Anticlassification results for groups acting freely on the line

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Caltech Logic Seminar

March 1, 2021

THE UNIVERSITY OF ILLINOIS AT CHICAGO

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G is a coutable infinite group.

Definition

G is **left-orderable** iff it admits a strict total order such that for all $f, g, h \in G$,

$$g < h \implies fg < fh.$$

- G left-orderable $\implies$ G torsion-free.
- If G is a finite-index subgroup of $\mathrm{SL}_n(\mathbb{Z})$ with $n \geq 3$, then G is not left-orderable. (Witte-Morris '94)

Proposition

*G is **left-orderable** iff there is $P \subseteq G$ such that*

1. $PP \subseteq P$;
2. $G = P \sqcup P^{-1} \sqcup \{1\}$.

Sketch.

If $<$ is a left-order on G then the **positive cone** $P_< = \{g \in G \mid 1 < g\}$ satisfies 1 and 2. Conversely, if $P \subseteq G$ satisfies 1 and 2, then we can define a left-order on G by

$$g <_P h \iff g^{-1}h \in P. \qquad \square$$

Theorem (Folklore)

When G is countable, the following are equivalent:

- *G acts faithfully on $\mathbb{R}$ by orientation preserving homeomorphism. I.e.,*

$$G \hookrightarrow \mathrm{Homeo}_+(\mathbb{R})$$

- *G is left-orderable.*

Definition

A left-order $<$ on G is **Archimedean** iff for all nonzero $g, h \in G$ there is $n \in \mathbb{Z}$ such that $g < h^n$.

Theorem (Hölder 1901)

When G is countable, the following are equivalent:

- *G has an Archimedean order.*
- *G is isomorphic to a subgroup of $(\mathbb{R}, +)$ equipped with the natural ordering on $\mathbb{R}$.*
- *G acts freely on $\mathbb{R}$ by orientation preserving homeomorphism.*

We explore Archimedean orders from the viewpoint of descriptive set theory.

1. Our work continues the analysis of the **Polish space of left-orderings** on a given countable group.
   (Sikora '04, Linnell '11, Navas '10, Rivas'12, etc...)
2. We address the **classification problem** for countable ordered Archimedean groups up to ismomorphism.

In both direction **Borel classification theory** is an indispensable tool.

Definition

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The space of left-orderings of G

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  is a closed subset of 2^G.
- Thus, $\mathrm{LO}(G)$ is a compact **Polish space** with the induced topology.

A DYNAMICAL VIEW OF $\mathrm{LO}(G)$

- We consider the **conjugacy action** $G \curvearrowright \mathrm{LO}(G)$ by letting $(g, P) \mapsto gPg^{-1}$.
- Let $E_{\mathsf{lo}}(G)$ be the **countable Borel equivalence relation** on $\mathrm{LO}(G)$ whose classes are the G-orbits.

Deroin, Navas, & Rivas asked:

> *Do there exist left-orderable groups G for which the quotient Borel space $\mathrm{LO}(G)/E_{\mathsf{lo}}(G)$ is not standard?*
>
> *(Groups, Orders, and Dynamics 2016)*

Clearly, this is not the case when G is abelian because inner automorphisms are trivial.

Definition

Let E and F equivalence relations on the standard Borel space X and Y, respectively. We say that E is **Borel reducible** to F (in symbols, $E \leq_B F$) if there is a Borel map $\phi\colon X \to Y$ such that

$$x_0 \, E \, x_1 \iff \phi(x_0) \, F \, \phi(x_1).$$

Proposition

*Let E be a countable Borel equivalence on the standard Borel space X. The quotient Borel space X/E is standard (with the quotient Borel structure) if and only if E is **smooth**, i.e. there exists a Borel $\phi\colon X \to \mathbb{R}$ such that*

$$x_0 \, E \, x_1 \iff \phi(x_0) = \phi(x_1).$$

$\mathrm{LO}(G)/G$ IS POSITIVELY COMPLICATED

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The answer is YES.

Theorem (C.–Clay 2020+)

The conjugacy relation $E_{\mathsf{lo}}(\mathbb{F}_2)$ on $\mathrm{LO}(\mathbb{F}_2)$ is a universal countable Borel equivalence relation.

Definition

For $(G,+)$ abelian we define the **space of Archimedean orderings** of G as

$$\mathrm{Ar}(G) := \{P \in \mathrm{LO}(G) \mid \forall x, y\, \exists k \in \mathbb{Z}\, (y \neq 0 \implies ky - x \in P)\},$$

which is a G_δ subset of $\mathrm{LO}(G)$, thus is a Polish space with the relative topology.

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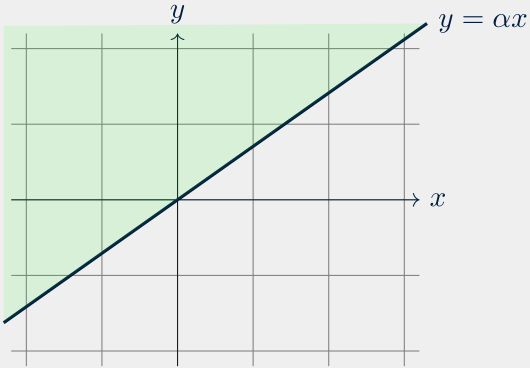
We denote by $\cong_{\mathrm{Ar}(\mathbb{Q}^2)}$ the countable Borel equivalence relation induced from the action

$$\mathrm{Aut}(\mathbb{Q}^2) = \mathrm{GL}_2(\mathbb{Q}) \curvearrowright \mathrm{Ar}(\mathbb{Q}^2).$$

Theorem (C.–Marker–Motto Ros–Shani 2020+)

$\cong_{\mathrm{Ar}(\mathbb{Q}^2)}$ *is not smooth. Hence* $\mathrm{Ar}(\mathbb{Q}^2)/\cong_{\mathrm{Ar}(\mathbb{Q}^2)}$ *is not standard.*

The proof uses the geometric interpretation of Archimedean orderings of $\mathbb{Q}^2$.



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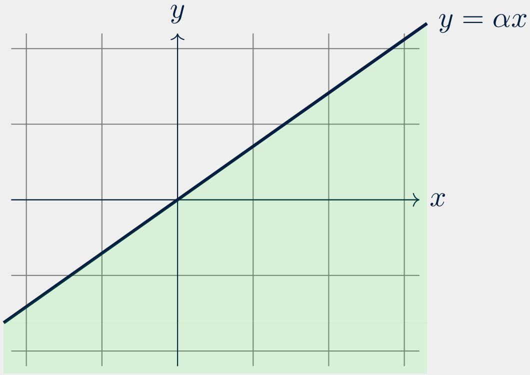

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Sketch of the proof.

Let $\mathrm{GL}_2(\mathbb{Z})$ be the group of 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integral coefficients and determinant $ad - bc = \pm 1$.

Let $\mathrm{GL}_2(\mathbb{Z}) \curvearrowright \mathbb{R} \cup \{\infty\}$ by Möbius transformation $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \frac{ax+b}{cx+d}$.

The induced equivalence relation $E_{\mathrm{GL}_2(\mathbb{Z})}^{\mathbb{R}\cup\{\infty\}}$ is not smooth. In fact,

$$E_{\mathrm{GL}_2(\mathbb{Z})}^{\mathbb{R}\cup\{\infty\}} \sim_B E_0.$$

Define

$$\begin{aligned} f \colon \mathbb{R} \smallsetminus \mathbb{Q} &\to \mathrm{Ar}(\mathbb{Q}^2) \\ \alpha &\mapsto \{\vec{x} \in \mathbb{Q}^2 \mid \vec{x} \cdot (1, \alpha) > 0\}. \end{aligned}$$

f is a **weak Borel reduction**, i.e., is a countable-to-one Borel function such that $x \, E_{\mathrm{GL}_2(\mathbb{Z})}^{\mathbb{R}\smallsetminus\mathbb{Q}} \, y \implies f(x) \cong_{\mathrm{Ar}(\mathbb{Q}^2)} f(y)$.

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Complete classifications

Let E be an equivalence relation on X. A **complete classification** for E is a map $c\colon X \to I$ such that for any $x, y \in X$,

$$x \, E \, y \iff c(x) = c(y).$$

The elements of I are called **complete invariants** for E.

For every $P, Q \in \mathrm{Ar}(\mathbb{Q}^2)$,

$$P \cong_{\mathrm{Ar}(\mathbb{Q}^2)} Q \quad \iff \quad (\mathbb{Q}^2, +, <_P) \cong (\mathbb{Q}^2, +, <_Q).$$

We cannot classify Archimedean ordered groups of the kind $(\mathbb{Q}^2, +, <)$ up to isomorphism using numerical invariants.

How complicated is the problem of classifying countable ordered Archimedean groups up to isomorphism?

We define the Polish space of countable ordered Archimedean groups in the usual way.

$$X_{\mathsf{ArGp}} := \{G = (\mathbb{N}, +^G, <^G) \models \varphi_{\mathsf{ArGp}}\}$$

where φ_{ArGp} is the $\mathcal{L}_{\omega_1\omega}$-axiom for ordered Archimedean groups.

In view of Hölder's theorem we can define:

$$\mathcal{A} := \{(x_i : i \in \mathbb{N}) \in \mathbb{R}^{\mathbb{N}} \mid \{x_i : i \in \mathbb{N}\} \text{ is a subgroup of } \mathbb{R}\}.$$

Proposition

There is a continuous function $X_{\mathsf{ArGp}} \to \mathcal{A}, G \mapsto \vec{x}_G$ such that

$$G \cong_{\mathsf{ArGp}} H \iff \vec{x}_G \cong_{\mathcal{A}} \vec{x}_H.$$

In particular, $\cong_{\mathsf{ArGp}}$ and $\cong_{\mathcal{A}}$ are Borel bi-reducible.

A famous consequence of Hölder's theorem.

Lemma (Hion 54')

Suppose that A and B are two (necessarily Archimedean) subgroups of $\mathbb{R}$ and $h\colon A \to B$ is an order preserving homomorphism. Then, there exists $\lambda \in \mathbb{R}^+$ such that $h(a) = \lambda a$, for every $a \in A$. In fact, such λ is computed as the ratio $\frac{h(a)}{a}$, for any nonzero $a \in A$.

Proposition

$\cong_{\mathcal{A}}$ is a $\mathbf{\Sigma}^0_4$ equivalence relation. Thus, so is $\cong_{\mathsf{ArGp}}$.

Definition (Friedman–Stanley 1989)

Let E be an equivalence relation on a standard Borel space X.
For $x = (x_i : i \in \mathbb{N})$ and $y = (y_i : i \in \mathbb{N})$ in $X^{\mathbb{N}}$ let

$$x \, E^+ \, y \iff \{[x_i]_E : i \in \mathbb{N}\} = \{[y_i]_E : i \in \mathbb{N}\}.$$

$$=^{(\alpha+1)+} \quad := \quad (=^{\alpha+})^+$$

$$=^{\lambda+} \quad := \quad \prod_{\alpha<\lambda} =^{\alpha+} \text{ for } \lambda \text{ limit.}$$

Proposition

Every Borel isomorphism relation is Borel reducible to $=^{\alpha+}$ for some $\alpha < \omega_1$.

The first Friedman-Stanley jump $=^+$ is defined on $\mathbb{R}^\mathbb{N}$ so that the map

$$(x_i : i \in \mathbb{N}) \mapsto \{x_i : i \in \mathbb{N}\} \in \mathcal{P}(\mathbb{R}).$$

is a complete classification of $=^+$ by countable sets of reals.

The second Friedman-Stanley jump $=^{++}$ is defined on $(\mathbb{R}^\mathbb{N})^\mathbb{N}$ and admits a complete classification by hereditarily countable elements in $\mathcal{P}_2(\mathbb{R})$.

If G is a nontrivial subgroup of $\mathbb{R}$ let $A_G := \left\{ \underbrace{\left\{ \frac{g}{r} : g \in G \right\}}_{G/r} : r \in G \setminus \{0\} \right\}$.

Proposition

Let G and H be non-trivial subgroups of $\mathbb{R}$. Then

$$G \text{ and } H \text{ are order isomorphic} \iff A_G = A_H.$$

Note that for $r \neq s$, the sets G/r and G/s are not at odds with each other.

We cannot use countable sets of reals to classify $\cong_{\mathsf{ArGp}}$.

Theorem (C.–Marker–Motto Ros–Shani 2020+)

$$=^+ <_B \cong_{\mathsf{ArGp}} <_B =^{++}$$

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In fact,

- $\cong_{\mathsf{ArGp}} \leq_B \cong^{*}_{3,1}$;
- $\cong_{\mathsf{ArGp}} \not\leq_B \cong^{*}_{3,0}$ *(it is known that* $=^{+} <_B \cong^{*}_{3,0}$*).*

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Corollary

It is not possible to define a Polish topology on X_{ArGp} *generating its usual Borel structure so that* $\cong_{\mathsf{ArGp}}$ *is a* $\mathbf{\Pi}^0_3$ *subset of the product space* $X_{\mathsf{ArGp}} \times X_{\mathsf{ArGp}}$.

(Hjorth–Kechris–Louveau 1998) defined a refinement of Friedman–Satanley hierarchy. In particular,

$$=^{+} <_B \cong^*_{3,0} <_B \cong^*_{3,1} <_B =^{++}$$

An invariant for $\cong^*_{3,1}$ is a hereditarily countable set $A \in \mathcal{P}_3(\mathbb{N})$ (i.e., a $=^{++}$-invariant) together with

- a ternary relation $R \subseteq A \times A \times \mathcal{P}(\mathbb{N})$, definable from A, such that given any $a \in A$, $R(a, -, -)$ is an injective function from A to $\mathcal{P}(\mathbb{N})$.

Definition

Let Γ be a complexity class closed by continuous preimages and suppose that E is Borel equivalence relation on a standard Borel space. We say that E is **potentially** Γ if and only if there is a Polish space Y and a Γ equivalence relation $F \subseteq Y \times Y$ such that $E \leq_B F$.

Theorem (Hjorth-Kechris-Louveau 1998)

Let E be an isomorphism relation. Then

- E *is potentially* $\boldsymbol{\Pi}^0_3$ *if and only if* $E \leq_B =^+$.
- E *is potentially* $\boldsymbol{\Sigma}^0_4$ *if and only if* $E \leq_B \cong^*_{3,1}$.

Since the complexity $\cong_{\mathsf{ArGp}}$ is exactly $\boldsymbol{\Sigma}^0_4$, it follows that

$$\cong_{\mathsf{ArGp}} \leq_B \cong^*_{3,1}$$

Lemma

Suppose that G, H are subfields of $\mathbb{R}$. Then

$$G \cong_{\mathcal{A}} H \iff G = H.$$

Proposition

$=^+ \leq_B \cong_{\mathsf{ArGp}}$

Sketch.

Let $T \subseteq \mathbb{R}$ be a perfect set of algebraic independent reals. Then the map $\mathcal{P}_{\mathsf{ctbl}}(T) \to \mathcal{A}, S \mapsto \mathbb{Q}(S)$ witnesses that $=^+ \leq_B \cong_{\mathcal{A}}$. □

$\cong_{\mathsf{ArGp}} \not\leq_B \cong^*_{3,0}$

Theorem (essentially Shani 2018)

Suppose E is a Borel equivalence relation on a standard Borel space X, and $x \mapsto A_x$ is an absolute classification of E by hereditarily countable sets. Let x be an element of X in some generic extension of V.
If $E \leq_B \cong^*_{3,0}$, then there is a set of sets of reals $B \in V(A_x)$ so that B is definable from A_x and parameters in V alone, $V(A_x) = V(B)$, and B is countable (in $V(A_x)$).

Over the Cohen model, we force the existence of a generic subgroup G of $\mathbb{R}$ so that every set of reals $B \in V(A_G)$ which is definable from A_G and parameters in V alone, we have $V(B) \neq V(A_G)$.

Let $(\mathfrak{c}_i : i \in \mathbb{N})$ be a sequence of Cohen reals.

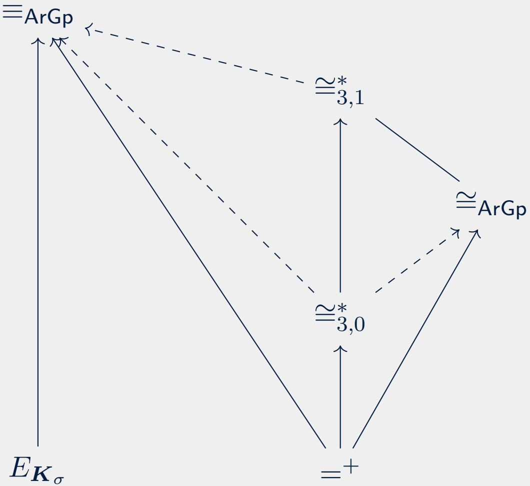
The **Basic Cohen model** $V(\{\mathfrak{c}_i : i \in \mathbb{N}\})$, can be defined as the closure of $\{\mathfrak{c}_i : i \in \mathbb{N}\}$ under definable set theoretic operation.

- is a model of ZF in which choice fails.
- $\{\mathfrak{c}_i : i \in \mathbb{N}\}$ is **Dekind-finite**, i.e., there are no infinite sequences in $\{\mathfrak{c}_i : i \in \mathbb{N}\}$.

Properties of the forcing

- Adds a generic subgroup G of the field $\mathbb{Q}(\mathfrak{c}_i : i \in \mathbb{N})$ generated by $\{\mathfrak{c}_i : i \in \mathbb{N}\}$.
- It does not add any real, so that all sets of reals in $V(A_G)$ live in the Cohen model.
- For every countable set of reals $B \in V(A_G)$ definable from A_G and parameters in V alone and such that B is countable,

$$V(B) = V(A_G) \quad \Longrightarrow \quad \text{there is an infinite sequence in } \{\mathfrak{c}_i : i \in \mathbb{N}\}.$$


$\equiv_{\mathsf{ArGp}}$
$\cong^*_{3,1}$
$\cong_{\mathsf{ArGp}}$
$\cong^*_{3,0}$
$E_{\boldsymbol{K}_\sigma}$
$=^+$

Theorem

When G is countable, the following are equivalent:

- *G has a circular Archimedean order.*
- *G acts freely on $\mathbb{S}^1$ by orientation preserving homeomorphism.*

Theorem (C.–Marker–Motto Ros–Shani 2020+)

The isomorphism relation for countable circular ordered Archimedean groups $\cong_{\mathrm{CO}}$ is Borel bi-reducible to $=^+$.

Theorem (Rast–Sahota 2016)

Let T be a complete first order o-minimal theory. Then

1. $\cong_T$ *is smooth; or*
2. $\cong_T$ *is Borel bi-reducible to* $=^+$*; or*
3. $\cong_T$ *is* S_∞*-universal.*

The theory of **ordered divisible abelian groups** (ODAG) is in case 3.

Theorem (C.–Marker–Motto Ros–Shani 2020+)

The bi-embeddability relation $\equiv_{\mathsf{ODAG}}$ *on countable ordered divisible abelian groups is a universal* $\mathbf{\Sigma}^1_1$ *equivalence relation.*

Thank You!