A new regularity property of the Haar null ideal

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($\Rightarrow$) Assume the contrary, pick K cpct with $\mu(K) > 0$. Let $(g_i K)_{i \in I}$ be a maximal family of pairwise disjoint translates of K. Then I is uncountable.

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Haar null sets

Definition (Christensen)

Let $(G, \cdot)$ be a Polish group. A universally measurable set $S \subset G$ is *Haar null*, if there exists a Borel probability measure μ on G such that for each $g, h \in G$ we have $\mu(gSh) = 0$.

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In contrast, the set $\mathbb{N}^\omega$ is not Haar null.

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3. (Christensen) If X is a separable Banach space, every $f : X \to \mathbb{R}$ Lipschitz functions is almost everywhere Gateaux differentiable.

Regularity properties

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There exists a Haar null Borel set in $\mathbb{Z}^\omega$, which is not contained in a Haar null G_δ set.

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There exists a Haar null co-analytic set in $\mathbb{Z}^\omega$, which is not contained in a Haar null Borel set.

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Lemma (Solecki, D. Nagy)

Let S be a Haar null set in $\mathbb{Z}^\omega$. There exists a $b \in \omega^\omega$ such that μ_b is a witness measure to $S \in \mathcal{HN}$.

Solecki's construction and dominating sets

For $b, b' \in \omega^\omega$ we say $b \leq^* b'$ if the set $\{n : b(n) \leq b'(n)\}$ co-finite. A set $D \subset \omega^\omega$ is called *dominating* if for every $b \in \omega^\omega$ there exists a $d \in D$ with $b \leq^* d$.

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Lemma

A set $S \subset \mathbb{Z}^\omega$ is Haar positive if and only if the set $\{b : \exists g \in \mathbb{Z}^\omega\ \mu_b(S + g) > 0\}$ is dominating.

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A dominating analytic set contains a dominating closed set. A Borel function $D \to \omega^\omega$, where D is dominating and Borel, is continuous on a dominating closed set.

Coding trick

Let $\mathcal{I}$-be an ideal on ω^ω, and assume that $A \subset \omega^\omega$ is analytic set $\notin \mathcal{I}$.

Suppose that $F \subset \omega^\omega$ is closed. Enough to construct $\phi : F \to A$ such that

- ϕ is continuous
- $\phi(F) \notin \mathcal{I}$
- $\forall x \in F \ (\phi(x) \geq^* x)$

Since then $\phi(F)$ is closed subset of A, with $\phi(F) \notin \mathcal{I}$.

Coding trick (General version)

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Corollary

$\{F \subset \mathcal{F}(\mathbb{Z}^\omega) : F$ is closed and Haar null$\}$ is Δ^1_2, but not $\Sigma^1_1 \cup \Pi^1_1$.

Game quantifier

Assume that $A \subset X \times \omega^\omega$, where X is Polish. Let

$$\Game A = \{x : I \text{ has a winning strategy in the game } G(A_x)\}.$$

If Γ is a class of sets, $\Game\Gamma = \{\Game A : A \subset X \times \omega^\omega, A \in \Gamma\}$.

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Open questions

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2. Is the poset of Haar positive Borel sets ordered under inclusion proper?
3. Is there a Haar null F_σ set that is not contained in a Haar null G_δ set?

Thank you for your attention!