Omnigenous Groups

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Outline

Hall's Group $\mathbb{H}$

The class of all finite groups is a Fraïssé class, with Fraïssé limit $\mathbb{H}$.

Theorem (Hall, 1959)

1. *Every finite group can be embedded in $\mathbb{H}$.*
2. *Any two isomorphic finite subgroups of $\mathbb{H}$ are conjugate in $\mathbb{H}$.*
3. *$\mathbb{H}$ is the unique countable locally finite group up to isomorphism with properties 1 and 2.*
4. *Every countable locally finite group can be embedded in $\mathbb{H}$, i.e., $\mathbb{H}$ is a universal countable locally finite group.*

Hall's Group $\mathbb{H}$

Fact

$\mathbb{H}$ *is the unique countable locally finite group with the property:*

For any finite subgroup $F \leq \mathbb{H}$, *finite group* Γ, *and isomorphic embedding* $\varphi : F \to \Gamma$, *there is a finite subgroup* G *with* $F \leq G \leq \mathbb{H}$ *and isomorphism* $\gamma : G \cong \Gamma$ *such that* $\gamma|_F = \varphi$.

Omnigenous groups

Definition

A countable locally finite group H is omnigenous if

For any finite subgroup $F \leq H$, finite group Γ, and isomorphic embedding $\varphi : F \to \Gamma$, there is a finite subgroup G with $F \leq G \leq H$ and surjective homomorphism $\gamma : G \to \Gamma$ such that $\gamma \restriction_F = \varphi$.

Theorem (EGLMM)

There are continuum many pairwise nonisomorphic countable universally locally finite groups that are omnigenous.

Urysohn Spaces $\mathbb{U}_\Delta$

Definition

Δ is a distance value set if it is a subset of $(0, +\infty)$ such that

$$\forall x, y \in \Delta \ \min(x + y, \sup(\Delta)) \in \Delta.$$

This is a particular case of Conant's distance monoids.

If Δ is a countable distance value set, the class of all finite Δ-metric spaces form a Fraïssé class, with Fraïssé limit $\mathbb{U}_\Delta$. The isometry group of $\mathbb{U}_\Delta$ is denoted as $\mathrm{Iso}(\mathbb{U}_\Delta)$.

E.g., when $|\Delta| = 1$, $\mathbb{U}_\Delta = K_\infty$ and $\mathrm{Iso}(\mathbb{U}_\Delta) = S_\infty$. When $\Delta = \{1, 2\}$, $\mathbb{U}_\Delta = R$ is the random graph and $\mathrm{Iso}(\mathbb{U}_\Delta) = \mathrm{Aut}(R)$.

Theorem (EGLMM)

Let H be a countable omnigenous locally finite group. Then for any countable distance value set Δ, $Iso(\mathbb{U}_\Delta)$ contains H as a dense subgroup.

Corollary (Vershik's Conjecture, 2008)

$Aut(R)$ and $Iso(\mathbb{U})$ contain $\mathbb{H}$ as a dense subgroup.

Lemma (Essentially Rosendal, 2011)

Let Δ be any countable distance value set. Let X be a finite Δ-metric space. Let $\Lambda \leq \Gamma$ be finite groups and $\pi : \Lambda \to Iso(X)$ be an isomorphic embedding. Then there is a finite Δ-metric space Y extending X and an isomorphic embedding $\pi' : \Gamma \to Iso(Y)$ such that for any $\gamma \in \Lambda$ and $x \in X$, $\pi'(\gamma)(x) = \pi(\gamma)(x)$.

Theorem (Solecki, 2009; Siniora-Solecki, 2020)

Let Δ be any distance value set and X be a finite Δ-metric space. Then there is a finite Δ-metric space Y extending X and a map ϕ such that

(a) *for any partial isometry p of X, $\phi(p) \in Iso(Y)$ extends p;*

(b) *for any partial isometries p and q of X with $rng(q) = dom(p)$, $\phi(p \circ q) = \phi(p) \circ \phi(q)$.*

The Converse Problem

To characterize dense countable (locally finite) subgroups of $\mathrm{Iso}(\mathbb{U}_\Delta)$ for all countable distance value sets Δ.

The MIF Properties

Definition

Let G be a group. Let F_n be the free group generated by variables $x_1, \ldots, x_n$. A nontrivial mixed identity in G is a word $w(x_1, \ldots, x_n) \in G * F_n \setminus G$ such that $w(g_1, \ldots, g_n) = 1$ for all $g_1, \ldots, g_n \in G$.

If there is no nontrivial mixed identity in G, we say G is mixed identity free (MIF).

E.g. in an abelian group G, $xyx^{-1}y^{-1}$ is a nontrivial mixed identity.

Theorem (EGLMM)

For any countable distance value set Δ with $|\Delta| > 1$, any dense subgroup of Iso$(\mathbb{U}_\Delta)$ is MIF.

The theorem fails for $|\Delta| = 1$.

Theorem (Hull-Osin, 2016)

Let G be a countable dense subgroup of S_∞. Then exactly one of the following holds:

(i) *G contains an isomorphic copy of Alt$(\mathbb{N})$ as a normal subgroup;*

(ii) *G is MIF.*

Definition

Let G be a locally finite group. We say that G is ∞-MIF if for any

$$w_1(x_1, \ldots, x_n; g_1, \ldots, g_k), w_2(x_1, \ldots, x_n; g_1, \ldots, g_k), \cdots\cdots \in G * F_n \setminus G$$

whenever there is a finite group Γ which is an overgroup of $\langle g_1, \ldots, g_k \rangle$ in which there are $\gamma_1, \ldots, \gamma_k \in \Gamma$ such that

$$w_i(\gamma_1, \ldots, \gamma_n; g_1, \ldots, g_n) \neq 1 \quad \forall i \geq 1,$$

there are $h_1, \ldots, h_n \in G$ such that

$$w_i(h_1, \ldots, h_n; g_1, \ldots, g_k) \neq 1 \quad \forall i \geq 1.$$

Theorem (EGLMM)

If G is a locally finite group, then G is ∞-MIF iff G is omnigenous.