] \). There is a perfect set \( X \subseteq 2^\omega \) coded in \( \mathbb{L} \) and continuous 1-1 maps \( f, g, h : X \to 2^\omega \) coded in \( \mathbb{L} \) such that \( a_0 \in X \) and \( x = f(a_0), \ y = g(a_0), \ z = h(a_0) \).

By A, there exist: a perfect \( Y \subseteq X \) coded in \( \mathbb{L} \) and some \( \alpha < \omega_1 \) such that \( a_0 \in Y \) and \( \langle E_\alpha, D_\alpha \rangle \) contains \( f \upharpoonright Y, \ g \upharpoonright Y, \ h \upharpoonright Y \). Thus

\[
\forall a \in X (a \ D_\alpha f(a) \ D_\alpha g(a) \ D_\alpha h(a))
\]

holds in \( \mathbb{L} \), hence, as \( \langle E_\alpha, D_\alpha \rangle \) is a DBP,

\[
\forall a \in X (f(a) \ E_\alpha g(a) \lor f(a) \ E_\alpha h(a) \lor g(a) \ E_\alpha h(a))
\]

in \( \mathbb{L} \). By Shoenfield this is absolute, hence

\[x \ E_\alpha y \lor x \ E_\alpha z \lor y \ E_\alpha z\]

as required.
Suppose to the contrary that $A, B$ are $\text{OD}$. Let $a_0 \in A$. But $A \setminus \mathcal{L}$ consists of Sacks reals. Hence there is a perfect set $X \subseteq 2^{\omega}$ coded in $\mathcal{L}$, such that $a_0 \in X \setminus \mathcal{L} \subseteq A$ in $\mathcal{L}$. By $B$, there exist: a perfect set $Y \subseteq X$ coded in $\mathcal{L}$, an ordinal $\alpha < \omega_1$, and a Borel 1-1 map $f : Y \to Y$ coded in $\mathcal{L}$, such that $a_0 \in Y$, $\langle E_\alpha, D_\alpha \rangle$ contains $f \upharpoonright Y$ negatively. Thus the reals $a_0$ and $x = f(a_0)$ in $Y \setminus \mathcal{L} \subseteq A$ satisfy $a_0 D_\alpha x$, but $a_0 \not\in E_\alpha x$. It follows that $a_0 \not\in E x$, which contradicts the fact that $a_0, x$ belong to the same $E$-class. Thus $A, B$ is not $\text{OD}$ in $\mathcal{L}[a_0]$. Therefore $A \cup B \subseteq 2^{\omega} \setminus \mathcal{L}$ and $A, B$ are $E$-classes inside $2^{\omega} \setminus \mathcal{L}$. But already asserts that there are $\leq 2$ $E$-classes touching $2^{\omega} \setminus \mathcal{L}$, hence we have $2^{\omega} \setminus \mathcal{L} = A \cup B$. 

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4. Suppose to the contrary that $A, B$ are OD. Let $a_0 \in A$.

But $A \setminus L$ consists of Sacks reals. Hence there is a perfect set $X \subseteq 2^\omega$, coded in $L$, such that $a_0 \in X \setminus L \subseteq A$ in $L[a_0]$. 
Suppose to the contrary that $A, B$ are OD. Let $a_0 \in A$.

But $A \setminus L$ consists of Sacks reals. Hence there is a perfect set $X \subseteq 2^\omega$, coded in $L$, such that $a_0 \in X \setminus L \subseteq A$ in $L[a_0]$.

By $B$, there exist: a perfect set $Y \subseteq X$ coded in $L$, an ordinal $\alpha < \omega_1$, and a Borel 1-1 map $f : Y \to Y$ coded in $L$, such that $a_0 \in Y$, $\langle E_\alpha, D_\alpha \rangle$ contains $f|_Y$ negatively.
4 Suppose to the contrary that $A, B$ are $\text{OD}$. Let $a_0 \in A$.

But $A \setminus L$ consists of Sacks reals. Hence there is a perfect set $X \subseteq 2^\omega$, coded in $L$, such that $a_0 \in X \setminus L \subseteq A$ in $L[a_0]$.

By $B$, there exist: a perfect set $Y \subseteq X$ coded in $L$, an ordinal $\alpha < \omega_1$, and a Borel 1-1 map $f : Y \to Y$ coded in $L$, such that $a_0 \in Y$, $\langle E_\alpha, D_\alpha \rangle$ contains $f \upharpoonright Y$ negatively.

Thus the reals $a_0$ and $x = f(a_0)$ in $Y \setminus L \subseteq A$ satisfy $a_0 \mathrel{D_\alpha} x$, but $a_0 \not\mathrel{E_\alpha} x$. 
4 Suppose to the contrary that $A, B$ are $\text{OD}$. Let $a_0 \in A$.

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Thus the reals $a_0$ and $x = f(a_0)$ in $Y \setminus L \subseteq A$ satisfy $a_0 D_\alpha x$, but $a_0 \not E_\alpha x$. It follows that $a_0 \not E x$, which contradicts the fact that $a_0, x$ belong to the same $E$-class.

Thus $A, B$ is not $\text{OD}$ in $L[a_0]$. 
Suppose to the contrary that $A, B$ are OD. Let $a_0 \in A$.

But $A \setminus L$ consists of Sacks reals. Hence there is a perfect set $X \subseteq 2^\omega$, coded in $L$, such that $a_0 \in X \setminus L \subseteq A$ in $L[a_0]$.

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Thus the reals $a_0$ and $x = f(a_0)$ in $Y \setminus L \subseteq A$ satisfy $a_0 D_\alpha x$, but $a_0 \not\in E_\alpha x$. It follows that $a_0 \not\in E x$, which contradicts the fact that $a_0, x$ belong to the same E-class.

Thus $A, B$ is not OD in $L[a_0]$.

Therefore $A \cup B \subseteq 2^\omega \setminus L$ and $A, B$ are E-classes inside $2^\omega \setminus L$. 
Suppose to the contrary that $A, B$ are OD. Let $a_0 \in A$.

But $A \setminus L$ consists of Sacks reals. Hence there is a perfect set $X \subseteq 2^\omega$, coded in $L$, such that $a_0 \in X \setminus L \subseteq A$ in $L[a_0]$.

By $B$, there exist: a perfect set $Y \subseteq X$ coded in $L$, an ordinal $\alpha < \omega_1$, and a Borel 1-1 map $f : Y \rightarrow Y$ coded in $L$, such that $a_0 \in Y$, $\langle E_\alpha, D_\alpha \rangle$ contains $f \restriction Y$ negatively.

Thus the reals $a_0$ and $x = f(a_0)$ in $Y \setminus L \subseteq A$ satisfy $a_0 D_\alpha x$, but $a_0 \not E_\alpha x$. It follows that $a_0 \not E x$, which contradicts the fact that $a_0, x$ belong to the same E-class.

Thus $A, B$ is not OD in $L[a_0]$.

Therefore $A \cup B \subseteq 2^\omega \setminus L$ and $A, B$ are E-classes inside $2^\omega \setminus L$.

But $3$ already asserts that there are $\leq 2$ E-classes touching $2^\omega \setminus L$, hence we have $2^\omega \setminus L = A \cup B$. 
To prove 5 make use of C.
The speaker thanks *everybody* for patience
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