An unpublished theorem of Solovay on OD partitions of the reals into two non-OD parts, revisited

Vladimir Kanovei (IITP, Moscow)

Common result with Ali Enayat, J. Math. Log. 2020

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Theorem (essentially Solovay 2002 back TOC)

Let $a \in 2^\omega$ *be* *Sacks generic* *over* $\mathbf{L}$. *Then it is true in* $\mathbf{L}[a]$ *that*

1) *there is a partition* $2^\omega \smallsetminus \mathbf{L} = A \cup B$, *of the* Π^1_2 *set* $2^\omega \smallsetminus \mathbf{L}$ *of all nonconstructible reals, such that*

2) *the associated equivalence relation on* $2^\omega \smallsetminus \mathbf{L}$ *is lightface* Π^1_2, *hence the partition is* **OD** *as an unordered pair*

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Problem

Figure out the cases of **Cohen, random, Silver** *etc*. forcing notions.

Example (early years of forcing)

Let $\langle a, b\rangle$ be a Sacks×Sacks generic pair of reals over $\mathbf{L}$. Then it is true in $\mathbf{L}[a]$ that the $\mathbf{L}$-degrees $[a]_{\mathbf{L}}$ and $[b]_{\mathbf{L}}$ are indiscernible non-**OD** sets but their unordered pair $\{[a]_{\mathbf{L}}, [b]_{\mathbf{L}}\}$ is **OD**.

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See *e.g.* FGH, GH on some modern research related to indiscernible sets.

Theorem (Silver)

Let E *be a Borel equivalence relation on a Borel uncountable set* X *in a Polish space. There is a perfect set* $Y \subseteq X$ *such that:*

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Suitable more complex canonization results known from KSZ are used for the cases of Miller and $\mathbb{E}_0$-large forcing.

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The two distinct but indiscernable members of the generic extension are the two equivalence classes of E.
The proof is a bit too involved to type in using a web-interface like yahoo. (Shades of Fermat's margin!) $[\dots]$

– Bob

Definition

A double-bubble pair, DBP, is a pair of *countable* Borel equivalence relations $\langle \mathsf{E}, \mathsf{D} \rangle$ on 2^ω, such that each D-class is the union of exactly two distinct E-classes (in particular $\mathsf{E} \subsetneqq \mathsf{D}$).

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Thus a DBP $\langle \mathsf{E}, \mathsf{D} \rangle$ can be seen as a *Borel* partition of 2^ω into *countable* parts by D, plus a *finer* Borel partition by E that splits each D-class in exactly two non-empty *half-classes*.

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Example

Define $\mathbb{E}_0^{\mathtt{even}}$ on 2^ω so that $x \mathbin{\mathbb{E}_0^{\mathtt{even}}} y$ iff $\{n : x(n) \neq y(n)\}$ has a finite even number of elements. Then $\langle \mathbb{E}_0^{\mathtt{even}}, \mathbb{E}_0 \rangle$ is a DBP.

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- $\mathsf{D} \subseteq \mathsf{D}'$ and $\mathsf{E} \subseteq \mathsf{E}'$, and
- $\mathsf{D} \smallsetminus \mathsf{E} \subseteq \mathsf{D}' \smallsetminus \mathsf{E}'$, so that, for any $x, y \in 2^\omega$, if $x \in [y]_{\mathsf{D}} \smallsetminus [y]_{\mathsf{E}}$ then we still have $x \in [y]_{\mathsf{D}'} \smallsetminus [y]_{\mathsf{E}'}$.

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- $D \subseteq D'$ and $E \subseteq E'$, and
- $D \smallsetminus E \subseteq D' \smallsetminus E'$, so that, for any $x, y \in 2^\omega$, if $x \in [y]_D \smallsetminus [y]_E$ then we still have $x \in [y]_{D'} \smallsetminus [y]_{E'}$.

Thus extension of a DBP $\langle E, D \rangle$ means coarsening
(that is merging classes into bigger classes)
of the D-partition and E-subpartition, that honors the original splitting of D-classes into E-halfclasses.

It follows that if $\lambda \in \mathtt{Ord}$ is limit and $\langle E_\alpha, D_\alpha \rangle_{\alpha<\lambda}$ is a $\preccurlyeq$-increasing sequence then the limit pair $\lim_{\alpha\to\lambda}\langle E_\alpha, D_\alpha\rangle = \langle \bigcup_{\alpha<\lambda} E_\alpha, \bigcup_{\alpha<\lambda} D_\alpha \rangle$ is a DBP extending each $\langle E_\alpha, D_\alpha \rangle$.

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This will allow us to define, in **L**, an increasing transfinite sequence $\langle \mathsf{E}_\alpha, \mathsf{D}_\alpha\rangle_{\alpha<\omega_1}$ of DBPs such that $\bigcup_{\alpha<\omega_1} \mathsf{D}_\alpha$ will be essentially the total equivalence while accordingly the union $\mathsf{E} = \bigcup_{\alpha<\lambda} \mathsf{E}_\alpha$ will lead to the proof of the Solovay theorem.

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But we have to spesify passages from $\langle \mathrm{E}_\alpha, \mathrm{D}_\alpha \rangle$ to $\langle \mathrm{E}_{\alpha+1}, \mathrm{D}_{\alpha+1} \rangle$.

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Lemma 1 (Containment Lemma 1)

Assume that $\langle \mathrm{E}, \mathrm{D} \rangle$ *is a DBP,* $X \subseteq 2^\omega$ *is a perfect set, and* $f : X \to 2^\omega$ *is Borel and 1-1. Then there exist:*

- *a perfect set* $Y \subseteq X$, *and*
- *a DBP* $\langle \mathrm{E}', \mathrm{D}' \rangle$ *which extends* $\langle \mathrm{E}, \mathrm{D} \rangle$ *and contains* $f \restriction Y$.

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Now cook up $\langle \mathsf{E}', \mathsf{D}' \rangle$.

- If $x \notin \Delta$ then no extension: $[x]_{\mathsf{D}'} = [x]_{\mathsf{D}}$ and $[x]_{\mathsf{E}'} = [x]_{\mathsf{E}}$.

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- *a DBP* $\langle \mathrm{D}', \mathrm{E}' \rangle$ *that extends* $\langle \mathrm{D}, \mathrm{E} \rangle$ *and negatively contains* $f \restriction Y$, *so that* $f(y) \in [y]_{\mathrm{D}'} \smallsetminus [y]_{\mathrm{E}'}$ *for all* $y \in Y$.

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But **3** already asserts that there are ≤ 2 E-classes touching $2^\omega \smallsetminus \mathbf{L}$, hence we have $2^\omega \smallsetminus \mathbf{L} = A \cup B$.

To prove 5 make use of C.

The speaker thanks **everybody** for patience

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