Continuous logic and equivalence relations (joint with Andreas Hallbäck and Todor Tsankov)

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Structures

A **structure** is a set M equipped with relations R_i, $i \in I$, functions f_j, $j \in J$, and constants c_k, $k \in K$.

Examples:

- ordered sets $(P, \leq)$,
- graphs (R, E),
- Boolean algebras $(B, \wedge, \vee, -, 0, 1)$,
- metric spaces $(M, \{d_r\}_{r \in R})$, $R \subseteq \mathbb{R}^+$.

The space of countable structures and the logic action

Let L be a relational signature L, with n_i the arity of relational symbol R_i, $i \in I$. Then $\mathrm{Mod}(L) = \prod_{i \in I} 2^{\mathbb{N}^{n_i}}$ is the space of codes of all countable L-structures with universe $\mathbb{N}$.

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The group S_∞ acts on $\mathrm{Mod}(L)$ in a natural way: for $M, N \in \mathrm{Mod}(L)$ we put $g.M = N$ if

$$R_i^N(k_1, \ldots, k_{n_i}) \leftrightarrow R_i^M(g^{-1}(k_1), \ldots, g^{-1}(k_{n_i})).$$

for any $i \in I$ and $(k_1, \ldots, k_{n_i}) \in \mathbb{N}^{n_i}$.

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This action, called the **logic action**, induces the isomorphism equivalence relation $\cong$ on $\mathrm{Mod}(L)$.

$\mathcal{L}_{\omega_1\omega}$ and its fragments

We will work in the setting of infinitary logic $\mathcal{L}_{\omega_1\omega}$, i.e., an extension of the finitary logic $\mathcal{L}_{\omega\omega}$ allowing for countably infinite conjunctions $\bigwedge_i \phi_i$, and disjunctions $\bigvee_i \phi_i$ of formulas.

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A (countable) **fragment** F is a countable set of $\mathcal{L}_{\omega_1\omega}$ formulas containing all $\mathcal{L}_{\omega\omega}$-formulas, and closed under $\wedge$, $\vee$, $\neg$, and $\exists$. We can talk about F-theories, spaces $\mathrm{Mod}(T) \subseteq \mathrm{Mod}(L)$ of models of a given F-theory, elementary F-embeddings etc.

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For an F-theory T, a (complete) *n*-**type** (in T) is a homomorphism from the (quotient) Boolean algebra $\mathcal{F}_T(\bar{x})$ of formulas with free variables among an n-tuple $\bar{x}$ into the two-element Boolean algebra. The space $S_n(T)$ of all n-types is naturally equipped with the logic topology with basis consisting of sets $[\phi]$ defined by $p \in [\phi]$ iff $p(\phi) = 1$, where $\phi \in \mathcal{F}_T(\bar{x})$.

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Warning: Not all the types defined in this way are realizable b/c the compactness theorem fails for $\mathcal{L}_{\omega_1\omega}$!

Topologies defined by fragments

For $\phi \in \mathcal{L}_{\omega_1\omega}$ and tuple $\bar{a} \in \mathbb{N}^{<\mathbb{N}}$, let

$$\mathrm{Mod}(\phi, \bar{a}) = \{M \in \mathrm{Mod}(L) : M \vDash \phi(\bar{a})\}.$$

A fragment F generates a Polish topology t_F on $\mathrm{Mod}(L)$ with basis

$$B_F = \{\mathrm{Mod}(\phi, \bar{a}) : \phi \in F, \bar{a} \in \mathbb{N}^{<\mathbb{N}}\}.$$

Complexity of equivalence relations

An equivalence relation E on a Polish space X is (Borel) **reducible** to an equivalence relation F on a Polish space Y if there is a Borel mapping $f : X \to Y$ such that, for any $x_1, x_2 \in X$,

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Important types of equivalence relations:

- smooth relations, e.g., relations reducible to the identity on a Polish space;
- essentially countable relations, e.g., relations reducible to a relation with countable classes.

Smooth and essentially countable isomorphism relations

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Theorem (Hjorth-Kechris)

Let T be a countable theory, and let $\cong_T$ be the isomorphism relation on $\mathrm{Mod}(T)$. TFAE:

1. $\cong_T$ *is potentially* Π^0_2*;*
2. *There exists a fragment* F *such that for every* $M \in \mathrm{Mod}(T)$*, the theory* $\mathrm{Th}_F(M)$ *is* $\aleph_0$*-categorical;*
3. $\cong_T$ *is smooth.*

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With the same assumptions as above, TFAE:

1. *$\cong_T$ is potentially Σ^0_2;*
2. *There exists a fragment F such that for every $M \in \mathrm{Mod}(T)$, there is $\bar{a} \in \mathbb{N}^{<\mathbb{N}}$ such that $\mathrm{Th}_F(M, \bar{a})$ is $\aleph_0$-categorical;*
3. *$\cong_T$ is essentially countable.*

Metric structures

A **metric structure** is a complete metric space (M, d) equipped with bounded uniformly continuous functions $R_i : M^{n_i} \to \mathbb{R}$, $i \in I$ (relations), uniformly continuous functions $f_j : M^{n_j} \to M$, $j \in J$, and constants c_k, $k \in K$.

A metric signature consists of relation (including the metric), function, and constant symbols, as well as arities, moduli of continuity $\Delta : [0, +\infty)^n \to [0, +\infty)$, and bounds $I \subseteq \mathbb{R}$ for relation symbols. Each of the relations and functions of a metric structure in a given signature must respect its modulus of continuity. Each of the relations must respect its bound.

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Examples:

- Complete metric spaces (M, d);
- Complete metric groups $(G, d, \cdot, \cdot^{-1}, e)$;
- Measure algebras $(B, d, \wedge, \vee, 0, 1)$;
- Banach spaces, C^*-algebras, etc.

The space of Polish metric structures

Let L be a countable relational signature L, with n_i the arity of relation R_i, $i \in I$, where $R_0 = d$. Then $\mathrm{Mod}(L) \subseteq \prod_{i \in I} \mathbb{R}^{\mathbb{N}^{n_i}}$ is the space of codes of all Polish metric structures with universe containing $\mathbb{N}$ as a (tail-)dense subset of M.

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Remark: There is no logic action, and thus no Vaught transforms! However, for any $M \in \mathrm{Mod}(L)$, we can consider a Polish space $D \subseteq M^{\mathbb{N}}$ of all tail-dense sequences in M, and a natural projection $\pi : D \to [M]$ from D to the isomorphism class $[M]$ of M in $\mathrm{Mod}(L)$. This gives a tool analogous to Vaught transforms.

Continuous $\mathcal{L}_{\omega\omega}$ and $\mathcal{L}_{\omega_1\omega}$

Formulas of (continuous) finitary logic $\mathcal{L}_{\omega\omega}$ are defined using

- inf and sup playing the role of quantifiers;
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We can also define fragments of $\mathcal{L}_{\omega_1\omega}$, and Polish topologies on $\mathrm{Mod}(L)$ defined by fragments.

Type spaces

For a given fragment F, and F-theory T, an n-type (in T) is a homomorphism into $\mathbb{R}$ from the quotient real Banach algebra $\mathcal{F}_T(\bar{x})$ of formulas with seminorm

$$\|\phi\| = \sup\{\phi^M(\bar{a}) : M \models T, \bar{a} \in M^n\}$$

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There is also a natural (complete) metric ∂ on $S_n(T)$. If $F = \mathcal{L}_{\omega\omega}$, because of compactness, it can be defined by

$$\partial(p,q) = \inf\{d^M(\bar{a},\bar{b}) : M \models T,\ \bar{a},\bar{b} \in M^n,\ \mathrm{tp}(\bar{a}) = p, \mathrm{tp}(\bar{b}) = q\ \}$$

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In general, we can put

$$\partial(p,q) = \sup_{\phi \in F_1} |p(\phi) - q(\phi)|\,,$$

where F_1 are 1-Lipschitz formulas in $\mathcal{F}_T(\bar{x})$.

Omitting types and atomic models

Definition: A type p is **isolated** if τ and ∂-topology coincide at p. A model is **atomic** if it realizes only isolated types.

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Theorem (Omitting types)

Let F be a fragment and let T be an F-theory. Suppose that for every n, we are given $O_n \subseteq S_n(T)$ a τ-meager and ∂-open set. Then there is a separable model $M \models T$ that omits all of the O_n.

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Theorem (Existence of atomic models)

Let T be a complete theory. Then the following are equivalent:

1. *T admits an atomic model;*
2. *There exist subsets $O_n \subseteq S_n(T)$ such that for all n, (O_n, ∂) is separable and $\forall^* M \in \mathrm{Mod}(T)\ \forall n\ \Theta[M] \subseteq O_n$.*

In particular, if $(S_n(T), \partial)$ is separable for every n, then T admits an atomic model.

$\aleph_0$-categorical and atomic models

Theorem

Let F be a fragment and let T be an F-theory. For any $M \in Mod(T)$,

1. *M is $\aleph_0$-categorical iff $[M]$ is closed in the topology t_F.*
2. *M is an atomic model of $\mathrm{Th}_F(M)$ iff $[M]$ is G_δ in the topology t_F.*

Smooth isomorphism relations

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1. *$\cong_T$ is potentially Π^0_2;*
2. *There exists a fragment F such that for every $M \in \mathrm{Mod}(T)$, the theory $\mathrm{Th}_F(M)$ is $\aleph_0$-categorical;*
3. *$\cong_T$ is smooth.*

Essentially countable isomorphism relations

Definition: A type p is $\aleph_0$-**rigid** if whenever $(M, \bar{a})$ and $(N, \bar{b})$ are two realizations of p with M and N separable, then $M \cong N$.

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Theorem

Let T be a countable theory such that all of its (separable) models are locally compact, and $\cong_T$ is Borel. TFAE:

1. $\cong_T$ *is potentially* Σ^0_2*;*
2. *There exists a fragment F such that for every $M \in \mathrm{Mod}(T)$, there is $k \in \mathbb{N}$ such that the set*

   $$\{\bar{a} \in M^k : \mathrm{Th}_F(M, \bar{a}) \text{ is } \aleph_0\text{-rigid}\}$$

   has non-empty interior in M^k;
3. $\cong_T$ *is essentially countable.*

Coding actions with Polish metric structures

Let $G \leq \mathrm{Homeo}(X)$ be a locally compact group with a proper right-invariant metric d_R, where X is compact with distance d bounded by 1, and let $\{a_i\}_{i\in\mathbb{N}}$ be a dense sequence in X. Let $L = \{P_i : i \in \mathbb{N}\}$ be the signature where each P_i is a unary predicate symbol bounded by 1. For each $x \in X$ define an L-structure $A(x)$ with universe (G, d_R) and predicates defined on G by

$$P_i^x(h) = d(h.x, a_i).$$

Note that the predicates P_i^x code x uniquely: if $P_i^x(1_G) = P_i^y(1_G)$ for all i, then $x = y$.

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Let A be a proper metric structure. Then (A, a) is $\aleph_0$-categorical (in $\mathcal{L}_{\omega\omega}$), and so $\mathrm{tp}(a)$ is $\aleph_0$-rigid, for every $a \in A$.

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Corollary (Kechris)

Let G be a locally compact Polish group continuously acting on a Polish space X. Then the orbit equivalence relation is essentially countable.

Thank You!