Realizations of countable Borel equivalence relations

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Abstract

We study topological realizations of countable Borel equivalence relations, including realizations by continuous actions of countable groups, with additional desirable properties. Some examples include minimal realizations on any perfect Polish space, realizations as K_σ relations, and realizations by continuous actions on the Baire space. We also consider questions related to realizations of specific important equivalence relations, like Turing and arithmetical equivalence. We focus in particular on the problem of realization by continuous actions on compact spaces and more specifically subshifts. This leads to the study of properties of subshifts, including universality of minimal subshifts, and a characterization of amenability of a countable group in terms of subshifts. Moreover we consider a natural universal space for actions and equivalence relations and study the descriptive and topological properties in this universal space of various properties, like, e.g., compressibility, amenability or hyperfiniteness.

Contents

1 Introduction

1.1 Topological and continuous action realizations

This paper is a contribution to the theory of countable Borel equivalence relations (CBER), a recent survey of which can be found in [Kec22]. One of our main concerns is the subject of well-behaved, in some sense, realizations of CBER. Given CBER E, F on standard Borel spaces X, Y, resp., a Borel isomorphism of E with F is a Borel bijection $f\colon X \to Y$ which takes E to F. If such f exists, we say that E, F are **Borel isomorphic**, in symbols $E \cong_B F$. Generally speaking a realization of a CBER E is a CBER $F \cong_B E$ with desirable properties.

To start with, a **topological realization** of E is an equivalence relation F on a *Polish space* Y such that $E \cong_B F$, in which case we say that F is a topological realization of E in the space Y. It is clear that every E admits a topological realization in some Polish space but we will look at topological realizations that have additional properties.

Recall here the Feldman-Moore Theorem that asserts that every CBER is induced by a Borel action of a countable group (see, e.g., [Kec22, 3.2]). By [Kec95, 13.11] there is a Polish topology with the same Borel structure in which this action is continuous. Thus every CBER admits a topological realization in some Polish space, which is induced by a continuous action of some countable (discrete) group. We will look again at such **continuous action realizations** for which the space and the action have additional properties.

To avoid uninteresting situations, unless it is otherwise explicitly stated or clear from the context, all the standard Borel or Polish spaces below will be uncountable and all CBER *will be* **aperiodic**, *i.e., have infinite classes. We will denote by* $\boldsymbol{\mathcal{AE}}$ *the class of all aperiodic* CBER *on uncountable standard Borel spaces.*

Concerning topological realizations, we first show the following (in Theorem 3.1.1):

Theorem 1.1.1. *For every equivalence relation $E \in \mathcal{AE}$ and every perfect Polish space Y, there is a topological realization of E in Y in which every equivalence class is dense.*

This has in particular as a consequence a stronger new version of a *marker lemma* (for the original form of the Marker Lemma see, e.g., [Kec22, Theorem 3.15]). Let E be a CBER on a standard Borel space X. A **Lusin marker scheme** for E is a family $\{A_s\}_{s\in\mathbb{N}^{<\mathbb{N}}}$ of Borel sets such that

(i) $A_\emptyset = X$;

(ii) $\{A_{sn}\}_n$ are pairwise disjoint and $\bigsqcup_n A_{sn} \subseteq A_s$;

(iii) Each A_s is a complete section for E (i.e., it meets every E-class).

We have two types of Lusin marker schemes:

(1) The Lusin marker scheme $\{A_s\}_{s\in\mathbb{N}^{<\mathbb{N}}}$ for E is of **type I** if in (ii) above we actually have that $\bigsqcup_n A_{sn} = A_s$ and moreover the following holds:

(iv) For each $x \in \mathbb{N}^\mathbb{N}$, $\bigcap_n A_{x|n}$ is a singleton.

(Then in this case, for each $x \in \mathbb{N}^\mathbb{N}$, $A^x_n = A_{x|n} \setminus \bigcap_n A_{x|n}$ is a vanishing sequence of markers (i.e., $\bigcap_n A^x_n = \emptyset$).)

(2) The Lusin marker scheme $\{A_s\}_{s\in\mathbb{N}^{<\mathbb{N}}}$ for E is of **type II** if it satisfies the following:

(v) If for each n, $B_n = \bigsqcup\{A_s \colon s \in \mathbb{N}^n\}$, then $\{B_n\}$ is a vanishing sequence of markers.

We now have (see Theorem 3.1.3):

Theorem 1.1.2. *Every $E \in \mathcal{AE}$ admits a Lusin marker scheme of type* I *and a Lusin marker scheme of type* II*.*

We next look at continuous action realizations. One such realization of $E \in \mathcal{AE}$ would be a realization F on a compact Polish space, where F is generated by a continuous action of a countable (discrete) group. We call these **compact action realizations**. Excluding the case of smooth relations (i.e., those that admit a Borel transversal), for which such a realization is impossible, we show the following (in Theorem 3.2.6). We use the following terminology: A CBER E on X is **compressible** if there is a Borel injection $f\colon X \to X$ with $f(C) \subsetneqq C$, for every E-class C. A CBER E is **hyperfinite** if $E = \bigcup_n E_n$, where each E_n is a **finite** CBER (i.e., all its classes are finite) and $E_n \subseteq E_{n+1}$. A **minimal, compact action realization** is a compact action realization in which the group acts minimally, i.e., all the orbits are dense. Finally, for each countable group Γ and topological space X consider the shift action of Γ on X^Γ. The restriction of this action to a nonempty invariant closed set is called a **subshift** of X^Γ. We often identify a subshift with the underlying closed set.

Theorem 1.1.3. *Every non-smooth hyperfinite equivalence relation in $\mathcal{AE}$ has a minimal, compact action realization. In fact this realization can be taken to be a subshift of $2^{\mathbb{F}_2}$ if the equivalence relation is compressible and a subshift of $2^\mathbb{Z}$ otherwise.*

We discuss other cases of CBER which admit such realizations in Section 3.3. For each infinite countable group Γ, let $E(\Gamma, 2^\mathbb{N})$ be the equivalence relation induced by the shift action of Γ on $(2^\mathbb{N})^\Gamma$. Let $Ap((2^\mathbb{N})^\Gamma)$ be the **aperiodic part** of $(2^\mathbb{N})^\Gamma$, i.e., the set of points x with infinite orbit, and let $E^{ap}(\Gamma, 2^{\mathbb{N}})$ be the restriction of $E(\Gamma, 2^{\mathbb{N}})$ to $Ap((2^{\mathbb{N}})^\Gamma)$. Let also $Fr((2^{\mathbb{N}})^\Gamma)$ be its **free part**, i.e., the set of points x such that $\gamma \cdot x \neq x, \ \forall \gamma \in \Gamma, \gamma \neq 1$. Denote by $F(\Gamma, 2^{\mathbb{N}})$ the restriction of $E(\Gamma, 2^{\mathbb{N}})$ to $Fr((2^{\mathbb{N}})^\Gamma)$. Every aperiodic (i.e., having infinite orbits) Borel action of Γ on a standard Borel space is Borel isomorphic to the restriction of the shift action to an invariant Borel subset of $Ap((2^{\mathbb{N}})^\Gamma)$ and similarly every free Borel action of Γ on a standard Borel space is Borel isomorphic to the restriction of the shift action to an invariant Borel subset of $Fr((2^{\mathbb{N}})^\Gamma)$

Also a CBER is **universal** if every CBER can be Borel reduced to it. As opposed to Theorem 1.1.3, the next results (see Theorem 3.3.1 and Corollary 3.6.6) show that some very complex CBER have compact action realizations.

Theorem 1.1.4. *(i) For every infinite countable group Γ, $F(\Gamma, 2^{\mathbb{N}})$ admits a compact action realization. If Γ is also finitely generated, then $E^{ap}(\Gamma, 2^{\mathbb{N}})$ admits a compact action realization. In fact in both cases such a realization can be taken to be a subshift of $(2^{\mathbb{N}})^\Gamma$.*

(ii) Every compressible, universal CBER *admits a compact action realization. In fact such a realization can be taken to be a subshift of $2^{\mathbb{F}_4}$.*

In particular, it follows that arithmetical equivalence $\equiv_A$ on $2^{\mathbb{N}}$ has a compact action realization but it is unknown if Turing equivalence $\equiv_T$ has such a realization. More generally, we do not know whether *every* non-smooth CBER has a compact action realization. We also do not know if *every* non-smooth CBER even admits some other kinds of realizations, for example transitive (i.e., having at least one dense orbit) continuous action realizations on arbitrary or special types of Polish spaces. These problems as well as the situation with smooth CBER in such realizations are discussed in Section 3.2.

Returning to Turing equivalence, in Section 3.4, we discuss topological realizations of Turing equivalence $\equiv_T$ and show that it admits a Baire class 2 isomorphism to an equivalence relation given by a continuous group action on the Baire space $\mathbb{N}^{\mathbb{N}}$. We do not know if this can be improved to Baire class 1 but we also show that no such isomorphism can be below the identity on a cone of Turing degrees.

In Section 3.5 we discuss some special properties of continuous actions of countable groups on compact Polish spaces, related to compressibility and paradoxical decompositions, that may be relevant to compact action realizations.

1.2 Subshifts

Related to Theorem 1.1.4, we call a countable group Γ **minimal subshift universal** if there is a minimal subshift of 2^Γ on which the restriction of the shift equivalence relation is universal. Then we have, see Corollary 3.6.5 and Corollary 3.6.6:

Theorem 1.2.1. *Let Γ and Λ be infinite groups, where Λ admits a Borel action on a standard Borel space whose induced equivalence relation is universal (e.g., any group containing $\mathbb{F}_2$). Then the wreath product $\Gamma \wr \Lambda$ is minimal subshift universal. In particular, $\mathbb{F}_3$ is minimal subshift universal.*

We do not know if $\mathbb{F}_2$ is minimal subshift universal.

It is well known that a countable group Γ is **amenable** iff every continuous action of Γ on a compact space admits an invariant Borel probability measure. Call a class $\mathcal{F}$ of such actions a **test for amenability** for Γ if Γ is amenable provided that every action in $\mathcal{F}$ admits an invariant Borel probability measure. In [GdlH97] it is shown that the class of actions on $2^{\mathbb{N}}$ is a test for amenability for any group. Equivalently this says that the class of all subshifts of $(2^{\mathbb{N}})^{\Gamma}$ is a test of amenability for Γ. It turns out that the strongest result along these lines is actually true, namely that the class of all subshifts of 2^{Γ} is a test of amenability for Γ, see Theorem 3.7.1. This gives another characterization of amenability.

Theorem 1.2.2. *Let Γ be a countable group. Then Γ is amenable iff every subshift of 2^{Γ} admits an invariant Borel probability measure.*

Our proof of Theorem 1.2.2 is based on an explicit construction of a subshift of 2^{Γ} with no invariant Borel probability measure for every non-amenable group Γ. Andy Zucker communicated subsequently to the authors a different proof based on ideas of abstract topological dynamics, especially the concept of strongly proximal flows.

We study in Section 3.8 a universal space for actions and equivalence relations and the descriptive or topological properties of various subclasses.

Fix a countable group Γ. For any Polish space X, define the standard Borel space of subshifts of X^{Γ} as follows:

$$\mathrm{Sh}(\Gamma, X) = \{F \in F(X^{\Gamma}) : F \text{ is } \Gamma\text{-invariant}\}.$$

If X is compact, we view this as a compact Polish space with the Vietoris topology.

Consider the Hilbert cube $\mathbb{I}^{\mathbb{N}}$. Every compact Polish space is (up to homeomorphism) a closed subspace of $\mathbb{I}^{\mathbb{N}}$, and thus every **Γ-flow** (i.e., a continuous action of Γ on a compact Polish space) is (topologically) isomorphic to a subshift of $(\mathbb{I}^{\mathbb{N}})^{\Gamma}$. We can thus consider the compact Polish space $\mathrm{Sh}(\Gamma, \mathbb{I}^{\mathbb{N}})$ as the universal space of Γ-flows.

Similarly consider the product space $\mathbb{R}^{\mathbb{N}}$. Every Polish space is (up to homeomorphism) a closed subspace of $\mathbb{R}^{\mathbb{N}}$, and thus every continuous Γ-action on a Polish space is (topologically) isomorphic to a subshift of $(\mathbb{R}^{\mathbb{N}})^{\Gamma}$. We can thus consider the standard Borel space $\mathrm{Sh}(\Gamma, \mathbb{R}^{\mathbb{N}})$ as the universal space of continuous Γ-actions.

In particular taking $\Gamma = \mathbb{F}_\infty$, the free group with a countably infinite set of generators, we see that every CBER is Borel isomorphic to the equivalence relation E_F induced on some subshift F of $(\mathbb{R}^{\mathbb{N}})^{\mathbb{F}_\infty}$ and so we can view $\mathrm{Sh}(\mathbb{F}_\infty, \mathbb{R}^{\mathbb{N}})$ also as the universal space of CBER and study the complexity of various classes of CBER (like, e.g., smooth, aperiodic, hyperfinite, etc.) as subsets of this universal space. Similarly we can view $\mathrm{Sh}(\mathbb{F}_\infty, \mathbb{I}^{\mathbb{N}})$ as the universal space of CBER that admit a compact action realization. In this case we can also consider complexity questions as well as generic questions of various classes.

Let Φ be a property of continuous Γ-actions on Polish spaces which is invariant under (topological) isomorphism. Let

$$\mathrm{Sh}_\Phi(\Gamma, X) = \{F \in \mathrm{Sh}(\Gamma, X) : F \models \Phi\},$$

where we write $F \models \Phi$ to mean that F has the property Φ.

We will consider below the following Φ, where for the definition of the concepts in 7)–10) below see Section 3.8, **(B)**.

1) fin: finite equivalence relation;

2) sm: smooth equivalence relation;

3) free: free action;

4) aper: aperiodic equivalence relation;

5) comp: compressible equivalence relation;

6) hyp: hyperfinite equivalence relation;

7) amen: amenable equivalence relation;

8) measHyp: measure-hyperfinite equivalence relation

9) freeMeasHyp: free action + measure-hyperfinite equivalence relation;

10) measAmen: measure-amenable action.

We summarize in the following table what we can prove concerning the descriptive or generic properties of the Φ above.

<table>
<tr><th>Φ</th><th>$\mathrm{Sh}_\Phi(\Gamma, \mathbb{I}^\mathbb{N})$</th><th colspan="2">$\mathrm{Sh}_\Phi(\Gamma, \mathbb{R}^\mathbb{N})$</th></tr>
<tr><td>fin</td><td rowspan="2">meager</td><td colspan="2" rowspan="2">$\mathbf{\Pi}^1_1$-complete</td></tr>
<tr><td>sm</td></tr>
<tr><td>free</td><td rowspan="3">comeager</td><td rowspan="2">G_δ</td><td rowspan="3">$\mathbf{\Pi}^1_1$-complete</td></tr>
<tr><td>aper</td></tr>
<tr><td>comp</td><td>open</td></tr>
<tr><td>hyp</td><td>?</td><td colspan="2" rowspan="2">$\mathbf{\Sigma}^1_2$, $\mathbf{\Pi}^1_1$-hard</td></tr>
<tr><td>amen</td><td rowspan="4">comeager</td></tr>
<tr><td>measHyp</td><td colspan="2">$\mathbf{\Pi}^1_1$-complete</td></tr>
<tr><td>freeMeasHyp</td><td rowspan="2">G_δ</td><td rowspan="2">$\mathbf{\Pi}^1_1$-complete</td></tr>
<tr><td>measAmen</td></tr>
</table>

In this table, Γ is an infinite group, Γ is residually finite in the "$\mathbf{\Pi}^1_1$-complete" entry of the first two rows, Γ is non-amenable in the "comeager" entry of the fifth row, Γ is non-amenable and residually finite in the "$\mathbf{\Pi}^1_1$-hard" and "$\mathbf{\Pi}^1_1$-complete" entries of the last five rows, and Γ is exact in the "comeager" entry of the last four rows (where a group is **exact** if it admits an amenable action on a compact Polish space; see [BO08, Chapter 5]). We show in Proposition 3.8.12 that hyperfiniteness is dense in $\mathrm{Sh}(\Gamma, \mathbb{I}^\mathbb{N})$ for any subgroup Γ of a hyperbolic group, but we do not know if hyperfiniteness is generic in $\mathrm{Sh}(\Gamma, \mathbb{I}^\mathbb{N})$ for every infinite Γ (or just $\mathbb{F}_2$), and we do not know the exact descriptive complexity of hyperfiniteness.

1.3 K_σ realizations

Clinton Conley also raised the question of whether every $E \in \mathcal{AE}$ admits a K_σ realization in a Polish space. We show in Theorem 3.10.1 that one can even obtain a transitive K_σ realization on $2^\mathbb{N}$, where an equivalence relation is **transitive** if it has at least one dense class. This raises the related question of whether every $E \in \mathcal{AE}$ admits a minimal K_σ (or even F_σ) realization in a Polish space, where an equivalence relation is called **minimal** if all its classes are dense. In view of Theorem 1.1.3, every non-smooth hyperfinite equivalence relation in $\mathcal{AE}$ has a minimal K_σ realization on a compact Polish space and Solecki in [Sol02] has shown that this fails for smooth relations, but this is basically the extent of our knowledge in this matter. Call a CBER on a compact Polish space X **compactly graphable** if there is a compact graphing of E, i.e., a compact graph (irreflexive, symmetric relation) $K \subseteq E$ so that the E-classes are the connected components of K. Clearly every such E is K_σ. We also show in Theorem 3.10.5 that every hyperfinite and every compressible CBER in $\mathcal{AE}$ has a **compactly graphable realization**, i.e., is Borel isomorphic to an equivalence relation on a compact Polish space that is compactly graphable. We do not know if this is true for every $E \in \mathcal{AE}$. Finally in Section 3.11 we study a σ-ideal associated with a K_σ CBER.

1.4 The Borel inclusion order

In connection with these realization problems, we were also led to consider the following quasi-order on CBER, which we call the **Borel inclusion order**. Given CBER E, F on standard Borel spaces, we put $E \subseteq_B F$ if there is $E' \cong_B E$ with $E' \subseteq F$.

Below, unless otherwise explicitly stated or understood from the context, by a **measure** *on a standard Borel space we will always mean a Borel probability measure.*

For each CBER E, we denote by EINV_E the set of ergodic, invariant measures for E and by $|\mathrm{EINV}_E| \in \{0, 1, 2, \dots \aleph_0, 2^{\aleph_0}\}$ its cardinality.

Recall here Nadkarni's Theorem (see, e.g., [Kec22, 5.3]) which asserts that for a CBER E the following are equivalent:

(i) E has no invariant measure;

(ii) $|\mathrm{EINV}_E| = 0$;

(iii) E is compressible.

We now have the following result (see Proposition 2.1.3, Theorem 2.2.3 and Corollary 2.2.6), where $\boldsymbol{\mathcal{AH}}$ is the class of hyperfinite relations in $\mathcal{AE}$.

Theorem 1.4.1. *(i) If $E \subseteq_B F$ are in $\mathcal{AE}$, then $|\mathrm{EINV_E}| \geq |\mathrm{EINV_F}|$ and if $E, F \in \mathcal{AH}$ are not smooth, then $E \subseteq_B F \iff |\mathrm{EINV_E}| \geq |\mathrm{EINV_F}|$.*

(ii) For any $E \in \mathcal{AE}$, there is $F \in \mathcal{AH}$ with $F \subseteq E$ such that moreover $\mathrm{EINV}_E = \mathrm{EINV}_F$.

Using this and the classification theorem for hyperfinite CBER from [DJK94, 9.1], one can then prove the next result (see Theorem 2.2.5 and Proposition 2.3.1), where we use the following terminology and notation:

For each CBER E and standard Borel space S, SE is the direct sum of "S" copies of E (see Section 2.1). We let E_0 be the equivalence relation on $2^{\mathbb{N}}$ given by $xE_0y \iff \exists m \forall n \geq m(x_n = y_n)$; E_t is the equivalence relation on $2^{\mathbb{N}}$ given by $xE_ty \iff \exists m \exists n \forall k(x_{m+k} = y_{n+k})$; $I_{\mathbb{N}} = \mathbb{N}^2$; E_∞ is a universal under Borel embeddability CBER; and $E \times F$ is the product of E and F. Finally $\subset_B$ is the strict part of $\subseteq_B$ and for any quasi-order $\preceq$ with strict part $\prec$ on a set Q and $q, r \in Q$, we say that r is a **successor** to q if $q \prec r$ and ($q \prec s \preceq r \implies s = r$). Finally, for each cardinal $\kappa \in \{0, 1, 2, 3, \dots, \aleph_0, 2^{\aleph_0}\}$, let $\mathcal{AE}_\kappa$ be the class of all $E \in \mathcal{AE}$ such that $|\mathrm{EINV}_E| = \kappa$. Thus by Nadkarni's Theorem, $\mathcal{AE}_0$ is the class of compressible relations. We also let for $\kappa > 0$, $\kappa E = SE$, where S is a standard Borel space of cardinality κ.

Theorem 1.4.2. *(i) $\mathbb{R}E_0 \subset_B \mathbb{N}E_0 \subset_B \cdots \subset_B 3E_0 \subset_B 2E_0 \subset_B E_0 \subset_B E_t$, each equivalence relation in this list is a successor in $\subseteq_B$ of the one preceding it and $\mathbb{N}E_0$ is the infimum in $\subseteq_B$ of the $nE_0, n \in \mathbb{N} \setminus \{0\}$.*

(ii) $\mathbb{R}I_{\mathbb{N}} \subset_B E_t$ and E_t is a successor of $\mathbb{R}I_{\mathbb{N}}$ in $\subseteq_B$.

(iii) $\mathbb{R}I_{\mathbb{N}}$ is $\subseteq_B$-minimum in $\mathcal{AE}_0$ and E_t is $\subseteq_B$-minimum among the non-smooth elements of $\mathcal{AE}_0$. (B. Miller) Also $E_\infty \times I_{\mathbb{N}}$ is $\subseteq_B$-maximum in $\mathcal{AE}_0$.

(iv) For each $\kappa > 0$, κE_0 is a $\subseteq_B$-minimum element of $\mathcal{AE}_\kappa$ but $\mathcal{AE}_\kappa$ has no $\subseteq_B$-maximum element.

(v) Let $\kappa \leq \lambda$. Then for every $E \in \mathcal{AE}_\lambda$, there is $F \in \mathcal{AE}_\kappa$ such that $E \subseteq_B F$.

In particular $\mathbb{R}E_0$ is $\subseteq_B$-minimum non-smooth in $\mathcal{AE}$ and $E_\infty \times I_{\mathbb{N}}$ is $\subseteq_B$-maximum in $\mathcal{AE}$. Thus one has the following version of the Glimm-Effros Dichotomy for $\subseteq_B$ (see Corollary 2.3.2):

Theorem 1.4.3. *Let $E \in \mathcal{AE}$. Then exactly one of the following holds:*

(i) E is smooth,

(ii) $\mathbb{R}E_0 \subseteq_B E$.

1.5 2-adequate groups

For each infinite countable group Γ and standard Borel space X consider the shift action of Γ on X^Γ and let $E(\Gamma, X)$ be the associated equivalence relation and $E^{ap}(\Gamma, X)$ be its aperiodic part, i.e., the restriction of $E(\Gamma, X)$ to the set of points with infinite orbits. Consider now a Borel action of Γ on an uncountable standard Borel space, which we can assume is equal to $\mathbb{R}$. Then the map $f\colon X \to \mathbb{R}^\Gamma$ given by $x \mapsto p_x$, where $p_x(\gamma) = \gamma^{-1} \cdot x$, is an equivariant Borel embedding of this action to the shift action on $\mathbb{R}^\Gamma$. Thus every aperiodic CBER E induced by a Borel action of Γ can be realized as (i.e., is Borel isomorphic to) the restriction of $E^{ap}(\Gamma, \mathbb{R})$ to an invariant Borel set. By a result in [JKL02, 5.5] we also have $E^{ap}(\Gamma, \mathbb{R}) \cong_B E^{ap}(\Gamma, \mathbb{N})$, so such realizations exist for $E^{ap}(\Gamma, \mathbb{N})$ as well. We consider here the question of whether these realizations can be achieved in the optimal form, i.e., replacing $E^{ap}(\Gamma, \mathbb{N})$ by $E^{ap}(\Gamma, 2)$. This is equivalent to the statement that $E^{ap}(\Gamma, \mathbb{R}) \cong_B E^{ap}(\Gamma, 2)$. If this happens then we call the group Γ **2-adequate**.

Using a recent result of Hochman-Seward, we show the following (see Theorem 4.0.4):

Theorem 1.5.1. *Every infinite countable amenable group is 2-adequate.*

This in particular answers in the negative a question of Thomas [Tho12, Page 391], who asked whether there are infinite countable amenable groups Γ for which $E(\Gamma, \mathbb{R})$ is not Borel reducible to $E(\Gamma, 2)$.

We also show the following (see Corollary 4.0.9 and Proposition 4.0.11):

Theorem 1.5.2. *(i) The free product of any countable group with a group that has an infinite amenable factor and thus, in particular, the free groups $\mathbb{F}_n, 1 \leq n \leq \infty$, are 2-adequate.*

(ii) Let Γ be n-generated, $1 \leq n \leq \infty$. Then $\Gamma \times \mathbb{F}_n$ is 2-adequate. In particular, all products $\mathbb{F}_m \times \mathbb{F}_n$, $1 \leq m, n \leq \infty$, are 2-adequate.

On the other hand there are groups which are not 2-adequate (see Theorem 4.0.12).

Theorem 1.5.3. *The group $\mathrm{SL}_3(\mathbb{Z})$ is not 2-adequate.*

We do not know if there is a characterization of 2-adequate groups.

1.6 Some other classes of groups

In the course of the previous investigations two other classes of groups have been considered. A countable group Γ is called **hyperfinite generating** if for every $E \in \mathcal{AH}$ there is a Borel action of Γ that generates E. We provide equivalent formulations of this property in Proposition 5.1.1 and show in Corollary 5.1.2 that all countable groups with an infinite amenable factor are hyperfinite generating, while no infinite countable group with property (T) has this property (see Proposition 5.1.3).

Finally we say that an infinite countable group Γ is **dynamically compressible** if every $E \in \mathcal{AE}$ generated by a Borel action of Γ can be Borel reduced to a compressible $F \in \mathcal{AE}$ induced by a Borel action of Γ. We show in Proposition 5.2.3 that every infinite countable amenable group is dynamically compressible and the same is true for any countable group that contains a non-abelian free group (see Proposition 5.2.4). However there are infinite countable groups that fail to satisfy these two conditions but they are still dynamically compressible (see Proposition 5.2.5). We do not know if *every* infinite countable group is dynamically compressible.

1.7 Organization

The paper is organized as follows. In Section 2, we study the structure of the Borel inclusion order on countable Borel equivalence relations. In Section 3, we consider topological realizations of countable Borel equivalence relations. In Section 4, we introduce and study the concept of 2-adequate groups, and in Section 5 we discuss results concerning the concepts of hyperfinite generating groups and dynamically compressible groups. In Section 6, we collect some of the main open problems discussed in this paper. Finally in Appendix A we discuss various notions of amenability for actions of countable groups that are relevant to the results in Section 3.8.

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2 The Borel inclusion order of countable Borel equivalence relations

2.1 General properties

Definition 2.1.1. Let E, F be CBER on standard Borel spaces X, Y, resp. We put $E \subseteq_B F$ if there is a Borel isomorphism $f\colon X \to Y$ with $f(E) \subseteq F$.

It is clear that $\subseteq_B$ is a quasi-order on CBER, which we call the **Borel inclusion order**. We also let $E \subset_B F \iff E \subseteq_B F \ \& \ F \nsubseteq_B E$ be the strict part of this order.

Recall that a **homomorphism** of an equivalence relation E on X to an equivalence relation F on Y is a map $f\colon X \to Y$ such that $xEy \implies f(x)Ff(y)$. Thus $E \subseteq_B F$ iff there is a bijective Borel homomorphism of E to F.

We will study in this section the structure of this inclusion order on aperiodic CBER in uncountable standard Borel spaces.

We first prove some basic facts concerning the Borel inclusion order that will be repeatedly used in the sequel. Recall that a CBER E on X is **smooth** if it admits a Borel selector and **compressible** if there is Borel injection $f\colon X \to X$ such that for each E-class C, $f(C) \subsetneqq C$. We also let $I_\mathbb{N}$ be the equivalence relation $\mathbb{N}^2$ on $\mathbb{N}$ and for each equivalence relation E on X and standard Borel space S, we let SE be the direct sum of "S" copies of E, i.e., the equivalence relation on $S \times X$ defined by $(s,x)SE(t,y) \iff s = t \ \& \ xEy$. It is clear that there is a unique up to Borel isomorphism (which we denote by $\cong_B$), smooth aperiodic CBER, namely $\mathbb{R}I_\mathbb{N}$.

Proposition 2.1.2. *(i) If $E \subseteq_B F$ and F is smooth, then E is smooth.*

(ii) E is compressible iff $\mathbb{R}I_\mathbb{N} \subseteq_B E$. Therefore if $E \subseteq_B F$ and E is compressible, then F is compressible.

Proof. (i) By the Feldman-Moore Theorem (see, e.g., [Kec22, 3.2]), there is a Borel action of a countable group $\Gamma = \{\gamma_n\}$ on X (the space of F) which induces F, i.e., $xFy \iff \exists\gamma \in \Gamma(\gamma \cdot x = y)$. Let f be a Borel selector for F and define for each $x \in X$, $n(x) =$ the least n with $\gamma_n \cdot f(x)Ex$. Then $g(x) = \gamma_{n(x)} \cdot f(x)$ is a Borel selector for E.

(ii) This follows from [Kec22, Theorem 3.23]. □

The number of ergodic, invariant probability Borel measures for a CBER E will play an important role in the sequel. We denote by $\mathrm{EINV_E}$ the set of ergodic, invariant probability Borel measures and by $|\mathrm{EINV_E}|$ its cardinality. Since $\mathrm{EINV_E}$ can be viewed in a canonical way as a standard Borel space (see, e.g., [Kec22, Theorem 5.11]) we have that $|\mathrm{EINV_E}| \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$. Moreover by Nadkarni's Theorem, see [Kec22, 5.3], we have that $|\mathrm{EINV_E}| = 0$ iff E is compressible.

We note here the following basic fact:

Proposition 2.1.3. *If $E \subseteq_B F$, then $|\mathrm{EINV_E}| \geq |\mathrm{EINV_F}|$.*

Proof. This is clear when $|\mathrm{EINV_F}| = 0$. Otherwise assume that $E \subseteq F$ live on a space X and F admits at least one invariant measure. Consider then the ergodic decomposition $\{X_e\}_{e\in \mathrm{EINV_F}}$ of F, see [Kec22, 5.4]. Then for each $e \in \mathrm{EINV_F}$, X_e is E-invariant and e is an invariant measure for $E|X_e$, thus X_e supports at least one ergodic, invariant measure for E, say e'. Since the map $e \mapsto e'$ is injective the proof is complete. □

We will next show that many subclasses of $\mathcal{AE}$, including $\mathcal{AE}$ itself, admit maximum under $\subseteq_B$ elements. This was proved for $\mathcal{AE}$ by Ben Miller, see [Kec22, 12.5], and the proof below is an adaptation of his argument to a more general context. Later we will show the existence of a minimum under $\subseteq_B$ non-smooth element of $\mathcal{AE}$ (see the paragraph following Corollary 2.3.2).

Below for equivalence relations E, F on spaces X, Y, resp., we let $E \sqsubseteq_B F$ iff there is a Borel injection $f \colon X \to Y$ such that $xEy \iff f(x)Ff(y)$. Again $\sqsubseteq_B$ is a quasi-order on CBER. Also we let $E \times F$ be the equivalence relation on $X \times Y$ given by $(x, y)E \times F(x', y') \iff (xEx' \;\&\; yFy')$. We now have:

Theorem 2.1.4. *Let $\mathcal{E} \subseteq \mathcal{AE}$ be a class of* CBER *such that $\mathcal{E}$ contains a maximum under $\sqsubseteq_B$ element E such that $E \times I_{\mathbb{N}} \in \mathcal{E}$. Then $E \times I_{\mathbb{N}} \in \mathcal{E}$ is $\subseteq_B$-maximum for $\mathcal{E}$.*

Proof. We start with the following fact, where for two equivalence relations F, G, $F \oplus G$ is their **direct sum**.

Lemma 2.1.5. *Let R be compressible. Then for any $S \in \mathcal{AE}$, $S \subseteq_B R \oplus S$.*

Proof. Suppose S lives on the space X. Then there is an S-invariant Borel set $X_0 \subseteq X$ such that $S|X_0 \cong_B \mathbb{R}I_{\mathbb{N}}$. Since $\mathbb{R}I_{\mathbb{N}} \oplus \mathbb{R}I_{\mathbb{N}} \cong_B \mathbb{R}I_{\mathbb{N}}$, we have, by Proposition 2.1.2, that $S \cong_B \mathbb{R}I_{\mathbb{N}} \oplus S \subseteq_B R \oplus S$. $\square$

Let now $F \in \mathcal{E}$ in order to show that $F \subseteq_B E \times I_{\mathbb{N}}$. Since $F \sqsubseteq_B E$, there is G such that $F \oplus G \subseteq_B E$. Recalling (see, e.g., [Kec22, Theorem 3.23]) that for any CBER R, $R \times I_{\mathbb{N}}$ is compressible, we have, by Lemma 2.1.5, that $F \subseteq_B F \oplus (F \times I_{\mathbb{N}}) \oplus (G \times I_{\mathbb{N}})$. Note now that $F \oplus (F \times I_{\mathbb{N}}) \subseteq_B F \times I_{\mathbb{N}}$, therefore $F \subseteq_B F \oplus (F \times I_{\mathbb{N}}) \oplus (G \times I_{\mathbb{N}}) \subseteq_B (F \times I_{\mathbb{N}}) \oplus (G \times I_{\mathbb{N}}) \cong_B (F \oplus G) \times I_{\mathbb{N}} \subseteq_B E \times I_{\mathbb{N}}$. $\square$

In particular this applies to the following classes $\mathcal{E}$: hyperfinite, α-amenable (see [Kec22, 9.2]), treeable, $\mathcal{AE}$.

2.2 Hyperfiniteness

We will discuss here the inclusion order on the hyperfinite equivalence relations. Recall first the following well-known fact (see, e.g., [Kec22, Theorem 8.23]):

Proposition 2.2.1. *If E is hyperfinite and $F \subseteq_B E$, then F is hyperfinite.*

Thus the class $\mathcal{AH}$ of hyperfinite aperiodic CBER forms an initial segment in $\subseteq_B$. It is also downwards cofinal in $\subseteq_B$ in view of the following standard result (see, e.g., [Kec22, Theorem 8.16]):

Theorem 2.2.2. *For any $E \in \mathcal{AE}$, there is $F \in \mathcal{AH}$ with $F \subseteq E$.*

We will actually need a more precise version of this result, see [Kec22, Theorem 8.16]. Since a proof of this result has not appeared in print before, we will include it below.

Theorem 2.2.3. *For any $E \in \mathcal{AE}$, there is $F \in \mathcal{AH}$ with $F \subseteq E$ such that moreover $\mathrm{EINV}_E = \mathrm{EINV}_F$.*

Proof. We will need the following lemma. Below E_0 is the equivalence relation on $2^{\mathbb{N}}$ defined by $xE_0y \iff \exists m \forall n \geq m(x_n = y_n)$ and μ_0 is the product measure on $2^{\mathbb{N}}$, where $2 = \{0,1\}$ is given the uniform $(\frac{1}{2}, \frac{1}{2})$ measure. Then μ_0 is the unique element of EINV_{E_0}.

Lemma 2.2.4. *Let E be a* CBER *on a standard Borel space X and let $\mu \in \mathrm{EINV}_E$. Then there is an E-invariant Borel set $X_0 \subseteq X$ with $\mu(X_0) = 1$, an E_0-invariant Borel set $C_0 \subseteq 2^{\mathbb{N}}$ with $\mu_0(C_0) = 1$ and a Borel isomorphism $f\colon C_0 \to X_0$ such that $f_*\mu_0 = \mu$ and $f(E_0|C_0) \subseteq E$.*

Proof. This follows from the proof of Dye's Theorem, see, e.g., [KM04, Section 7] and [Kec94, 5.26]. □

If E is compressible, then the result follows from Proposition 2.1.2, (ii). Otherwise by Nadkarni's Theorem (see, e.g., [Kec22, 5.3]) EINV_E is nonempty. Consider then the ergodic decomposition $\{X_e\}_{e\in \mathrm{EINV_E}}$ of E (see, e.g., [Kec22, 5.4]). For each $e \in \mathrm{EINV_E}$, by Lemma 2.2.4, there is an E-invariant Borel set $X_{0,e} \subseteq X_e$ with $e(X_{0,e}) = 1$, an E_0-invariant Borel set $C_{0,e} \subseteq 2^{\mathbb{N}}$ with $\mu_0(C_{0,e}) = 1$ and a Borel isomorphism $f_e \colon C_{0,e} \to X_{0,e}$ such that $(f_e)_*\mu_0 = e$ and $F_e = f_e(E_0|C_{0,e}) \subseteq E$. Note that F_e admits a unique ergodic, invariant measure, namely e.

The proof of Lemma 2.2.4 is effective enough (see, e.g., the proof of [DJK94, 9.6]), so that $X_0 = \bigcup_e X_{0,e}$ is Borel and $F_0 = \bigcup_e F_e$, which lives on X_0, is also Borel and hyperfinite. Let $X' = X \setminus X_0$. Then by the properties of the ergodic decomposition $F|X'$ is compressible, so by the compressible case above there is a hyperfinite compressible equivalence relation $F' \subseteq E|X'$. Finally put $F = F_0 \cup F'$. This clearly works. □

Recall that the classification theorem for hyperfinite CBER, see [DJK94, 9.1], shows that, up to Borel isomorphism, $\mathcal{AH}$ consists exactly of the following equivalence relations, where E_t is the equivalence relation on $2^{\mathbb{N}}$ given by $xE_ty \iff \exists m\exists n\forall k(x_{m+k} = y_{n+k})$:

$$\mathbb{R}I_{\mathbb{N}}, E_t, E_0, 2E_0, 3E_0, \ldots, \mathbb{N}E_0, \mathbb{R}E_0.$$

Moreover $|\mathrm{EINV}_E|$, for E in this list, is respectively $0, 0, 1, 2, 3, \ldots, \aleph_0, 2^{\aleph_0}$.

Below for a quasi-order $\preceq$ with strict part $\prec$ on a set Q and $q, r \in Q$, we say that r is a successor to q if $q \prec r$ and $(q \prec s \preceq r \implies s = r)$.

We now have:

Theorem 2.2.5. *(i) $\mathbb{R}E_0 \subset_B \mathbb{N}E_0 \subset_B \cdots \subset_B 3E_0 \subset_B 2E_0 \subset_B E_0 \subset_B E_t$, each equivalence relation in this list is a successor in $\subseteq_B$ of the one preceding it and $\mathbb{N}E_0$ is the infimum in $\subseteq_B$ of the $nE_0, n \in \mathbb{N} \setminus \{0\}$.*

(ii) $\mathbb{R}I_{\mathbb{N}} \subset_B E_t$ and E_t is a successor of $\mathbb{R}I_{\mathbb{N}}$ in $\subseteq_B$.

Proof. (i) Clearly $E_0 \subseteq E_t$ and thus $E_0 \subset_B E_t$ as E_0 is not compressible. To see that $2E_0 \subseteq_B E_0$, note that $2^{\mathbb{N}} = X_0 \sqcup X_1$, where $X_i = \{x \in 2^{\mathbb{N}} : x_0 = i\}$, and $E_0|X_i \cong_B E_0$. From this it follows immediately that $(n+1)E_0 \subseteq_B nE_0$, for each $n \in \mathbb{N}, n \geq 1$.

To show that $\mathbb{N}E_0 \subseteq_B nE_0$, for each $n \in \mathbb{N} \setminus \{0\}$, it is enough to show that $\mathbb{N}E_0 \subseteq_B E_0$. Let $s_n = 1^n0$ be the finite sequence staring with n 1's followed by one

0, for $n \in \mathbb{N}$. Let X_n be the subset of $2^{\mathbb{N}}$ consisting of all sequences starting with s_n, let $\bar{1}$ be the constant 1 sequence and put $X = 2^{\mathbb{N}} \setminus \{\bar{1}\}$. Then $X = \bigsqcup_n X_n$ and $E_0 \cong_B E|X \cong_B E|X_n$, for each $n \in \mathbb{N}$, which completes the proof that $\mathbb{N}E_0 \subseteq_B E_0$.

Finally to show that $\mathbb{R}E_0 \subseteq_B \mathbb{N}E_0$, it is enough to show that $\mathbb{R}E_0 \subseteq_B E_0$. To prove this, let for each $y \in 2^{\mathbb{N}}$, $X_y = \{x \in 2^{\mathbb{N}} : \forall n \in \mathbb{N}(x_{2n} = y_n)\}$. Then $2^{\mathbb{N}} = \bigsqcup_y X_y$ and $E_0|X_y \cong_B E_0, \forall y \in 2^{\mathbb{N}}$, which immediately implies that $\mathbb{R}E_0 \subseteq_B E_0$.

This establishes the non-strict orders in the list of (i). The strict orders and the last two statements of (i) now follow from Proposition 2.1.3.

(ii) Since E_t is compressible and not smooth, by Proposition 2.1.2, $\mathbb{R}I_{\mathbb{N}} \subset_B E_t$. It is also clear that E_t is a successor of $\mathbb{R}I_{\mathbb{N}}$. □

The following is an immediate corollary of Theorem 2.2.5:

Corollary 2.2.6. *Let $E, F \in \mathcal{AH}$ be non-smooth. Then*

$$E \subseteq_B F \iff |\mathrm{EINV}_E| \geq |\mathrm{EINV}_F|.$$

2.3 A global decomposition

For each cardinal $\kappa \in \{0, 1, 2, 3, \dots, \aleph_0, 2^{\aleph_0}\}$, let $\mathcal{AE}_\kappa$ be the class of all $E \in \mathcal{AE}$ such that $|\mathrm{EINV}_E| = \kappa$. Clearly $\mathcal{AE} = \bigsqcup_\kappa \mathcal{AE}_\kappa$ and each $\mathcal{AE}_\kappa$ is invariant under the equivalence relation associated with the quasi-order $\subseteq_B$, by Proposition 2.1.3. We also let for $\kappa > 0$, $\kappa E = SE$, where S is a standard Borel space of cardinality κ.

Proposition 2.3.1. *(i) $\mathbb{R}I_{\mathbb{N}}$ is $\subseteq_B$-minimum in $\mathcal{AE}_0$ and E_t is $\subseteq_B$-minimum among the non-smooth elements of $\mathcal{AE}_0$. (B. Miller) Also $E_\infty \times I_{\mathbb{N}}$ is $\subseteq_B$-maximum in $\mathcal{AE}_0$.*

(ii) For each $\kappa > 0$, κE_0 is a $\subseteq_B$-minimum element of $\mathcal{AE}_\kappa$ but $\mathcal{AE}_\kappa$ has no $\subseteq_B$-maximum element.

(iii) Let $\kappa \leq \lambda$. Then for every $E \in \mathcal{AE}_\lambda$, there is $F \in \mathcal{AE}_\kappa$ such that $E \subseteq_B F$.

(iv) (with R. Chen) The map $E \mapsto E \oplus E_0$ is an order embedding of the non-smooth elements of $\mathcal{AE}$ into $\mathcal{AE}$, i.e., for non-smooth $E, F \in \mathcal{AE}$, $E \subseteq_B F \iff E \oplus E_0 \subseteq_B F \oplus E_0$. It maps $\mathcal{AE}_\kappa$ into $\mathcal{AE}_{\kappa+1}$, if κ is finite,and $\mathcal{AE}_\kappa$ into itself, if κ is infinite.

Proof. (i) That $\mathbb{R}I_{\mathbb{N}}$ is $\subseteq_B$-minimum in $\mathcal{AE}_0$ follows from Proposition 2.1.2 and that $E_\infty \times I_{\mathbb{N}}$ is $\subseteq_B$-maximum in $\mathcal{AE}_0$ follows from Theorem 2.1.4. Finally we have to show that if $E \in \mathcal{AE}_0$ is not smooth, then $E_t \subseteq_B E$.

Since E is not smooth, we have that $E_t \sqsubseteq_B E$ (see [Kec22, 6.5 and 8.5]), so, as E_t is compressible, $E_t \sqsubseteq_B^i E$ (see [Kec22, Theorem 3.27]), i.e., E_t is Borel isomorphic to the restriction of E to an E-invariant Borel set. So if E lives on X, we have a Borel partition $X = Y \sqcup Z$ into E-invariant Borel sets such that $E|Y \cong_B E_t$. Since $E|Z$ is compressible, we see, using Lemma 2.1.5, that $E_t \subseteq_B E_t \oplus E|Z \cong_B E|Y \oplus E|Z \cong_B E$.

(ii) The fact that κE_0 is a $\subseteq_B$-minimum element of $\mathcal{AE}_\kappa$ is clear from Theorem 2.2.3. That $\mathcal{AE}_\kappa$ has no $\subseteq_B$-maximum element can be seen as follows.

Assume that E is such a $\subseteq_B$-maximum, towards a contradiction. Say E lives on the space X. Fix an invariant measure μ for E. We will show that every infinite countable group Γ embeds algebraically into $[E]$, the measure theoretic full group of E with respect to μ, contradicting a result of Ozawa, see [Kec10, page 29].

The group Γ admits a free Borel action on a standard Borel space Y, with associated equivalence relation G that has exactly κ ergodic, invariant measures. To see this, consider the free part of the shift action of Γ on 2^Γ, which has $2^{\aleph_0}$ ergodic components, and restrict the action to κ many ergodic components. Since E is $\subseteq_B$-maximum in $\mathcal{AE}_\kappa$, let $f \colon Y \to X$ be a Borel isomorphism such that $f(G) = F \subseteq E$. Then Γ acts freely in a Borel way on X inducing F, so that Γ can be algebraically embedded in $[F]$, the measure theoretic full group of F with respect to μ (which is clearly invariant for F). But $[F] \leq [E]$, so Γ embeds algebraically into $[E]$.

(iii) We can of course assume that $\kappa > 0$. Let $E \in \mathcal{AE}_\lambda$. Let $\{X_e\}_{e \in \mathrm{EINV}_E}$ be the ergodic decomposition of E, which has λ many components. If E lives on X, let Y be a Borel E-invariant subset of X consisting of exactly κ many ergodic components. Put $Z = X \setminus Y$. Then let $E' = E|Y$ and let G be a compressible equivalence relation on Z with $G \supseteq E|Z$. Let $F = E' \cup G$. Then $E \subseteq F$ and $F \in \mathcal{AE}_\kappa$.

(iv) We show that $E \mapsto E \oplus E_0$ is an order embedding on non-smooth aperiodic CBERs (on uncountable standard Borel spaces). (Note that the only failure is that $E_t \oplus E_0 \cong_B \mathbb{R}I_\mathbb{N} \oplus E_0$.)

Clearly, if $E \subseteq_B F$, then $E \oplus E_0 \subseteq_B F \oplus E_0$. Conversely, suppose that $E \oplus E_0 \subseteq_B F \oplus E_0$. We want to show that $E \subseteq_B F$.

We can write $E \cong_B R \oplus R'$ and $E_0 \cong_B S \oplus S'$ with $R \oplus S \subseteq_B F$ and $R' \oplus S' \subseteq_B E_0$. Note that R', S, S' are all aperiodic hyperfinite (maybe on a countable space), and since $E_0 \cong_B S \oplus S'$, exactly one of S or S' must be E_0, and the other is compressible hyperfinite. Also since E is non-smooth, we have $E \cong_B E \oplus E_t$, and similarly for F.

We have two cases:

1. If $S = E_0$ and S' is compressible, then since $R' \oplus S' \subseteq_B E_0$, we must have $R' \subseteq_B E_0 = S$.

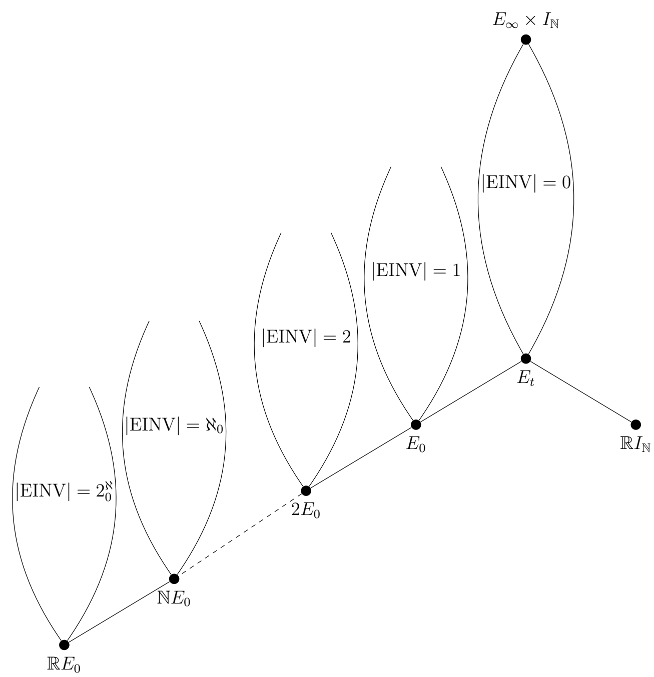
2. If S is compressible, then we have $R' \oplus E_t \subseteq_B S \oplus E_t$, since $R' \oplus E_t$ is hyperfinite and $S \oplus E_t \cong_B E_t$.

In both cases, we get:

$$E \cong_B E \oplus E_t \cong_B R \oplus R' \oplus E_t \subseteq_B R \oplus S \oplus E_t \subseteq_B F \oplus E_t \cong_B F$$

□

The following picture illustrates parts (i) and (ii) of Proposition 2.3.1.



The next result, which is an immediate corollary of the above, is a version of the Glimm-Effros Dichotomy, see [Kec22, Theorem 6.5], for the inclusion order $\subseteq_B$ instead of $\sqsubseteq_B$.

Corollary 2.3.2. *Let $E \in \mathcal{AE}$. Then exactly one of the following holds:*
(i) E is smooth,
(ii) $\mathbb{R}E_0 \subseteq_B E$.

Denote by E_∞ a universal CBER, in the sense that every CBER F satisfies $F \sqsubseteq_B E_\infty$, see, e.g., [Kec22, 6.3]. Then, by Corollary 2.3.2, $\mathbb{R}E_0$ is a $\subseteq_B$-minimum among all the non-smooth relations in $\mathcal{AE}$ and, by Theorem 2.1.4, $E_\infty \times I_{\mathbb{N}}$ is a $\subseteq_B$-maximum relation in $\mathcal{AE}$.

It is interesting to consider the problem of existence of $\subseteq_B$-maximum elements in $\mathcal{E}_\kappa = \mathcal{AE}_\kappa \cap \mathcal{E}$ for other classes $\mathcal{E} \subseteq \mathcal{AE}$. This is clearly the case if $\kappa = 0$ and $\mathcal{E}$ satisfies the conditions of Theorem 2.1.4, so we will consider $\kappa \geq 1$.

Clearly κE_0 is $\subseteq_B$-maximum in $\mathcal{AH}_\kappa$. Denote by $\boldsymbol{\mathcal{AT}}$ the subclass of $\mathcal{AE}$ consisting of the treeable equivalence relations.

Problem 2.3.3. *Let $\kappa \geq 1$. Does $\mathcal{AT}_\kappa$ have a $\subseteq_B$-maximum element?*

If E is $\subseteq_B$-maximum in $\mathcal{AT}_1$, then κE is $\subseteq_B$-maximum in $\mathcal{AT}_\kappa$, for every $1 \leq \kappa \leq \aleph_0$, so we will concentrate in the case $\kappa = 1$, i.e., the class of **uniquely ergodic** elements of $\mathcal{AT}$. We do not know the answer to this problem but we would like to point out that a positive answer has an implication in the context of the theory of measure preserving CBER, see [Kec21].

Fix a standard Borel space X and a measure μ on X. We will consider as in [Kec21] **pmp** CBER on X, i.e., μ-measure preserving CBER on X, where we identify two such relations if they agree μ-a.e. Inclusion of pmp relations is also understood in the μ-a.e. sense. Such a relation is treeable if it has this property μ-a.e. We also denote by $\mathrm{Aut}(X, \mu)$ the group of measure preserving automorphisms of (X, μ).

Proposition 2.3.4. *If E on a standard Borel space X is a $\subseteq_B$-maximum uniquely ergodic, equivalence relation in $\mathcal{AT}$, with (unique) invariant measure μ, then for every treeable pmp relation F on (X, μ), there is an automorphism $T \in \mathrm{Aut}(X, \mu)$ such that $T(F) \subseteq E$.*

Proof. We will use the following lemma.

Lemma 2.3.5. *Let G be a treeable pmp* CBER *on (X, μ). Then there is an ergodic, treeable pmp* CBER *H on (X, μ) with $G \subseteq H$.*

Proof. For each $T \in \mathrm{Aut}(X, \mu)$ denote by E_T the equivalence relation induced by T. By [CM14, Theorem 8] the set of $T \in \mathrm{Aut}(X, \mu)$ such that E_T is independent of G (see [KM04, Section 27] for the notion of independence) is comeager in $\mathrm{Aut}(X, \mu)$, equipped with the usual weak topology. So is the set of all ergodic $T \in \mathrm{Aut}(X, \mu)$, see [Kec10, Theorem 2.6]. Thus there is an ergodic $T \in \mathrm{Aut}(X, \mu)$ such that E_T is independent of G. Then put $H = E_T \vee G$, the smallest equivalence relation containing E_T and G. □

By Lemma 2.3.5, we can assume that F is ergodic. We can also assume that there is $F' \in \mathcal{AT}$ which agrees with F μ-a.e. By considering the ergodic decomposition of F', we can also assume that μ is the unique invariant measure for F'. Fix then a Borel automorphism $T \colon X \to X$ such that $T(F') \subseteq E$. Then both $T_*\mu$ and μ are $T(F')$-invariant. Since $T(F')$ is uniquely ergodic, it follows that $T_*\mu = \mu$, i.e., $T \in \mathrm{Aut}(X, \mu)$ and the proof is complete. □

Remark 2.3.6. We note here that an analog of the conclusion of Proposition 2.3.4 is valid for the class $\mathcal{AH}$. More precisely, let $X = 2^{\mathbb{N}}$ and let μ be the usual product measure on X. Then for every hyperfinite pmp relation F on (X, μ), there is an automorphism $T \in \mathrm{Aut}(X, \mu)$ such that $T(F) \subseteq E_0$. This can be seen as follows: By [Kec10, 5.4] (in which the aperiodicity of E is not needed), we can find an ergodic hyperfinite pmp relation F' such that $F \subseteq F'$. By Dye's Theorem (see, e.g., [Kec10, 3.13]) there is an automorphism $T \in \mathrm{Aut}(X, \mu)$ such that $T(F') = E_0$ and thus $T(F) \subseteq E_0$.

3 Topological realizations

3.1 Dense realizations and Lusin marker schemes

We will first use the results in Section 2 to prove the following:

Theorem 3.1.1. *For every equivalence relation $E \in \mathcal{AE}$ and every perfect Polish space Y, there is a minimal topological realization of E in Y.*

Proof. First, since for every perfect Polish space Y there is a continuous bijection from the Baire space $\mathbb{N}^{\mathbb{N}}$ onto Y (see [Kec95, 7.15]), we can assume that $Y = \mathbb{N}^{\mathbb{N}}$. Moreover by Corollary 2.3.2, it is enough to prove this result for $E = \mathbb{R}E_0$ and $E = \mathbb{R}I_{\mathbb{N}}$.

Case 1: $\mathbb{R}E_0$.

Consider the shift map of $\mathbb{Z}$ on $2^{\mathbb{Z}}$ with associated equivalence relation F'. Let $Y = \{x \in 2^{\mathbb{Z}} \colon [x]_{F'} \text{ is dense in } 2^{\mathbb{Z}}\}$, Clearly Y is a dense, co-dense G_δ set in $2^{\mathbb{Z}}$, so, in particular, it is a zero-dimensional Polish space (with the relative topology from $2^{\mathbb{Z}}$). We next check that every compact set in Y has empty interior. Indeed let $K \subseteq Y$ be compact in Y. Then K is compact in $2^{\mathbb{Z}}$. If now V is open in $2^{\mathbb{Z}}$ and $\emptyset \neq V \cap Y \subseteq K$, then since Y is dense in $2^{\mathbb{Z}}$, by looking at $V \setminus K$ we see that $V \subseteq K$, contradicting that Y is also co-dense in $2^{\mathbb{Z}}$.

By [Kec95, 7.7] Y is homeomorphic to $\mathbb{N}^{\mathbb{N}}$. Moreover if $F = F'|Y$, F has dense classes and $|\mathrm{EINV}_F| = 2^{\aleph_0}$, so $F \cong_B \mathbb{R}E_0$.

Case 2: $\mathbb{R}I_{\mathbb{N}}$.

Consider the equivalence relation R on $\mathbb{N}^{\mathbb{N}}$ given by

$$xRy \iff \exists m \forall n \geq m (x_n = y_n).$$

Let $A \subseteq \mathbb{N}^{\mathbb{N}}$ be an uncountable Borel partial transversal for R (i.e., no two distinct elements of A are in R). Then, as R is not smooth, denoting by $B = [A]_R$ the

R-saturation of A, we also have that $Y = \mathbb{N}^{\mathbb{N}} \setminus B$ is uncountable. Fix then a Borel bijection $f\colon A \to Y$ and let F be the equivalence relation obtained by adding to each $[a]_R$, $a \in A$, the point $f(a)$. Then F is a smooth CBER, so $F \cong_B \mathbb{R}I_{\mathbb{N}}$, and every F-class is dense in $\mathbb{N}^{\mathbb{N}}$. □

A **complete section** of an equivalence relation E on X is a subset $Y \subseteq X$ which meets every E-class. Recall that a **vanishing sequence of markers** for a CBER E is a decreasing sequence of complete Borel sections $\{A_n\}$ for E such that $\bigcap_n A_n = \emptyset$. A very useful result in the theory of CBER is the Marker Lemma, which asserts that every $E \in \mathcal{AE}$ admits a vanishing sequence of markers, see, e.g., [Kec22, Theorem 3.15]. We will see next that Theorem 3.1.1 implies a strong new version of a marker lemma.

Definition 3.1.2. Let E be a CBER on a standard Borel space X. A **Lusin marker scheme** for E is a family $\{A_s\}_{s\in\mathbb{N}^{<\mathbb{N}}}$ of Borel sets such that

(i) $A_\emptyset = X$;

(ii) $\{A_{sn}\}_n$ are pairwise disjoint and $\bigsqcup_n A_{sn} \subseteq A_s$;

(iii) Each A_s is a complete section for E.

We have two types of Lusin marker schemes:

1) The Lusin marker scheme $\{A_s\}_{s\in\mathbb{N}^{<\mathbb{N}}}$ for E is of **type I** if in (ii) above we actually have that $\bigsqcup_n A_{sn} = A_s$ and moreover the following holds:

   (iv) For each $x \in \mathbb{N}^{\mathbb{N}}$, $\bigcap_n A_{x|n}$ is a singleton.

   Then in this case, for each $x \in \mathbb{N}^{\mathbb{N}}$, $A^x_n = A_{x|n} \setminus \bigcap_n A_{x|n}$ is a vanishing sequence of markers.

2) The Lusin marker scheme $\{A_s\}_{s\in\mathbb{N}^{<\mathbb{N}}}$ for E is of **type II** if it satisfies the following:

   (v) If for each n, $B_n = \bigsqcup\{A_s\colon s \in \mathbb{N}^n\}$, then $\{B_n\}$ is a vanishing sequence of markers.

Theorem 3.1.3. *Every $E \in \mathcal{AE}$ admits a Lusin marker scheme of type* I *and a Lusin marker scheme of type* II.

Proof. Type I: By Theorem 3.1.1, we can assume that E lives on $\mathbb{N}^\mathbb{N}$ and that every equivalence class is dense. Let then for each $s \in \mathbb{N}^n$, $A_s = \{x : x|n = s\}$.

Type II: By Theorem 3.1.1, we can assume that E lives on $\mathbb{R}$ and that every equivalence class is dense. By induction on n, we can easily construct open sets $A_s, s \in \mathbb{N}^n$, such that $\{A_s\}_{s\in\mathbb{N}^{<\mathbb{N}}}$ is a Lusin marker scheme for E and moreover it has the following properties:

(a) Each $A_s, s \in \mathbb{N}^n, n \geq 1$, is contained in (n, ∞);

(b) Each $A_s, s \in \mathbb{N}^n, n \geq 1$, has non-empty intersection with the interval $(k, k+1)$ for every $k \geq n$.

Then clearly $\{A_s\}_{s\in\mathbb{N}^{<\mathbb{N}}}$ is of type II. □

Remark 3.1.4. (a) We can also easily see that every $E \in \mathcal{AE}$ admits a **Cantor marker scheme** $\{A_s\}_{s\in 2^{<\mathbb{N}}}$ of each type, which is defined in an analogous way.

(b) By applying Theorem 3.1.3 to $\mathbb{R}E$, and using the ccc property for category, we can see that every $E \in \mathcal{AE}$ admits a variant of a Lusin marker scheme of type I, where condition (iv) in Definition 3.1.2 is replaced by the following condition:

(iv)' For each $x \in \mathbb{N}^\mathbb{N}$, $\bigcap_n A_{x|n}$ has at most one element and for a comeager set of x it is empty.

3.2 Continuous action realizations

Any CBER has a continuous action realization, i.e., a topological realization induced by a continuous action of a countable group on a Polish space. We will consider what additional properties of the action and the Polish space of the realization are possible. For example, we have the following:

Proposition 3.2.1. *Every $E \in \mathcal{AE}$ has a continuous action realization in the Baire space $\mathbb{N}^\mathbb{N}$.*

Proof. By the usual change of topology arguments, we can assume that E is induced by a continuous action of a countable group on a 0-dimensional space X. Let $P \subseteq X$ be the perfect kernel of X, which is clearly invariant under the action. Since $X \setminus P$ is countable, it is easy to see that $E|P \cong_B E$, so we can assume that X is perfect. Let then D be a countable dense subset of X which is also invariant under the action and put $Y = X \setminus D$. Then again $E \cong_B E|Y$. The space Y is a nonempty, 0-dimensional Polish space in which every compact set has empty interior and thus is homeomorphic to the Baire space (see [Kec95, Theorem 7.7]). □

Definition 3.2.2. (i) A **transitive action realization**, resp., **minimal action realization** of a CBER is a topological realization induced by a continuous, topologically transitive action of a countable group (i.e., one which has a dense orbit), resp., induced by a continuous, topologically minimal action of a countable group (i.e., one for which all orbits are dense).

(ii) A **σ-compact action realization**, resp., **locally compact action realization**, resp., **compact action realization** of a CBER is a topological realization induced by a continuous action of a countable group on a σ-compact, resp., locally compact, resp., compact Polish space.

(iii) A **transitive, σ-compact action realization** is a topological realization induced by a continuous, topologically transitive action of a countable group on a σ-compact Polish space. Similarly we define the concepts of

transitive, locally compact action realization,
transitive, compact action realization,
minimal, σ-compact action realization,
minimal, locally compact action realization,
minimal, compact action realization.

We first note the following fact:

Proposition 3.2.3. *If $E \in \mathcal{AE}$ has a compact action realization or a transitive action realization on a perfect Polish space or a minimal action realization, then E is not smooth.*

Proof. Suppose a smooth E has a compact action realization F, towards a contradiction. Then there is a compact invariant subset K in which the action is minimal. Since $F|K$ is also smooth, by [Kec95, 8.46] some orbit in K is non-meager in K, thus consists of isolated points in K. Minimality then implies that K consists of a single infinite orbit, contradicting compactness.

The proof of the case of a transitive action realization on a perfect Polish space or a minimal action realization follows also from [Kec95, 8.46]. □

We first note here that the hypothesis of perfectness in Proposition 3.2.3 is necessary.

Proposition 3.2.4. *Every smooth equivalence relation in $\mathcal{AE}$ has a transitive locally compact action realization (in some non-perfect space).*

Proof. Let $\mathbb{N} = \bigsqcup_{q\in\mathbb{Q}} N_q$ be a decomposition of $\mathbb{N}$ into infinite sets indexed by the rationals. Define then recursively $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$, with $\mathrm{Im}\ z_n > 0, \mathrm{Im}\ z_{n+1} < \mathrm{Im}\ z_n$, $\mathrm{Im}\ z_n \to 0$, and pairwise disjoint closed squares S_n with center z_n with $\mathrm{Im}\ S_n > 0$ as follows:

If $0 \in N_q$, choose $z_0 \in \{q\} \times \mathbb{R}$ and let S_0 be a very small square around z_0. At stage $n+1$, if $n+1 \in N_q$, choose $z_{n+1} \in \{q\}\times\mathbb{R}$ so that $0 < \mathrm{Im}\ z_{n+1} < \frac{1}{n+1}, \mathrm{Im}\ z_{n+1} <$

Im z_n, $z_{n+1} \notin \bigcup_{m \leq n} S_m$, and then choose S_{n+1} to be a small square around z_{n+1} so that it has empty intersection with all $S_m, m \leq n$.

Put $X = \mathbb{R} \cup \{z_n\}_{n \in \mathbb{N}}$. Then X is closed in $\mathbb{C}$, so it is locally compact. Next define $T \colon X \to X$ as follows:

If $x \in \mathbb{R}$, then $T(x) = x + 1$. If $x = z_n$ with $n \in N_q$, so that $x \in \{q\} \times \mathbb{R}$, and if in the increasing enumeration of N_q, n is the ith element, then put $T(x) = z_m$, where m is the ith element in the increasing enumeration of N_{q+1}. It is not hard to check that T is a homeomorphism of X. For example, to check that T is continuous (a similar argument works for T^{-1}) let $w_n, w \in X$, with $w_n \to w$, in order to show that $T(w_n) \to T(w)$. We can assume of course that $w_n \notin \mathbb{R}, w \in \mathbb{R}, \text{Im } w_n \to 0$. Now $\text{Re } T(w_n) = \text{Re } w_n + 1$ and $\text{Im } T(w_n) \to 0$, thus $T(w_n) = \text{Re } w_n + 1 + i \text{ Im } T(w_n) \to w + 1 = T(w)$.

Next for each pair $(m, n) \in \mathbb{N}^2$, let $T_{m,n}$ be the homeomorphism of X that switches z_m with z_n and keeps every other point of X fixed. Then the group generated by all $T_{m,n}$ and T acts continuously on X. One of its orbits is $\{z_n\}$ which is dense in X, thus the action is topologically transitive. The equivalence relation F it generates has as classes the set $\{z_n\}$ and the sets of the form $x + \mathbb{Z}$, for $x \in \mathbb{R}$, so it is aperiodic and smooth, with transversal $\{z_0\} \cup [0, 1)$. □

Also the hypothesis of compactness in Proposition 3.2.3 is necessary.

Proposition 3.2.5. *Every smooth equivalence relation in $\mathcal{AE}$ has a locally compact action realization on a perfect space, in fact one in the space $2^{\mathbb{N}} \setminus \{\bar{1}\}$, where $\bar{1}$ is the constant 1 sequence.*

Proof. We use an example in [DJK94, page 200, (b)]. Consider the space $X = 2^{\mathbb{N}} \setminus \{\bar{1}\}$. For each $m \neq n$, let $h_{m,n}$ be the homeomorphism of X defined by: $h_{m,n}(1^m 0 \hat{\ } y) = 1^n 0 \hat{\ } y, h_{m,n}(1^n 0 \hat{\ } y) = 1^m 0 \hat{\ } y, h_{m,n}(x) = x$, otherwise. Then the group generated by these homeomorphisms acts continuously on X and generates the equivalence relation F given by: $xFy \iff \exists z(x = 1^m 0 \hat{\ } z \ \& \ y = 1^n 0 \hat{\ } z)$, which is smooth aperiodic. □

We next show that non-smooth hyperfinite equivalence relations in $\mathcal{AE}$ have the strongest kind of topological realization. For a countable group Γ, recall that a subshift of 2^Γ is the restriction of the shift action of Γ to a nonempty closed invariant subset.

Theorem 3.2.6. *Every non-smooth hyperfinite equivalence relation in $\mathcal{AE}$ has a minimal, compact action realization on the Cantor space $2^{\mathbb{N}}$. In fact, we have the following:*

(i) If it is compressible, then it can be realized by a minimal subshift of $2^{\mathbb{F}_2}$.
(ii) If it is not compressible, then it can be realized by a minimal subshift of $2^{\mathbb{Z}}$.

Proof. (i) Consider E_t. Then E_t is generated by a continuous action of $\mathbb{F}_2$, see [Kec22, 3.2], defined as follows: The first generator acts via $i\hat{\ }x \mapsto (1-i)\hat{\ }x$, and the second generator acts via

$$\begin{aligned} 0\hat{\ }x &\mapsto 00\hat{\ }x \\ 11\hat{\ }x &\mapsto 1\hat{\ }x \\ 10\hat{\ }x &\mapsto 01\hat{\ }x \end{aligned}$$

This action has a clopen 2-generator, namely the partition given by $\{X_0 = 0\hat{\ }2^{\mathbb{N}}, X_1 = 1\hat{\ }2^{\mathbb{N}}\}$. This means that the sets $\gamma \cdot X_i, \gamma \in \mathbb{F}_2, i \leq 1$, separate points. This implies that this action is (topologically) isomorphic to a subshift of $2^{\mathbb{F}_2}$.

(ii) Assume that $E \in \mathcal{AH}$ is non-compressible and let $\kappa = |\mathrm{EINV}_E| > 0$. By a theorem of Downarowicz [Dow91, Theorem 5], for every metrizable Choquet simplex K there is a minimal subshift of $2^{\mathbb{Z}}$ such that K is affinely homeomorphic to the simplex of invariant measures for this subshift. In particular the cardinality of the set of ergodic, invariant measures for this subshift is the same as the cardinality of the set of extreme points of K. Fix now a compact Polish space X of cardinality κ and let K be the Choquet simplex of measures on X. The extreme points are the Dirac measures, so there are exactly κ many of them. Thus we can find a minimal subshift of $2^{\mathbb{Z}}$ with exactly κ many ergodic, invariant measures and therefore if F is the equivalence relation induced by this subshift, we have that $E \cong_B F$. □

Although E_t does not have a minimal, compact action realization where the acting group is amenable (otherwise it would have an invariant measure), we have the following:

Proposition 3.2.7. *A compressible, non-smooth, hyperfinite* CBER *has a minimal, locally compact action realization where the acting group is $\mathbb{Z}$.*

Proof. It is known that there are minimal homeomorphisms on uncountable locally compact spaces with no invariant measure, which thus generate a compressible non-smooth hyperfinite CBER; see, e.g., [Dan01, Section 2]. Below we give a simple example:

Let $A = \mathbb{Z}/4\mathbb{Z}$ as an abelian group, and let $X \subseteq A^{\mathbb{N}}$ be the set of sequences which eventually lie in $\{1, 2\}$. Let $X_n = A^n \times \{1, 2\}^{\mathbb{N}}$, so that $X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$ and $X = \bigcup_n X_n$. We give X_n the usual product topology, so that X_n is clopen in X_{n+1}, and X the inductive limit topology, so that $U \subseteq X$ is open iff $\forall n(U \cap X_n$ is open in X_n). This is Hausdorff, locally compact and second countable, with basis $\bigcup_n \mathcal{B}_n$, where $\mathcal{B}_n$ is a countable basis for X_n. Thus X is a locally compact Polish space, see, e.g., [Kec95, 5.3].

Let now $\varphi : A^{\mathbb{N}} \to A^{\mathbb{N}}$ be the odometer map, i.e., addition by 1 with carry, which is a homeomorphism of $A^{\mathbb{N}}$. Note that $\varphi(X) \subseteq X$ and $\varphi^{-1}(X) \subseteq X$. We next check that $\varphi \restriction X$ is a homeomorphism of X. It enough to check that $\varphi \restriction X_n \colon X_n \to X$ and $\varphi^{-1} \restriction X_n \colon X_n \to X$ are continuous. This follows from noticing that $\varphi(X_n) \subseteq X_{n+1}$ and $\varphi^{-1}(X_n) \subseteq X_{n+1}$.

Let E be the equivalence relation on X induced by $\varphi \restriction X$. Denote by E_0' the equivalence relation on $A^{\mathbb{N}}$ defined by $x E_0' y \iff \exists m \forall n \geq m (x_n = y_n)$. Then $E = E_0' \restriction X$ and $E \restriction X_n = E_0' \restriction X_n$, so $\varphi \restriction X$ is minimal.

Finally, we show that E is compressible. For every $x \in X$, let n_x be least such that $x \in X_{n_x}$, and define the Borel map $f \colon X \to X$ as follows:

$$f(x)_n = \begin{cases} x_n + 2 & n = n_x \\ x_n & n \neq n_x \end{cases}$$

Then f is a compression of E. □

Remark 3.2.8. Here are also some other minimal, locally compact action realizations of a compressible, non-smooth, hyperfinite CBER (but where the acting group is not $\mathbb{Z}$).

(i) Let X be the locally compact space constructed in the proof of Proposition 3.2.4, whose notation we use below. For each $q \in \mathbb{Q}$, let $T_q \colon X \to X$ be the homeomorphism which is translation by q on $\mathbb{R}$ and defined on $\{z_n\}$ in a way similar to translation by 1 in the proof of Proposition 3.2.4. Also define a homeomorphism $T \colon X \to X$ as follows: T is the identity on $\mathbb{R}$. Next let for each $q \in \mathbb{Q}$, $N_q = \{n_0^q < n_1^q < n_2^q < \dots\}$ be the increasing enumeration of N_q and define $T(z_{n^q_{2n+3}}) = z_{n^q_{2n+1}}, T(z_{n^q_1}) = z_{n^q_0}, T(z_{n^q_{2n}}) = z_{n^q_{2n+2}}, n \in \mathbb{N}$.

The group generated by $T, T_q, q \in \mathbb{Q}$ is abelian and acts continuously on X. The orbits consist of $\{z_n\}$ and the sets of the form $x + \mathbb{Q}$ for $x \in \mathbb{R}$, so the action is minimal. Finally there is clearly no invariant measure for this action.

(ii) Another construction, where the acting group is actually $\mathbb{Z}^2$ is the following: Let S be a minimal homeomorphism on an uncountable compact metric space K, inducing the equivalence relation F, and let $X = K \times \mathbb{Z}$. Then let $\mathbb{Z}^2$ act by homeomorphisms on X, where one of the generators acts like S on K and the other as translation by 1 on $\mathbb{Z}$. The associated equivalence relation of this action is Borel isomorphic to $F \times I_N$ so it is compressible, non-smooth and hyperfinite by [Kec22, Theorem 8.33].

Below for a Borel action of a countable group Γ on a standard Borel space X and a probability measure ζ on Γ, whose support generates Γ as a semigroup, we say that a measure μ on X is **ζ-stationary** if $\mu = \int \gamma_* \mu \, \mathrm{d}\zeta(\gamma)$.

It is easy to see that μ is quasi-invariant under the action, i.e., the action sends μ-null sets to μ-null sets. Next we check that if the action has infinite orbits, then μ is non-atomic. Let $x \in X$ be such that $\mu(\{x\}) > 0$, towards a contradiction. Since $\mu(\{x\}) = \int \mu(\gamma^{-1} \cdot \{x\}) \, \mathrm{d}\zeta(\gamma)$, if $\mu(\gamma^{-1} \cdot \{x\}) \leq \mu(\{x\}), \forall \gamma$, then as $\mu(\gamma^{-1} \cdot \{x\}) > 0$, we must have that $\mu(\gamma^{-1} \cdot \{x\}) = \mu(\{x\}), \forall \gamma$, a contradiction. Thus we see that for every $x \in X$ with $\mu(\{x\}) > 0$, there is $x' \in \Gamma \cdot x$, with $\mu(\{x'\}) > \mu(\{x\})$. So we can find $x_0, x_1, x_2, \dots$ with $\mu(\{x_0\}) < \mu(\{x_1\}) < \mu(\{x_2\}) < \dots$, a contradiction.

We use these facts and Theorem 3.2.6 to prove the following:

Proposition 3.2.9. *Let $E \in \mathcal{AE}$ be an equivalence relation on a standard Borel space X. Then the following are equivalent:*

(i) E is not smooth;

(ii) There is a Borel action of a countable group Γ on X generating E, such that for every *measure ζ on Γ, whose support generates Γ as a semigroup, there is a ζ-stationary, ergodic for this action measure on X.*

(iii) There is a Borel action of a countable group Γ on X generating E, such that for some *measure ζ on Γ, whose support generates Γ as a semigroup, there is a ζ-stationary, ergodic for this action measure on X.*

Proof. If (iii) holds, then E admits a non-atomic, ergodic, quasi-invariant measure, so it is not smooth. We next prove that (i) implies (ii).

Since E is not smooth, by the Glimm-Effros dichotomy, there is an E-invariant Borel set $Y \subseteq X$ such that $E|Y$ is non-smooth, hyperfinite. Then, by Theorem 3.2.6, there is a continuous action of $\Gamma = \mathbb{F}_\infty$ on a compact space Z inducing an equivalence relation $F \cong_B E|Y$. Let ζ by any measure on Γ, whose support generates Γ as a semigroup. Then there is a ζ-stationary for this action measure on Z, see, e.g., [CKM13, Corollary 9]. The set of ζ-stationary for this action measures is thus a non-empty compact, convex set of measures, so it has an extreme point which is therefore ergodic. Transferring this back to Y and extending the Γ action to X so that it generates $E|(X \setminus Y)$ on $X \setminus Y$, we see that (ii) holds. □

The following question is open:

Problem 3.2.10. *Does* every *non-smooth $E \in \mathcal{AE}$ have any of the topological realizations stated in Definition 3.2.2? In particular, does* every *non-smooth $E \in \mathcal{AE}$ admit a compact action realization?*

We will consider the case of compact action realizations in the next two sections. The answer to the following is also unknown:

Problem 3.2.11. *If a* CBER *admits a compact action realization, does it admit one in which the underlying space is* $2^{\mathbb{N}}$*?*

Recall that a **reduction** of an equivalence relation E on X to an equivalence relation F on Y is a map $f\colon X \to Y$ such that $xEy \iff f(x)Ff(y)$. If such a Borel reduction exists, we say that E is **Borel reducible** to F and write $E \leq_B F$. If $E \leq_B F$ and $F \leq_B E$, then E, F are **Borel bireducible**, in symbols $E \sim_B F$. We note here that the following weaker version of Problem 3.2.10 is also open:

Problem 3.2.12. *Is* every *non-smooth* $E \in \mathcal{AE}$ *Borel bireducible to some* $F \in \mathcal{AE}$ *which has any of the topological realizations stated in Definition 3.2.2? In particular, can one find such an* F *that admits a compact action realization?*

3.3 Compact action realizations

(A) We have seen in Theorem 3.2.6 that the answer to Problem 3.2.10 is affirmative in the strongest sense for hyperfinite E but the situation for general E is unclear. The following results provide some cases of non-hyperfinite equivalence relations that admit compact action realizations.

Theorem 3.3.1. *For every infinite countable group* Γ*,* $F(\Gamma, 2^{\mathbb{N}})$ *admits a compact action realization. If* Γ *is also finitely generated, then* $E^{ap}(\Gamma, 2^{\mathbb{N}})$ *admits a compact action realization. In fact in both cases such a realization can be taken to be a subshift of* $(2^{\mathbb{N}})^{\Gamma}$*.*

Proof. The result will follow easily from the following lemma, which is an extended version of the result in [Ele18], who dealt with the case of free actions. Denote below by $\mathbf{s} = \mathbf{s}_{\Gamma, 3^{\mathbb{N}}}$ the shift action of Γ on $(3^{\mathbb{N}})^{\Gamma}$ and for each $y \in (3^{\mathbb{N}})^{\Gamma}$, let $\mathrm{stab}(y) \leq \Gamma$ be the stabilizer of y in this action.

Lemma 3.3.2. *Let* $\mathbf{a}$ *be a Borel action of an infinite countable group* Γ *on a standard Borel space* X*. Then there is an equivariant Borel embedding* $p\colon X \to (3^{\mathbb{N}})^{\Gamma}$ *of the action* $\mathbf{a}$ *to the shift action* $\mathbf{s}$ *of* Γ *on* $(3^{\mathbb{N}})^{\Gamma}$ *such that if* $y \in \overline{p(X)}$*, then there are* $y_0, y_1, \dots \in p(X)$ *with*

$$stab(y) \subseteq \bigcup_m \bigcap_{n \geq m} stab(y_n).$$

Proof. Let $\Gamma = \{\gamma_n\}$ and let $\varphi_n \colon X \to \{0,1,2\}$ be a Borel coloring of the graph of $\gamma_n^{\mathbf{a}}$, where $\gamma_n^{\mathbf{a}}(x) = \mathbf{a}(\gamma_n, x)$ (see [KST99, 4.6]). By changing the topology, we can assume that X is 0-dimensional, so a G_δ subset of $3^{\mathbb{N}}$, and $\mathbf{a}, \varphi_n$ are continuous.

Let $\theta \colon X \to 3^{\mathbb{N}}$ be defined by

$$\theta(x) = (x(0), \varphi_0(x), x(1), \varphi_1(x), \dots).$$

Then θ is a homeomorphism of X with a G_δ subset of $3^{\mathbb{N}}$, and we have that if $\gamma_m^{\mathbf{a}} \cdot x = y \neq x$, then $\theta(y)(2m+1) \neq \theta(x)(2m+1)$, where we write $\gamma_m^{\mathbf{a}} \cdot x$ for $\gamma_m^{\mathbf{a}}(x)$.

Thus identifying x with $\theta(x)$, we can thus assume that

(i) X is a G_δ subset of $3^{\mathbb{N}}$,
(ii) The action $\mathbf{a}$ is continuous,
(iii) $\gamma_m^{\mathbf{a}} \cdot x = y \neq x \implies y(2m+1) \neq x(2m+1)$.

Consider now the standard equivariant Borel embedding of the action $\mathbf{a}$ into the shift action on $(3^{\mathbb{N}})^\Gamma$ given by

$$p \colon X \to (3^{\mathbb{N}})^\Gamma$$
$$x \mapsto (\gamma \mapsto (\gamma^{-1})^{\mathbf{a}} \cdot x).$$

Let now $y \in \overline{p(X)}$ and take $x_n \in X$ with $p(x_n) \to y$.

Let $\gamma = \gamma_m \in \mathrm{stab}(y)$. Then, as $p(x_n) \to y, p(x_n)(1) \to y(1)$, i.e., $x_n \to y(1)$. Also $\gamma^{\mathbf{s}} \cdot p(x_n) \to \gamma^{\mathbf{s}} \cdot y$, so $(\gamma^{\mathbf{s}} \cdot p(x_n))(1) \to (\gamma^{\mathbf{s}} \cdot y)(1)$, i.e., $p(x_n)(\gamma^{-1}) \to y(\gamma^{-1})$ or $\gamma^{\mathbf{a}} \cdot x_n \to (\gamma^{\mathbf{s}} \cdot y)(1) = y(1)$, so both $x_n(2m+1)$ and $(\gamma^{\mathbf{a}} \cdot x_n)(2m+1)$ converge to $y(1)$, in the discrete space $\{0,1,2\}$, thus they are eventually equal. Then by (iii) above $x_n, \gamma^{\mathbf{a}} \cdot x_n$ are eventually equal, i.e., $\gamma \in \mathrm{stab}(x_n) = \mathrm{stab}(p(x_n))$, for all large enough n. □

From Lemma 3.3.2 it is clear that if $\mathbf{a}$ is a free action, then $\overline{p(X)}$ is contained in $Fr((3^{\mathbb{N}})^\Gamma)$ and thus in particular by taking $\mathbf{a}$ to be the restriction of the shift action to $Fr((3^{\mathbb{N}})^\Gamma)$, we see that this action is Borel isomorphic to a subshift contained in $Fr((3^{\mathbb{N}})^\Gamma)$. Of course we can trivially replace $3^{\mathbb{N}}$ by $2^{\mathbb{N}}$ here, so this proves the first statement of the theorem.

Assume now that Γ is finitely generated. In a similar way, to prove the second statement of the theorem, it is enough to show, in the notation of Lemma 3.3.2, that if the action $\mathbf{a}$ is aperiodic, so that the stabilizers of the points y_n have infinite index, then the stabilizer of the point y also has infinite index. This follows from the conclusion of Lemma 3.3.2 and the fact that a finite index subgroup of a finitely generated group is also finitely generated. □

Problem 3.3.3. *Is it true that for an arbitrary infinite countable group Γ, $E^{ap}(\Gamma, 2^{\mathbb{N}})$ admits a compact action realization?*

We next note the following fact, which can be used to provide more examples of CBER that admit compact action realizations.

Proposition 3.3.4. *Let F be an aperiodic* CBER *on a standard Borel space X. Let $Z \subseteq X$ be a Borel invariant set and put $Y = X \setminus Z$ and $E = F|Y$. If E is not smooth and $F|Z$ is hyperfinite, compressible, then $E \cong_B F$. So if F has a compact action realization, so does E.*

Proof. If $F|Z$ is smooth, then $F|Z \cong_B \mathbb{R}I_{\mathbb{N}}$ is Borel isomorphic to a direct sum of copies of $I_{\mathbb{N}}$, while if it is not smooth $F|Z \cong_B E_t$. Thus, by the Glimm-Effros Dichotomy and [Kec22, Theorem 8.3 and Proposition 3.27] in the second case, we can find a decomposition $Y = Y_0 \sqcup Y_1 \sqcup Y_2 \sqcup \cdots \sqcup Y_\infty$ into invariant Borel sets such that $F|Z \cong_B F|Y_n, \forall n \in \mathbb{N}$. Let π_0 be a Borel isomorphism of $F|Z$ with $F|Y_0$ and for $n > 0$, let π_n be a Borel isomorphism of $F|Y_{n-1}$ with $F|Y_n$. Finally let π_∞ be the identity on Y_∞. Then $\bigcup_{n\in\mathbb{N}} \pi_n \cup \pi_\infty$ is a Borel isomorphism of F and E. □

Corollary 3.3.5. *Let F be an aperiodic* CBER *on a Polish space X. Then there is meager, invariant Borel set $M \subseteq X$ such that for any invariant Borel set $Y \supseteq M$, if $E = F|Y$ is not smooth, then $E \cong_B F$.*

Proof. By [KM04, 12.1 and 13,3], there is an invariant, comeager Borel set $C \subseteq X$ such that $F|C$ is compressible, hyperfinite. Put $M = X \setminus C$. If $Y \supseteq M$ is invariant Borel such that $E = F|Y$ is not smooth and $Z = X \setminus Y$, then we can apply Proposition 3.3.4. □

For example, let Γ be a countable group and consider a continuous, topologically transitive action of Γ on a compact Polish space X with infinite orbits. Then there is an invariant dense G_δ set $C \subseteq X$ consisting of points with dense orbits in X and such that if F is the equivalence relation induced by the action, then $F|C$ is compressible, hyperfinite and non-smooth (as the action of Γ on C is topologically transitive). So $F|C \cong_B E_t \cong_B \mathbb{R}E_t$. Then by the countable chain condition for category, some copy of E_t in $E|C$ is meager, so can subtract it from C and assume that if $M = X \setminus C$, then $F|M$ is not smooth. It follows that for any invariant Borel set $Y \supseteq M$, if $E = F|Y$, then $E \cong_B F$, so that E has a compact action realization

Since for every $E \in \mathcal{AE}$ on a Polish space X there is an invariant comeager Borel set $Y \subseteq X$ such that $E|Y$ is hyperfinite, it follows that if $E \in \mathcal{AE}$ is not smooth when restricted to any invariant comeager Borel set, then there is an invariant comeager Borel set $Y \subseteq X$ such that $E|Y$ admits a minimal, compact action realization. Whether this holds for measure instead of category is an open problem.

Problem 3.3.6. *Let $E \in \mathcal{AE}$ be on a standard Borel space X and let μ be a measure on X such that the restriction of E to any invariant Borel set of measure 1 is not smooth. Is there is an invariant Borel set $Y \subseteq X$ with $\mu(Y) = 1$ such that $E|Y$ admits a compact action realization?*

(B) We next describe a "gluing" construction of two continuous actions of groups on compact Polish spaces at an orbit of one of the actions. We thank Aristotelis Panagiotopoulos for a useful discussion on this construction.

Let the countable group Γ act continuously on the compact Polish space X and let $X_0 \subseteq X$ be an infinite orbit of this action. Let also the countable group Δ act continuously on the compact Polish space Y with a fixed point $y_0 \in Y$. Fix compatible metrics $d_X \leq 1$ and $d_Y \leq 1$ for X and Y, respectively. Fix also a map $x \mapsto |x|$ from X_0 to $\mathbb{R}^+$ such that $\lim_{x\to\infty} |x| = +\infty$, i.e., for every $M \in \mathbb{R}^+$, there is finite $F \subseteq X_0$ such that $x \notin F \implies |x| > M$. For each $x \in X_0$, let Y_x be a set and let π_x be a bijection $\pi_x \colon Y \to Y_x$ such that $\pi_x(y_0) = x$ and $x_1 \neq x_2 \implies Y_{x_1} \cap Y_{x_2} = \emptyset$. Put $Y'_x = Y_x \setminus \{x\}$ and let $Z = X \sqcup \bigsqcup_{x\in X_0} Y'_x$. Define a metric d_x on Y_x as follows:

$$d_x(y_1, y_2) = \frac{d_Y(\pi_x^{-1}(y_1), \pi_x^{-1}(y_2))}{|x|}.$$

Then define a metric d_Z on Z as follows:

$$d_Z(x_1, x_2) = d_X(x_1, x_2), \text{if } x_1, x_2 \in X,$$

$$d_Z(y_1, y_2) = d_x(y_1, y_2), \text{if } y_1, y_2 \in Y_x, x \in X_0,$$

$$d_Z(y, x') = d_x(y, x) + d_X(x, x'), \text{if } y \in Y_x, x \in X_0, x' \in X,$$

$$d_Z(y_1, y_2) = d_{x_1}(y_1, x_1) + d_X(x_1, x_2) + d_{x_2}(x_2, y_2), \text{if } y_1 \in Y_{x_1}, y_2 \in Y_{x_2}, x_1 \neq x_2.$$

Remark 3.3.7. We note here that in the preceding "gluing" construction, if the spaces X, Y are 0-dimensional, so is the space Z. To see this we start with metrics d_X, d_Y as above which are actually ultrametrics (these exist since X, Y are 0-dimensional). Then it is enough to show that for every $z \in Z$, there is an $\varepsilon_z > 0$ such that every open ball (in the metric d_Z) $B_z(\varepsilon)$, for $\varepsilon < \varepsilon_z$, is closed. Below recall that open balls in ultrametrics are closed.

Consider first the case where $z \in X$ and fix $z_1, z_2, \dots \in B_z(\varepsilon)$ with $z_n \to z_\infty$. If infinitely many z_n are in X, then clearly $z_\infty \in B_z(\varepsilon)$ as d_X is an ultrametric. Otherwise, we can assume that all z_n are in $Z \setminus X$. If now there is some $x \in X_0$ such that infinitely many $z_n \in Y'_x$, so that $z_\infty \in Y_x$, we have $d_Z(z_n, z) = d_x(z_n, x) + d_X(x, z)$, so $d_x(z_n, x) < \varepsilon - d_X(x, z)$, thus, since d_x is an ultrametric, $d_x(z_\infty, x) <$

$\varepsilon - d_X(x,z)$ and thus $d_Z(z_\infty, z) < \varepsilon$. Otherwise there is a subsequence (z_{n_i}) and $x_i \in X_0$ with $z_{n_i} \in Y'_{x_i}$ and x_i converges to $x \in X$ and thus $z_{n_i} \to z_\infty = x$ (since $d_Z(z_{n_i}, x_i) < \frac{1}{|x_i|}$). Now $d_Z(z, z_{n_i}) = d_X(z, x_i) + d_{x_i}(x_i, z_{n_i}) < \varepsilon$, so $d_X(z, x_i) < \varepsilon$ and, since d_X is an ultrametric, $d_Z(z, z_\infty) = d_X(z, z_\infty) < \varepsilon$.

The other case is when $z \in Y'_x$, for some $x \in X_0$. Take $\varepsilon_z = d_x(z, x)$. Then for $\varepsilon < \varepsilon_z$ the open ball $B_z(\varepsilon)$ is the same as the open ball of radius ε in the metric d_x, so the proof is complete.

Proposition 3.3.8. *(Z, d_Z) is a compact metric space.*

Proof. It is routine to check that d_Z is a metric on Z. We next verify compactness. Let (z_n) be a sequence in Z in order to find a converging subsequence. The other cases being obvious, we can assume that $z_n \in Y_{x_n}$ with $x_n \in X_0$ distinct and therefore $|x_n| \to \infty$, in which case, by going to a subsequence, we can also assume that $x_n \to x \in X$. Since $d_Z(z_n, x_n) \le \frac{1}{|x_n|}$, it follows that $z_n \to x$. □

We next define an action of Δ on Z. Given $\delta \in \Delta$ and $z \in Z$ we define $\delta \cdot z$ as follows:

$$\delta \cdot z = \pi_x(\delta \cdot \pi_x^{-1}(z)), \text{if } z \in Y_x, x \in X_0,$$
$$\delta \cdot z = z, \text{if } z \in X.$$

If we identify each Y_x with Y, then this action "extends" the action of Δ on Y.

We finally extend the action of Γ from X to all of Z. Given $\gamma \in \Gamma$ and $z \in Y_x, x \in X_0$, define $\gamma \cdot z$ as follows:

$$\gamma \cdot z = \pi_{\gamma \cdot x}(\pi_x^{-1}(z)), \text{if } z \in Y_x, x \in X_0.$$

It is easy to see that these two actions commute, so they give an action of $\Gamma \times \Delta$ on Z.

Proposition 3.3.9. *The action of $\Gamma \times \Delta$ on Z is continuous.*

Proof. It is enough to check that the action of Γ on Z is continuous and so is the action of Δ.

Let first $\gamma \in \Gamma$ and $z_n \in Z$ be such that $z_n \to z$, in order to show that $\gamma \cdot z_n \to \gamma \cdot z$. It is enough to find a subsequence (n_i) such that $\gamma \cdot z_{n_i} \to \gamma \cdot z$. Again, the other cases being trivial, we can assume that $z_n \in Y_{x_n}$ with $x_n \in X_0$ distinct, so that also $|x_n| \to \infty$, in which case, by going to a subsequence, we can also assume that $x_n \to x \in X$. Then $\gamma \cdot x_n \to \gamma \cdot x$ and $d_Z(\gamma \cdot z_n, \gamma \cdot x_n) \le \frac{1}{|\gamma \cdot x_n|} \to 0$, as the $\gamma \cdot x_n$ are also distinct and thus $|\gamma \cdot x_n| \to \infty$. Since $d_Z(z_n, x_n) \leq \frac{1}{|x_n|}$, clearly $x = z$, and thus $\gamma \cdot z_n \to \gamma \cdot z$.

Let now $\delta \in \Delta$ and $z_n \in Z$ be such that $z_n \to z$, in order to show that $\delta \cdot z_n \to \delta \cdot z$. It is enough again to find a subsequence (n_i) such that $\delta \cdot z_{n_i} \to \delta \cdot z$ and as before we can assume that $z_n \in Y_{x_n}$ with $x_n \in X_0$ distinct, so that also $|x_n| \to \infty$, in which case, by going to a subsequence, we can also assume that $x_n \to x \in X$. Then $\delta \cdot x_n = x_n \to \delta \cdot x = x$. Now $\delta \cdot z_n \in Y_{x_n}$, so that $\delta_Z(\delta \cdot z_n, x_n) \to 0$ and $d_Z(z_n, x_n) \to 0$. Thus $z = x$ and $\delta \cdot z_n \to \delta \cdot z = z$. □

Let now E be the equivalence relation induced by the action of Γ on X, let F be the equivalence relation induced by the action of Δ on $Y \setminus \{y_0\}$ and finally let G be the equivalence relation induced by the action of $\Gamma \times \Delta$ on Z. Then it is easy to check the following;

Proposition 3.3.10. $G \cong_B E \oplus (F \times I_{\mathbb{N}})$

We present now an application of this construction to the problem of compact action realizations.

Theorem 3.3.11. *Let the non-smooth* CBER F *be induced by a continuous action of a countable group on a locally compact Polish space. Then $F \times I_{\mathbb{N}}$ admits a compact action realization. In particular, if F is compressible, F admits a compact action realization.*

Moreover, if the locally compact space is 0-dimensional, $F \times I_{\mathbb{N}}$ admits a compact action realization on the Cantor space $2^{\mathbb{N}}$.

Proof. In the preceding "gluing" construction, take $X = 2^{\mathbb{N}}$ and a continuous action of $\Gamma = \mathbb{F}_2$ such that $E = E_t$. Fix also a countable group Δ and a continuous action of Δ on a locally compact space Y' which induces F. Let $Y = Y' \sqcup \{y_0\}$ be the one-point compactification of Y' (if Y' is already compact, we obtain Y by adding an isolated point to Y'). Then the action of Δ can be continuously extended to Y by fixing y_0. Thus we have by Proposition 3.3.10 that $E \oplus (F \times I_{\mathbb{N}})$ admits a compact action realization. Since F is not smooth, we have, as in the proof of Proposition 3.3.4, that $E \oplus (F \times I_{\mathbb{N}}) \cong_B (F \times I_{\mathbb{N}})$ and the proof is complete.

In the case that Y' is 0-dimensional, by Remark 3.3.7 $F \times I_{\mathbb{N}}$ admits a compact action realization on a 0-dimensional space Z. By going to the perfect kernel of Z, we can assume that Z is perfect (see the proof of Proposition 3.2.1), thus homeomorphic to the Cantor space. □

For a sequence (E_n) of CBER, we let $\bigoplus_n E_n$ be the direct sum of this sequence. If E_n is on the space X_n, then $E = \bigoplus_n E_n$ is the equivalence relation on the space

$\bigsqcup_n X_n$, where $xEy \iff \exists n(x, y \in X_n \text{ and } xE_ny)$. The following is an immediate consequence of Theorem 3.3.11.

Corollary 3.3.12. *Let each $E_n \in \mathcal{AE}$ admit a compact action realization. Then $\bigoplus_n E_n \times I_{\mathbb{N}}$ also admits a compact action realization. In particular, if also every E_n is compressible, $\bigoplus_n E_n$ admits a compact action realization.*

Recall that a CBER E is **universal** if for every CBER F we have $F \leq_B E$. Note that by [CK18, Proposition 5.23 (d)], there is a unique, up to Borel isomorphism, compressible, universal CBER. The following are immediate consequences of Theorem 3.3.11.

Corollary 3.3.13. *Let E be a compressible, universal* CBER. *Then E admits a transitive, compact action realization on the Cantor space $2^{\mathbb{N}}$.*

Proof. Let us first note that there exists a compressible, universal CBER F that is generated by a continuous action of a countable group on $2^{\mathbb{N}}$. Indeed, let $E(\mathbb{F}_2, 2)$ be the equivalence relation generated by the canonical action of $\mathbb{F}_2$ on $2^{\mathbb{F}_2}$. Consider the equivalence relation $F = E(\mathbb{F}_2, 2) \times I_{\mathbb{N}}$. This equivalence relation is compressible, universal. By Theorem 3.3.11, F has a continuous action realization on $2^{\mathbb{N}}$. An inspection of the "gluing" construction involved in the proof of Theorem 3.3.11 shows that this action is topologically transitive. □

Corollary 3.3.14. *Let E be a compressible, universal* CBER. *Then E admits a minimal action realization on the Baire space $\mathbb{N}^{\mathbb{N}}$.*

Proof. By Corollary 3.3.13 consider a continuous action a of a countable group G on $2^{\mathbb{N}}$, which induces an equivalence relation F which is Borel isomorphic to E. Then consider the Borel map f that sends $x \in 2^{\mathbb{N}}$ to the closure of its orbit (which is a member of the space of all compact subsets of $2^{\mathbb{N}}$), By [MSS16, Theorem 3.1], there is some K such that $F|f^{-1}(K)$ is universal. But clearly $Z = f^{-1}(K)$ is a G_δ set, so a Polish, 0-dimensional space, invariant under the action a. Moreover this action restricted to Z is minimal. As in the proof of Proposition 3.2.1, we can find a subspace Y of Z homeomorphic to $\mathbb{N}^{\mathbb{N}}$ invariant under the action, such that $F|Z \cong_B F|Y$. Thus $F|Z$ is induced by a minimal action on the Baire space and is compressible, universal. As in the proof of Corollary 3.3.13, this shows that every compressible, universal CBER admits a minimal action realization on the Baire space. □

The following is an open problem:

Problem 3.3.15. *Does an arbitrary (not necessarily compressible) aperiodic, universal* CBER *admit a compact action realization?*

In the next Section 3.6 we will consider realizations of equivalence relations by subshifts and in particular prove a considerable strengthening of Corollary 3.3.13.

3.4 Turing and arithmetical equivalence

Below let $\equiv_T$ denote **Turing equivalence** and $\equiv_A$ **arithmetical equivalence**.

The following is an immediate consequence of Corollary 3.3.13, since $\equiv_A$ is compressible and universal by [MSS16]:

Corollary 3.4.1. *Arithmetical equivalence $\equiv_A$ on $2^{\mathbb{N}}$ admits a compact action realization on $2^{\mathbb{N}}$, which is in fact a topologically transitive subshift of $2^{\mathbb{F}_4}$.*

On the other hand the following is open:

Problem 3.4.2. *Does Turing equivalence $\equiv_T$ on $2^{\mathbb{N}}$ admit a compact action realization?*

A negative answer to this question will on the one hand provide a new proof of the non-hyperfiniteness of $\equiv_T$ but, more importantly, give a negative answer to the long-standing problem of the universality of $\equiv_T$, see [DK00].

Concerning Turing equivalence, we know from Proposition 3.2.1 that it admits a continuous action realization on the Baire space $\mathbb{N}^{\mathbb{N}}$, i.e., that there is a Borel isomorphism of $2^{\mathbb{N}}$ with $\mathbb{N}^{\mathbb{N}}$ which sends $\equiv_T$ to an equivalence relation induced by a continuous action of a countable group on $\mathbb{N}^{\mathbb{N}}$. We calculate below an upper bound for the Baire class of such a Borel isomorphism. A version of the next theorem was first proved by Andrew Marks, in response to an inquiry of the authors, with "Baire class 3" instead of "Baire class 2". The proof of Theorem 3.4.3 below uses some of his ideas along with other additional arguments.

Theorem 3.4.3. *There exists a Baire class 2 map $\Phi\colon 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ that is an isomorphism between $\equiv_T$ and an equivalence relation given by a continuous group action on $\mathbb{N}^{\mathbb{N}}$.*

The most natural construction of the isomorphism will yield Proposition 3.4.4 below. We will show later that it in fact implies Theorem 3.4.3.

Proposition 3.4.4. *There exists a Baire class 2 map Ψ that is an isomorphism between $\equiv_T$ on $2^{\mathbb{N}}$ and an equivalence relation given by a continuous group action on a 0-dimensional Polish space.*

Proof. Let φ^i denote the partial function computed by the ith Turing machine, in some recursive enumeration of all the Turing machines. That is, we consider Turing machines with oracle and input tapes, and $\varphi^i(x) = y$ iff for each n the ith Turing machine with oracle x and input n halts with the output $y(n)$.

We start with an easy observation. Below, for $s \in 2^{<\mathbb{N}}$, put $[s] = \{x \in 2^{\mathbb{N}} : s \subseteq x\}$.

Lemma 3.4.5. *Assume that $x \equiv_T y$. There exists an i with $\varphi^i(x) = y$ and $\varphi^i(y) = x$.*

Proof. We can assume that $x \neq y$. Pick $j, k \in \mathbb{N}$ with $\varphi^j(x) = y$ and $\varphi^k(y) = x$, and n with $x \restriction n \neq y \restriction n$. Then, an i with $\varphi^i \restriction [x \restriction n] = \varphi^j \restriction [x \restriction n]$ and $\varphi^i \restriction [y \restriction n] = \varphi^k \restriction [y \restriction n]$ clearly works. $\square$

The idea is to define a coding function $\Psi = (\alpha, \beta, \gamma)$ that will serve as an isomorphism. The crucial property of $\Psi(x)$ is that α encodes for each i whether φ^i is an involution on x (and does this for every $y \equiv_T x$), β will ensure that Ψ is continuous, while γ will be the identity, in order to encode x.

Let us now give the precise definitions. Fix a function $\iota : \mathbb{N}^2 \to \mathbb{N}$ such that for each $i, j \in \mathbb{N}$ we have $\varphi^{\iota(i,j)} = \varphi^i \circ \varphi^j$.

Let $\beta : 2^{\mathbb{N}} \to (\mathbb{N} \cup \{*\})^{\mathbb{N}^3}$ be defined by $\beta(x)(i, j, m) = n$, if n is least such that both the ith and the jth Turing machines with oracle x and input m halt with the same output in at most n steps, and let $\beta(x)(i, j, m) = *$, if such an n does not exist.

Define a map $\alpha : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}^2}$ by letting

$$\alpha(x)(i, j) = 0 \iff \beta(x)(i, j) \in \mathbb{N}^{\mathbb{N}},$$

and

$$\alpha(x)(i, j) = m + 1 \iff m \text{ is least with } \beta(x)(i, j, m) = *.$$

Finally, let $\Psi(x) = (\alpha(x), \beta(x), x)$. Let us denote the space $\mathbb{N}^{\mathbb{N}^2} \times (\mathbb{N} \cup \{*\})^{\mathbb{N}^3} \times 2^{\mathbb{N}}$ by X, where $\mathbb{N} \cup \{*\}$ is endowed with the discrete topology.

Lemma 3.4.6. *$\Psi(2^{\mathbb{N}})$ is closed in X.*

Proof. Assume that $(\alpha(x_k), \beta(x_k), x_k)_k$ is a convergent sequence, let x be the limit of $(x_k)_k$. Take any $i, j, m \in \mathbb{N}$. It is clear from the definition of β that $\beta(x)(i, j, m) = n$ holds for some $n \in \mathbb{N}$ if and only if $\beta(x_k)(i, j, m) = n$ is true for every large enough k. This shows that $\beta(x_k) \to \beta(x)$. Using this, it is easy to check that $\alpha(x_k) \to \alpha(x)$ holds as well. $\square$

For $i \in \mathbb{N}$ define a map δ_i from X to itself as follows:

$$\delta_i(\Psi(x)) = \Psi(x) \text{ if } \alpha(x)(\iota(i, i), 0) \neq 0,$$

otherwise
$$\delta_i(\Psi(x)) = \Psi(\varphi^i(x)).$$

Lemma 3.4.7. *The maps $(\delta_i)_{i\in\mathbb{N}}$ are $\Psi(2^{\mathbb{N}}) \to \Psi(2^{\mathbb{N}})$ homeomorphisms.*

Proof. Fix $i \in \mathbb{N}$. First we check that δ_i is continuous on the set $S = \{\Psi(x) : \alpha(x)(\iota(i,i),0) = 0\}$. For each $i', j', m \in \mathbb{N}$ we have that

$$\delta_i(\alpha(x), \beta(x), x)(0)(i', j') = (\alpha(x))(\iota(i', i), \iota(j', i)),$$

$$\delta_i(\alpha(x), \beta(x), x)(1)(i', j', m) = (\beta(x))(\iota(i', i), \iota(j', i), m),$$

thus, δ_i selects and permutes some of the coordinates of $\Psi(x)$. Moreover, if $\Psi(x) \in S$, as Ψ is injective, we have $\alpha(x)(\iota(i,i),0) = 0$. Therefore, $x \in dom(\varphi^i)$ and as φ^i is continuous on its domain, this shows the continuity of δ_i.

Note that the set S is relatively clopen set and on $\Psi(2^{\mathbb{N}}) \setminus S$ the function δ_i is the identity, showing that δ_i is continuous on the entire $\Psi(2^{\mathbb{N}})$.

Finally, it follows from the definition of α and δ_i that $\delta_i(\delta_i(\Psi(x))) = \Psi(x)$ holds for each x: indeed, $\Psi^{-1}(S)$ is the collection of $x \in 2^{\mathbb{N}}$ for which $\varphi^i \circ \varphi^i(x) = x$. Hence, δ_i is a continuous involution $\Psi(2^{\mathbb{N}}) \to \Psi(2^{\mathbb{N}})$. $\square$

Let E_Δ be the equivalence relation on $\Psi(2^\omega)$ generated by the maps $\{\delta_i : i \in \mathbb{N}\}$.

Lemma 3.4.8. *Ψ is an isomorphism between $\equiv_T$ and E_Δ.*

Proof. First, it is clear from the definition of δ_i that $\delta_i(\Psi(x)) = \Psi(y)$ implies that $x \equiv_T y$. So Ψ^{-1} is a homomorphism.

Second, assume that $x \equiv_T y$. Then by Lemma 3.4.5 there exists an i with $\varphi^i(x) = y$ and $\varphi^i(y) = x$. Then $\alpha(x)(\iota(i,i),0) = 0$, so $\delta_i(\Psi(x)) = \Psi(\varphi^i(x)) = \Psi(y)$, so $\Psi(x) E_\Delta \Psi(y)$. $\square$

Now we turn to the calculation of the complexity of the map Ψ.

Lemma 3.4.9. *The map β is Baire class 1 and the map α is Baire class 2. Consequently, the map Ψ is Baire class 2.*

Proof. For β, take any $i, j, m \in \mathbb{N}$. Then for each natural number n, the set $\{x : \beta(x)(i,j,m) = n\}$ is open. Thus, the set $\{x : \beta(x)(i,j,m) = *\}$ is closed. This shows that β preimages of basic clopen sets are $\mathbf{\Delta}^0_2$.

For α, for a given i, the set $\{x : \alpha(x)(i,j) = 0\} = \{x : \forall m(\beta(x)(i,j,m) \in \omega)\}$ is $\mathbf{\Pi}^0_2$, and also, for $m \neq 0$ we have that $\{x : \alpha(x)(i,j) = m\} = \{x : \forall m' < m\ (\beta(x)(i,j,m') \in \mathbb{N} \wedge \beta(x)(i,j,m) = *)\}$, which shows that these sets are $\mathbf{\Pi}^0_2$ as well, and thus α is indeed Baire class 2. $\square$

This completes the proof of Proposition 3.4.4 □

In order to finish the proof of Theorem 3.4.3 we need a last observation.

Lemma 3.4.10. *Assume that Γ acts continuously on an uncountable and zero-dimensional Polish space X, so that the induced equivalence relation E_Γ^X is aperiodic. Then there exist invariant under the action $X' \subseteq X$ that is homeomorphic to $\mathbb{N}^\mathbb{N}$, and an isomorphism φ between E_Γ^X and $E_\Gamma^X \restriction X'$ that moves only countably many points.*

Proof. As in the proof of Proposition 3.2.1. □

Proof of Theorem 3.4.3. By Proposition 3.4.4 there exists a Baire class 2 isomorphism between $\equiv_T$ and some equivalence relation of the form E_Γ^X, where Γ acts continuously on a zero-dimensional Polish space X. Applying Lemma 3.4.10 we get an isomorphism with an equivalence relation on the Baire space. Moreover, as countable modifications of Baire class 2 functions do not change their class, we are done. □

We do not know if the complexity of the Borel isomorphism in Theorem 3.4.3 is optimal.

Problem 3.4.11. *Is there a Baire class 1 map that is an isomorphism between $\equiv_T$ and an equivalence relation given by a continuous group action on $\mathbb{N}^\mathbb{N}$?*

Consider now *any* map $\Phi\colon 2^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ satisfying the conditions of Theorem 3.4.3. Then for some $p \in \mathbb{N}^\mathbb{N}$ we have $x \equiv_T y \implies \langle \Phi(x), p\rangle \equiv_T \langle \Phi(y), p\rangle$. Thus if Martin's Conjecture is true, we have that on a cone $\langle \Phi(x), p\rangle$ is Turing equivalent to one of x, x', x'' and thus the same is true for $\Phi(x)$. For the *particular* Φ that was constructed in the proof of Theorem 3.4.3, it is easy to see that $\Phi(x) \equiv_T x''$, since x'' can be easily computed from the map α defined in the proof of Proposition 3.4.4. Similar to the problem Problem 3.4.11, we have the following:

Problem 3.4.12. *Is there a Borel map $\Phi : 2^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ that is an isomorphism between $\equiv_T$ and an equivalence relation given by a continuous group action on $\mathbb{N}^\mathbb{N}$ such that $\Phi(x) \equiv_T x'$ on a cone?*

On the other hand we have the following result, where for $x, y \in \mathbb{N}^\mathbb{N}$, $x \leq_T y$ iff x is recursive in y.

Proposition 3.4.13. *There is no Borel map $\Phi\colon 2^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ that is an isomorphism between $\equiv_T$ and an equivalence relation given by a continuous group action on $\mathbb{N}^\mathbb{N}$ such that $\Phi(x) \leq_T x$ on a cone.*

Proof. Recall that a **pointed perfect tree** is a perfect binary tree $S \subseteq 2^{<\mathbb{N}}$ such that $x \in [S] \implies S \leq_T x$, where $[S] \subseteq 2^{\mathbb{N}}$ is the set of infinite branches of S. Below we will use certain properties of pointed perfect trees due to Martin, whose proofs can be found, for example, in [Kec88]. Assume that $\Phi(x) \leq_T x$ on a cone, towards a contradiction. Then by [Kec88, Theorem 1.3] there is a perfect pointed tree T such that $x \in [T] \implies \Phi(x) \leq_T x$. Then by [Kec88, Lemma 1.4], there is a perfect pointed subtree $S \subseteq T$ and $i \in \mathbb{N}$ such that if $x \in [S]$, then $\varphi^i(x)$ is defined and $\varphi^i(x) = \Phi(x)$. Thus Φ is continuous on $[S]$. It follows that $\equiv_T$ restricted to $[S]$ is $\mathbf{\Sigma^0_2}$. Let now Ψ be the canonical homeomorphism of $2^{\mathbb{N}}$ with $[S]$, so that $\Psi(x) \equiv_T x$, if $S \leq_T x$. It follows that for $S \leq_T x, y$, we have $x \equiv_T y \iff \Psi(x) \equiv_T \Psi(y)$, thus, in particular, for some $z \in 2^{\mathbb{N}}$, the Turing degree of z, i.e., the set $\{w \in 2^{\mathbb{N}} : w \equiv_T z\}$ is $\Sigma^0_2(z)$. This is false in view of the following well-known fact:

Lemma 3.4.14. *For any $z \in 2^{\mathbb{N}}$, the Turing degree of z is in $\Sigma^0_3(z)$ but not in $\Pi^0_3(z)$.*

Proof. It is easy to check that the Turing degree of z is $\Sigma^0_3(z)$. Assume now that it is in $\Pi^0_3(z)$, towards a contradiction. Then if $A = \{w \in 2^{\mathbb{N}} \colon w \leq_T z\}$, we have that A is also $\Pi^0_3(z)$, since $w \in A \iff \langle w, z\rangle \equiv_T z$. But then $2^{\mathbb{N}} \setminus A$ is a comeager $\Sigma^0_3(z)$ set, so by the relativized version of the basis theorem of Shoenfield [Sho58] it contains a recursive in z real, a contradiction. □

□

3.5 Continuous actions on compact spaces, compressibility and paradoxicality

(A) In connection with Problem 3.2.10, *for the case of compact action realizations*, we discuss some special properties of continuous actions of countable groups on compact Polish spaces that may have some relevance to this question.

Let Γ be a countable group and let $\mathbf{a}$ be a Borel action of Γ on a standard Borel space X. Put $\gamma \cdot x = \mathbf{a}(\gamma, x)$. We denote by $\langle \mathbf{a} \rangle$ the set of all Borel maps $T \colon X \to X$ such that $\forall x \exists \gamma \in \Gamma(T(x) = \gamma \cdot x)$. Equivalently this means that there is a Borel partition $X = \bigsqcup_{\gamma \in \Gamma} X_\gamma$ such that $T(x) = \gamma \cdot x$ for $x \in X_\gamma$. We also let $\langle \mathbf{a} \rangle^f$ consist of all Borel maps $T \colon X \to X$ for which there is a *finite* subset $F \subseteq \Gamma$ such that $\forall x \exists \gamma \in F(T(x) = \gamma \cdot x)$. Equivalently this means that there is a Borel partition $X = \bigsqcup_{\gamma \in F} X_\gamma$ such that $T(x) = \gamma \cdot x$ for $x \in X_\gamma$.

We say that the action $\mathbf{a}$ is **compressible** (resp., **finitely compressible**) if there is an injective Borel map in $T \in \langle \mathbf{a} \rangle$ (resp., $T \in \langle \mathbf{a} \rangle^f$) such that for every orbit C of $\mathbf{a}$, $T(C) \subsetneqq C$ or equivalently $\Gamma \cdot (X \setminus T(X)) = X$. Clearly the action $\mathbf{a}$

is compressible iff the associated equivalence relation is compressible. The action $\mathbf{a}$ is **paradoxical** (resp., **finitely paradoxical**) if there are two injective Borel maps T_1, T_2 in $\langle a \rangle$ (resp., in $\langle \mathbf{a} \rangle^f$) such that $T_1(X) \cap T_2(X) = \emptyset, T_1(X) \cup T_2(X) = X$.

Clearly if $\mathbf{a}$ is paradoxical (resp., finitely paradoxical), then $\mathbf{a}$ is compressible (resp., finitely compressible). It is also known that if $\mathbf{a}$ is compressible, then $\mathbf{a}$ is paradoxical; see, e.g., [Kec22, 3.23].

Remark 3.5.1. It is easy to see that finite compressibility does not imply imply finite paradoxicality. Take for example $\mathbb{Z}$ acting on itself by translation. Since $\mathbb{Z}$ is amenable this action is not finitely paradoxical. On the other hand the map $T : \mathbb{Z} \to \mathbb{Z}$ such that $T(n) = n$, if $n < 0$, and $T(n) = n + 1$, if $n \geq 0$, shows that this action is finitely compressible.

Remark 3.5.2. One can easily see that finite paradoxicality is equivalent to the following strengthening of finite compressibility: There is an injective Borel map $T \in \langle \mathbf{a} \rangle^f$ and a finite subset $F \subseteq \Gamma$ such that $F \cdot (X \setminus T(X)) = X$.

For $n \geq 1$, let $[n] = \{1, 2, \dots, n\}$. The ***n*-amplification** of $\mathbf{a}$ is the action $\mathbf{a}_n$ of the group $\Gamma \times S_n$ on $X \times [n]$ given by $(\gamma, \pi) \cdot (x, i) = (\gamma \cdot x, \pi(i))$, where S_n is the group of permutations of $[n]$. An **amplification** of $\mathbf{a}$ is an n-amplification of $\mathbf{a}$, for some n.

Theorem 3.5.3. *Let* $\mathbf{a}$ *be a continuous action of a countable group* Γ *on a compact Polish space* X*. Then the following are equivalent:*

(i) $\mathbf{a}$ *is compressible;*

(ii) $\mathbf{a}$ *is paradoxical;*

(iii) an amplification of $\mathbf{a}$ *is finitely compressible;*

(iv) an amplification of $\mathbf{a}$ *is finitely paradoxical.*

Proof. (1) The proof will be based on Nadkarni's Theorem and the following two results. We first recall some standard terminology:

Let X be a standard Borel space and let $B(X)$ be the σ-algebra of its Borel sets. A **finitely additive Borel probability measure** is a map $\mu : B(X) \to [0, 1]$ such that $\mu(\emptyset) = 0, \mu(X) = 1$, and $\mu(A \cup B) = \mu(A) + \mu(B)$, if $A \cap B = \emptyset$. It is countably additive if moreover $\mu(\bigcup_n A_n) = \sum_n \mu(A_n)$, for any pairwise disjoint family (A_n). Recall that we call these simply *measures*. If $\mathbf{a}$ is a Borel action of a countable group Γ on X, then μ is invariant if for any Borel set A and $\gamma \in \Gamma$, $\mu(\gamma \cdot A) = \mu(A)$.

Theorem 3.5.4 ([Tse15, 5.3])**.** *Let* Γ *be a countable group and let* $\mathbf{a}$ *be a continuous action of* Γ *on a compact Polish space* X*. If* $\mathbf{a}$ *admits an invariant finitely additive Borel probability measure, then it admits an invariant measure.*

Remark 3.5.5. The hypothesis that X is compact Polish is necessary here. From Remark 3.5.9 we see that there is a counterexample to this statement even with X Polish locally compact.

Theorem 3.5.6 ([TW16, 11.3])**.** *Let Γ be a countable group and let $\mathbf{a}$ be a Borel action of Γ on a standard Borel space X. Then the following are equivalent:*

(i) there is no invariant finitely additive Borel probability measure on X;

(ii) there is a finitely paradoxical amplification of $\mathbf{a}$.

(2) We now prove Theorem 3.5.3. We have already mentioned (in the paragraph preceding Remark 3.5.1) the equivalence of (i) and (ii).

(i) $\implies$ (iv): If $\mathbf{a}$ is compressible, then by Nadkarni's Theorem it does not admit an invariant measure, so by Theorem 3.5.4 it does not admit an invariant finitely additive Borel probability measure. Then by Theorem 3.5.6 some amplification of $\mathbf{a}$ is finitely paradoxical.

(iv) $\implies$ (iii) is obvious.

(iii) $\implies$ (i): Assume that for some n the amplification $\mathbf{a}_n$ is finitely compressible but, towards a contradiction, $\mathbf{a}$ is not compressible. Then by Nadkarni's Theorem, $\mathbf{a}$ admits an invariant measure and thus so does $\mathbf{a}_n$, contradicting the compressibility of $\mathbf{a}_n$. □

Problem 3.5.7. *In Theorem 3.5.3, can one replace (iii) by "$\mathbf{a}$ is finitely compressible" and similarly for (iv).*

Remark 3.5.8. It follows from Theorem 3.5.3 that for a continuous action $\mathbf{a}$ of a countable group on a compact Polish space, the property "$\mathbf{a}$ has a finitely compressible (or finitely paradoxical) amplification" is a property of the induced equivalence relation $E_{\mathbf{a}}$. More precisely, if $\mathbf{a}, \mathbf{b}$ are two continuous actions of groups Γ, Δ on compact metrizable spaces X, Y, resp., and $E_{\mathbf{a}} \cong_B E_{\mathbf{b}}$, i.e., $E_{\mathbf{a}}, E_{\mathbf{b}}$ are Borel isomorphic, then $\mathbf{a}$ admits a finitely compressible (or finitely paradoxical) amplification iff $\mathbf{b}$ admits a finitely compressible (or finitely paradoxical) amplification. In view of Problem 3.5.7, this may not be true for the property "$\mathbf{a}$ is finitely compressible" or "$\mathbf{a}$ is finitely paradoxical". In fact one way to try to prove Problem 3.5.7 is to search for two continuous actions $\mathbf{a}, \mathbf{b}$ of countable groups Γ, Δ on a compact metrizable space X with $E_{\mathbf{a}} = E_{\mathbf{b}}$, for which $\mathbf{a}$ is finitely compressible (or finitely paradoxical) but $\mathbf{b}$ is not.

Remark 3.5.9. Theorem 3.5.3 fails if the space X is not compact. In fact there are even counterexamples with X Polish locally compact. Recall that an action of a group Γ on a set X is **amenable** if there is a finitely additive probability measure defined on all subsets of X and invariant under the action. Any action of a countable amenable group is amenable. Take now Γ to be a locally finite, infinite group and consider the left-translation action of Γ on itself. This action is not finitely compressible. Let then $X = 2^{\mathbb{N}} \times \Gamma$ (with Γ discrete). This is Polish locally compact and Γ acts on it continuously by the action a given by $\gamma \cdot (x, \delta) = (x, \gamma\delta)$. This action is clearly compressible via the map $T(x, \gamma) = (x, f(\gamma))$, where $f \colon \Gamma \to \Gamma$ is an injection with $f(\Gamma) \neq \Gamma$, so (i) in Theorem 3.5.3 holds. On the other hand, all amplifications a_n are amenable, so not finitely paradoxical and (iv) in Theorem 3.5.3 fails. Also all the actions a_n are not finitely compressible and (iii) in Theorem 3.5.3 also fails.

In this counterexample the action $\mathbf{a}$ is smooth. One can find another counterexample where the action $\mathbf{a}$ is not smooth as follows: Let Γ be as before and consider again the translation action of Γ on itself. Let also Γ act on 2^{Γ} by shift and consider the action $\mathbf{b}$ of $\Delta = \Gamma^2$ on $X = 2^{\Gamma} \times \Gamma$ given by $(\gamma, \delta) \cdot (x, \varepsilon) = (\gamma \cdot x, \delta\varepsilon)$. This action is not smooth and is compressible but it is also amenable, since the action of each factor of Δ is amenable on the corresponding space and therefore the action of Δ is amenable by taking the product of finitely additive probability measures witnessing the amenability of these two actions. (By the product of a finitely additive probability measure μ defined on all subsets of a set A and a finitely additive probability measure ν defined on all subsets of a set B, we mean the finitely additive probability measure $\mu \times \nu$ on $A \times B$ defined by $\mu \times \nu(C) = \int_A \nu(C_x) \mathrm{d}\mu(x)$.) Also all the actions $\mathbf{a}_n$ are not finitely compressible.

Remark 3.5.10. Using Remark 3.5.1 it is easy to see that finite compressibility does not imply finite paradoxicality even for continuous actions of countable groups on compact Polish spaces. To see this, let G be a compact metrizable group containing a copy of $\mathbb{Z}$ (e.g., the unit circle under multiplication) and consider the left-translation action of $\mathbb{Z}$ on G.

Remark 3.5.11. Let E be a countable Borel equivalence relation on a standard Borel space X. We say that E is compressible (resp., finitely compressible, paradoxical, finitely paradoxical) if there is a Borel action $\mathbf{a}$ of a countable group Γ on X with $E = E_{\mathbf{a}}$ and $\mathbf{a}$ is compressible (resp., finitely compressible, paradoxical, finitely paradoxical). Then it is easy to check that these conditions are equivalent. Indeed if E is compressible, then there is a smooth, aperiodic (i.e., having infinite classes) Borel equivalence relation F with $F \subseteq E$. Then $F \cong_B \mathbb{R} \times I_{\mathbb{N}}$, where $H = \mathbb{R} \times I_{\mathbb{N}}$ is the equivalence relation on $\mathbb{R} \times \mathbb{N}$ given by $(x, m) H (y, n) \iff x = y$. There is a transitive action of the free group with two generators $\mathbb{F}_2$ on $\mathbb{N}$ which is finitely paradoxical and thus a Borel action $\mathbf{b}$ of $\mathbb{F}_2$ on $\mathbb{R} \times \mathbb{N}$ with $F = E_b$ which is finitely paradoxical. Fix also a Borel action $\mathbf{c}$ of a countable group Γ with $E_{\mathbf{c}} = E$. Then the action $\mathbf{a}$ of $\Gamma * \mathbb{F}_2$ that is equal to $\mathbf{c}$ on Γ and $\mathbf{b}$ on $\mathbb{F}_2$ is finitely paradoxical and $E_{\mathbf{a}} = E$.

Remark 3.5.12. Ronnie Chen pointed out that (iv) $\implies$ (ii) in Theorem 3.5.3 can be also proved by using the cardinal algebra $K(E \times I_{\mathbb{N}})$ as in [Che21] and the cancellation law for cardinal algebras.

Recall also that a CBER E admits an invariant measure iff *some* Borel action of a countable group that generates E has an invariant measure iff *every* Borel action of a countable group that generates E has an invariant measure (iff E is not compressible). On the other hand, there are aperiodic CBER E such that some Borel action of a countable group that generates E has an invariant finitely additive Borel probability measure but some other Borel action of a countable group that generates E has no invariant finitely additive Borel probability measure. For example, let $E = E_t$. There is a continuous action of $\mathbb{F}_2$ on $2^{\mathbb{N}}$ that generates E (see the proof of Theorem 3.2.6) and this action has no invariant finitely additive Borel probability measure by Theorem 3.5.4. On the other hand, E_t is induced by a Borel action of $\mathbb{Z}$ and this action has in fact an invariant finitely additive probability measure defined on all subsets of $2^{\mathbb{N}}$.

However in view of Remark 3.5.11 we have the following equivalent formulation of existence of invariant measures for a CBER:

Proposition 3.5.13. *For every aperiodic* CBER E*,* E *admits an invariant measure iff every Borel action of a countable group that generates* E *admits an invariant finitely additive Borel probability measure.*

(B) The preceding results in part **(A)** of this subsection can be generalized as follows.

Let Γ be a countable group and let $\mathbf{a}$ be an action of Γ on a set X. Let also $\mathcal{A}$ be an algebra of subsets of X invariant under this action. For $A, B \in \mathcal{A}$, let $A \sim_{\mathcal{A}} B$ iff there are partitions $A = \bigsqcup_{i=1}^n A_i, B = \bigsqcup_{i=1}^n B_i$, where $A_i, B_i \in \mathcal{A}$, and $\gamma_i \in \Gamma$ such that $\gamma_i \cdot A_i = B_i$. We say that the action is **$\mathcal{A}$-finitely compressible** if $X \sim_{\mathcal{A}} Y$ with witnesses X_i, Y_i, γ_i as above, so that if $T \colon X \to X$ is such that $T(x) = \gamma_i \cdot x$, for $x \in X_i$, then for every orbit C of the action, $T(C) \subsetneqq C$. Also the action is **$\mathcal{A}$-finitely paradoxical** if there is a partition $X = Y \sqcup Z$, with $Y, Z \in \mathcal{A}$ and $X \sim_{\mathcal{A}} Y \sim_{\mathcal{A}} Z$.The concept of an invariant finitely additive probability measure on $\mathcal{A}$ is defined as before.

We extend the algebra $\mathcal{A}$ to an algebra $\mathcal{A}_n$ of subsets of $X \times [n]$ by letting $A \in \mathcal{A}_n \iff A = \bigcup_{i=1}^n A_i \times \{i\}$, where $A_i \in \mathcal{A}$. We say that $\mathbf{a}_n$ is $\mathcal{A}$-finitely compressible if it is $\mathcal{A}_n$-finitely compressible. Similarly we define what it means for $\mathbf{a}_n$ to be $\mathcal{A}$-finitely paradoxical.

We now have the following generalization of Theorem 3.5.3:

Theorem 3.5.14. *Let* $\mathbf{a}$ *be a continuous action of a countable group* Γ *on a compact Polish space* X*. Let* $\mathcal{A}$ *be an algebra of Borel subsets of* X *which is invariant under the action and contains a basis for* X*. Then the following are equivalent:*

(i) there is no invariant finitely additive probability measure μ *on* $\mathcal{A}$*;*

(ii) there is no invariant measure ν*;*

(iii) an amplification of $\mathbf{a}$ *is* $\mathcal{A}$*-finitely compressible;*

(iv) an amplification of $\mathbf{a}$ *is* $\mathcal{A}$*-finitely paradoxical.*

The proof of Theorem 3.5.14 is similar to the proof of Theorem 3.5.3 using the following generalizations of Theorem 3.5.4 and Theorem 3.5.6.

Theorem 3.5.15 ([Tse15, 5.3])**.** *Let* Γ *be a countable group and let* $\mathbf{a}$ *be a continuous action of* Γ *on a second countable Hausdorff space* X*. Let* $\mathcal{A}$ *be an algebra of subsets of* X *which is invariant under the action and contains a basis for* X *and a compact set* K*. If there is an invariant finitely additive probability measure* μ *on* $\mathcal{A}$ *with* $\mu(K) > 0$*, then there is an invariant (Borel probability, countably additive) measure* ν*.*

Theorem 3.5.16 ([TW16, 11.3])**.** *Let* Γ *be a countable group and let* $\mathbf{a}$ *be an action of* Γ *on a set* X*. Let* $\mathcal{A}$ *be an algebra of subsets of* X *invariant under this action. Then the following are equivalent:*

(i) there is no invariant finitely additive probability measure on $\mathcal{A}$*;*

(ii) there is a $\mathcal{A}$*-finitely paradoxical amplification of* $\mathbf{a}$*.*

As a particular case of Theorem 3.5.14 we have the following. Let $\mathbf{a}$ be a continuous action of a countable group Γ on a zero-dimensional compact Polish space X (e.g., the Cantor space). Let $\mathcal{C}$ be the algebra of clopen subsets of X. Then the following are equivalent:

(i) there is no invariant finitely additive probability measure μ on $\mathcal{C}$;

(ii) there is no invariant measure ν;

(iii) an amplification of $\mathbf{a}$ is $\mathcal{C}$-finitely compressible;

(iv) an amplification of $\mathbf{a}$ is $\mathcal{C}$-finitely paradoxical;

(v) $\mathbf{a}$ is compressible;

(vi) $\mathbf{a}$ is paradoxical;

(vii) an amplification of $\mathbf{a}$ is finitely compressible;

(viii) an amplification of $\mathbf{a}$ is finitely paradoxical.

Thus, rather surprisingly, for a continuous action of a countable group on a zero-dimensional compact Polish space, existence of a (countable Borel) paradoxical decomposition is equivalent to the existence of an amplification with a finite paradoxical decomposition using Borel pieces and also equivalent to the existence of an amplification with a finite paradoxical decomposition using *clopen* pieces.

3.6 Realizations by subshifts

In this and the next two sections, unless it is otherwise stated or clear from the context, we assume all groups to be countable.

If Γ is a group a **Γ-flow** is a continuous action of Γ on a compact Polish space. A **subflow** of a Γ-flow is the restriction of the action to a nonempty closed invariant subset.

Recall that for a group Γ, a set L equipped with a Γ-action, and a set X, the canonical **shift action** of Γ on X^L is given by

$$(\gamma \cdot x)_l = x_{\gamma^{-1} \cdot l}.$$

When X is a topological space, the restriction of the shift action to a nonempty closed invariant set $F \subseteq X^\Gamma$ is called a **subshift**. We often refer to F itself as a subshift.

If X is compact Polish, this is a Γ-flow and we denote it by $\mathbf{s}_{L,X}$. In particular, $\mathbf{s}_{\Gamma,X}$ is the shift action of Γ, where Γ acts on itself by left multiplication.

For Γ-flows $\mathbf{a}$ and $\mathbf{b}$ on X and Y respectively, a **Γ-map $\mathbf{a} \to \mathbf{b}$** is a Γ-equivariant continuous function $X \to Y$. Let $\mathrm{Hom}_\Gamma(\mathbf{a}, \mathbf{b})$ denote the set of Γ-maps $\mathbf{a} \to \mathbf{b}$.

Below for any action $\mathbf{a}$, we denote by $E_\mathbf{a}$ the induced orbit equivalence relation.

(A) Coinduction and generators

Let $\Gamma \leq \Delta$ be groups. Given a Δ-flow $\mathbf{b}$, we denote the Γ-restriction of $\mathbf{b}$ by $\mathbf{b}\restriction_\Gamma$.

Given a Γ-flow $\mathbf{a}$ on X, the **coinduced Δ-flow** of $\mathbf{a}$, denoted by $\mathrm{CInd}_\Gamma^\Delta(\mathbf{a})$, is the Δ-subflow of $\mathbf{s}_{\Delta,X}$ on the subspace

$$\{x \in X^\Delta : \forall \gamma \in \Gamma \, \forall \delta \in \Delta \, [x_{\delta\gamma} = \gamma^{-1} \cdot x_\delta]\}.$$

In particular, $\mathbf{s}_{\Gamma,X}$ is isomorphic to $\mathrm{CInd}_1^\Gamma(\mathbf{s}_{1,X})$, where 1 is the trivial group (note that $\mathbf{s}_{1,X}$ is the 1-flow on X).

There is a natural bijection

$$\mathrm{Hom}_\Delta(\mathbf{b}, \mathrm{CInd}_\Gamma^\Delta(\mathbf{a})) \cong \mathrm{Hom}_\Gamma(\mathbf{b}\restriction_\Gamma, \mathbf{a})$$

taking f to the map $y \mapsto (f(y))_1$.

Let $\mathbf{a}$ and $\mathbf{b}$ be flows for Γ and Δ respectively. A Γ-map $f : \mathbf{b}\restriction_\Gamma \to \mathbf{a}$ is an **a-generator** for $\mathbf{b}$ if its corresponding Δ-map $\mathbf{b} \to \mathrm{CInd}_\Gamma^\Delta(\mathbf{a})$ is injective. Explicitly, f is an $\mathbf{a}$-generator for $\mathbf{b}$ if for every $x, x' \in X_\mathbf{a}$, if $f(\delta \cdot x) = f(\delta \cdot x')$ for every δ, then $x = x'$.

We note the following facts:

(1) Let $\mathbf{a}$ be a Γ-flow on X, and let $n \geq 2$. Considering n as a discrete space, an $\mathbf{s}_{1,n}$-generator for $\mathbf{a}$ coincides with the usual notion of a **clopen n-generator** for $\mathbf{a}$, that is, a partition $(A_i)_{i<n}$ of X into clopen sets such that for every $x, x' \in X$, if for every $\gamma \in \Gamma$ and every $i < n$ we have

$$\gamma \cdot x \in A_i \iff \gamma \cdot x' \in A_i,$$

then $x = x'$. Equivalently $\mathbf{a}$ admits a clopen n-generator iff it is (topologically) isomorphic to a subshift of n^Γ.

(2) Every injective Γ-map $\mathbf{b}\restriction_\Gamma \hookrightarrow \mathbf{a}$ is an $\mathbf{a}$-generator for $\mathbf{b}$.

(3) If $\mathbf{b} = \mathrm{CInd}_\Gamma^\Delta(\mathbf{a})$, then the map $y \mapsto y_1$ is an $\mathbf{a}$-generator for $\mathbf{b}$, since it corresponds to the identity on $\mathrm{CInd}_\Gamma^\Delta(\mathbf{a})$.

(4) Let $\Gamma \leq \Delta \leq \Lambda$ be groups with flows $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. If $\mathbf{c}$ has a $\mathbf{b}$-generator f, and $\mathbf{b}$ has an $\mathbf{a}$-generator g, then the composition $f \circ g$ is an $\mathbf{a}$-generator for $\mathbf{c}$. To see this, let $x, x' \in X_\mathbf{a}$ and suppose that $f(g(\lambda \cdot x)) = f(g(\lambda \cdot x'))$ for every λ. Then for every δ and every λ, we have $f(\delta \cdot g(\lambda \cdot x)) = f(\delta \cdot g(\lambda \cdot x'))$. Thus since f is a generator, we have $g(\lambda \cdot x) = g(\lambda \cdot x')$ for every λ. Since g is a generator, we have $x = x'$.

Below we call a flow **compressible** iff the induced equivalence relation is compressible. Equivalently by Nadkarni's Theorem (see, e.g., [Kec22, 5.3]) this means that the flow admits no invariant Borel probability measure.

Proposition 3.6.1. *Let $\Gamma \leq \Delta$ be groups, let $\mathbf{a}$ be a Γ-flow, and let $\mathbf{b}$ be a Δ-flow.*

(i) Suppose there is a Γ-map $\mathbf{b}\restriction_\Gamma \to \mathbf{a}$. If $\mathbf{a}$ is compressible, then $\mathbf{b}$ is compressible.

(ii) Suppose $\mathbf{b}$ has an $\mathbf{a}$-generator. If $\mathbf{a}$ has a clopen n-generator, then $\mathbf{b}$ has a clopen n-generator.

(ii) Suppose $\mathbf{b}\restriction_\Gamma = \mathbf{a}$. If $\mathbf{a}$ is minimal, then $\mathbf{b}$ is minimal.

Proof. (i) If μ is an invariant Borel probability measure for $\mathbf{b}$, then it is invariant for $\mathbf{b}\restriction_\Gamma$, so its pushforward to $\mathbf{a}$ is invariant.

(ii) If $\mathbf{a}$ has a clopen n-generator, then it has an $\mathbf{s}_{1,n}$-generator. Composing them gives a clopen n-generator for $\mathbf{b}$.

(iii) This is obvious. $\square$

Corollary 3.6.2. *Let $\Gamma \leq \Delta$ be groups. The following properties of a Γ-flow pass to its coinduced Δ-flow:*

(i) Compressibility.

(ii) Existence of a clopen n-generator.

Proof. Take $\mathbf{b} = \mathrm{CInd}_{\Gamma}^{\Delta}(\mathbf{a})$ in Proposition 3.6.1. $\square$

(B) Jumps

Let Γ and Λ be groups, and let L be a countable Λ-set. The **unrestricted wreath product** is the group $\Gamma \wr_L \Lambda$ defined by

$$\Gamma \wr_L \Lambda := \Gamma^L \rtimes \Lambda = \langle \Gamma^L, \Lambda : \lambda\gamma\lambda^{-1} = \lambda \cdot \gamma \rangle$$

If $L = \Lambda$ with the left-translation action, then we omit the subscript and write $\Gamma \wr \Lambda$. Denote by $\Gamma^{\oplus L}$ the subgroup of Γ^L consisting of those elements which are the identity on cofinitely many coordinates. Note that the shift Λ-action on Γ^L preserves $\Gamma^{\oplus L}$. The **restricted wreath product** is the subgroup $\Gamma \wr_L^{\oplus} \Lambda$ of $\Gamma \wr_L \Lambda$ generated by $\Gamma^{\oplus L}$ and Λ. If L is a transitive Λ-set, and S and T are generating sets for Γ and Λ respectively, then $S \cup T$ generates $\Gamma \wr_L^{\oplus} \Lambda$ (see [HR94, 2.3]).

Let E be a CBER on X. The **unrestricted L-jump** of E, denoted $E^{[L]}$ is the Borel equivalence relation on X^L defined by

$$x \; E^{[L]} \; y \iff \exists \lambda \, [\lambda \cdot x \; E^L \; y].$$

(see [CC20] for more uses of this jump). Let $E^{\oplus L}$ be the subequivalence relation of the product equivalence relation E^L consisting of pairs which are equal on cofinitely many coordinates. The **restricted L-jump** of E, is the subequivalence relation $E^{\oplus[L]}$ of $E^{[L]}$ obtained by using $E^{\oplus L}$ instead of E^L in the definition of $E^{[L]}$.

Given a Borel embedding $E \sqsubseteq_B F$ via a map $X \to Y$, the induced map $X^L \to Y^L$ witnesses the Borel embeddings $E^{[L]} \sqsubseteq_B F^{[L]}$ and $E^{\oplus[L]} \sqsubseteq_B F^{\oplus[L]}$.

Let $\mathbf{a}$ be a Γ-flow on X. Let $\mathbf{a}^L$ be the Γ^L-flow on X^L defined by $(\boldsymbol{\gamma} \cdot \boldsymbol{x})_l = \gamma_l \cdot x_l$. We have $E_{\mathbf{a}^L} = (E_{\mathbf{a}})^L$. The **unrestricted L-jump** of $\mathbf{a}$, denoted $\mathbf{a}^{[L]}$, is the unique $\Gamma \wr_L \Lambda$-flow on X^L which simultaneously extends both the Γ^L-flow $\mathbf{a}^L$ and the Λ-flow $\mathbf{s}_{L,X}$ on X^L. Explicitly, the action is given by $(\boldsymbol{\gamma}\lambda \cdot \boldsymbol{x})_l = \gamma_l x_{\lambda^{-1}l}$. We have $E_{\mathbf{a}^{[L]}} = (E_{\mathbf{a}})^{[L]}$, since

$$x \; E_{\mathbf{a}^{[L]}} \; y \iff \exists \boldsymbol{\gamma} \, \exists \lambda \, [\boldsymbol{\gamma}\lambda \cdot x = y] \iff \exists \lambda \, [\lambda \cdot x \; E_{\mathbf{a}^L} \; y] \iff \exists \lambda \, [\lambda \cdot x \; E_{\mathbf{a}}^L \; y]$$

Let $\mathbf{a}^{\oplus L}$ be the $\Gamma^{\oplus L}$-flow $\mathbf{a}^L \restriction_{\Gamma^{\oplus L}}$. We have $E_{\mathbf{a}^{\oplus L}} = E_{\mathbf{a}}^{\oplus L}$. The **restricted L-jump** of $\mathbf{a}$, denoted $\mathbf{a}^{\oplus[L]}$, is the restriction $\mathbf{a}^{[L]} \restriction_{\Gamma \wr_L^{\oplus} \Lambda}$. We have $E_{\mathbf{a}^{\oplus[L]}} = (E_{\mathbf{a}})^{\oplus[L]}$.

If L is a transitive Λ-set, then for any $l_0 \in L$, the map $\boldsymbol{x} \mapsto x_{l_0}$ is an $\mathbf{a}$-generator for $\mathbf{a}^{\oplus[L]}$, since if $(\lambda \cdot \boldsymbol{x})_{l_0} = (\lambda \cdot \boldsymbol{y})_{l_0}$ for every λ, then by transitivity, we have $x_l = y_l$ for every $l \in L$, and thus $x = y$.

Proposition 3.6.3. *If L is a transitive Λ-set, then the following properties of a Γ-flow pass to its restricted L-jump:*
(i) Compressibility.
(ii) Existence of a clopen n-generator.
(iii) Minimality.

Proof. Let $\mathbf{a}$ be a Γ-flow.

Since $\mathbf{a}^{\oplus[L]}$ has an $\mathbf{a}$-generator, the first two properties follow from Proposition 3.6.1.

If $\mathbf{a}$ is minimal Γ-flow, then $\mathbf{a}^{\oplus L}$ is a minimal $\Gamma^{\oplus L}$-flow. Since $\mathbf{a}^{\oplus[L]}\restriction_{\Gamma^{\oplus L}} = \mathbf{a}^{\oplus L}$, we have by Proposition 3.6.1 that $\mathbf{a}^{\oplus[L]}$ is minimal. ◻

(C) Realizations by minimal subshifts

A flow is **orbit-universal** if its orbit equivalence relation is a universal CBER.

Let $E(L,\mathbb{R})$ denote the orbit equivalence relation of the shift action $\Lambda \curvearrowright \mathbb{R}^L$.

Theorem 3.6.4. *Let Γ and Λ be countable groups. Let L be a countable Λ-set, and let $\mathbf{a}$ be a Γ-flow on X, with X uncountable. Then there is a Borel injection $f : \mathbb{R}^L \to X^L$ which simultaneously witnesses $E(L,\mathbb{R}) \sqsubseteq_B E_{\mathbf{a}^{\oplus[L]}}$ and $E(L,\mathbb{R}) \sqsubseteq_B E_{\mathbf{a}^{[L]}}$. In particular, for every group G (no definability condition required) with $\Gamma \wr_L^{\oplus} \Lambda \leq G \leq \Gamma \wr_L \Lambda$, the map f witnesses $E(L,\mathbb{R}) \sqsubseteq_B E_{\mathbf{a}^{[L]}\restriction_G}$.*

In particular, if $E(L,\mathbb{R})$ is universal and Δ is a countable group with $\Gamma \wr_L^{\oplus} \Lambda \leq \Delta \leq \Gamma \wr_L \Lambda$, then $\mathbf{a}^{[L]}\restriction_\Delta$ is orbit-universal.

Proof. Since X is uncountable, there is a Borel map $\mathbb{R} \to X$ witnessing $\mathrm{id}_\mathbb{R} \sqsubseteq_B E_\mathbf{a}$. Let $f : \mathbb{R}^L \to X^L$ be the induced map. Then f witnesses

$$E(L,\mathbb{R}) = (\mathrm{id}_\mathbb{R})^{\oplus[L]} \sqsubseteq_B (E_\mathbf{a})^{\oplus[L]} = E_{\mathbf{a}^{\oplus[L]}}$$

and also

$$E(L,\mathbb{R}) = (\mathrm{id}_\mathbb{R})^{[L]} \sqsubseteq_B (E_\mathbf{a})^{[L]} = E_{\mathbf{a}^{[L]}}$$

◻

Corollary 3.6.5. *Let Γ and Λ be groups, and suppose that there is a countable transitive Λ-set L such that $E(L,\mathbb{R})$ is a universal* CBER. *Let Δ be a countable group with a factor Δ' such that $\Gamma \wr_L^{\oplus} \Lambda \leq \Delta' \leq \Gamma \wr_L \Lambda$. Then there is an orbit-universal minimal Δ-flow with a clopen 2-generator. If Γ is non-amenable, then this flow can be taken to be compressible.*

Proof. It suffices to consider the case where $\Delta' = \Delta$. Let $\mathbf{a}$ be an uncountable minimal Γ-flow with a 2-generator; for instance, take a minimal subshift of a free subshift of 2^Γ (these exist by [GJS09]; see also [Ber17]). If Γ is non-amenable, by Theorem 3.7.1 below we can take $\mathbf{a}$ to be compressible and then pass to a subflow to ensure minimality.

Now consider the $\Gamma \wr_L^\oplus \Lambda$-flow $\mathbf{a}^{\oplus[\Lambda]}$. By Proposition 3.6.3, this is minimal and has a 2-generator, and is compressible if Γ is non-amenable. Orbit-universality follows from Theorem 3.6.4. □

Corollary 3.6.6. *(i) There is an orbit-universal minimal subshift of* $2^{\mathbb{F}_3}$.

(ii) There is a compressible orbit-universal minimal subshift of $2^{\mathbb{F}_4}$.

In particular any compressible, universal CBER *admits a minimal, compact action realization which is in fact a minimal subshift of* $2^{\mathbb{F}_4}$.

Proof. Recall that $E(\mathbb{F}_2, 2)$ is a universal CBER.

(i) $\mathbb{F}_3$ has the factor $\mathbb{Z} \wr^\oplus \mathbb{F}_2$, so by Corollary 3.6.5, $\mathbb{Z} \wr^\oplus \mathbb{F}_2$ admits an orbit-universal minimal flow with a clopen 2-generator.

(ii) $\mathbb{F}_4$ has the factor $\mathbb{F}_2 \wr^\oplus \mathbb{F}_2$, so since $\mathbb{F}_2$ is non-amenable, by Corollary 3.6.5, $\mathbb{F}_2 \wr^\oplus \mathbb{F}_2$ admits a compressible orbit-universal minimal flow with a clopen 2-generator, □

Problem 3.6.7. *Does Corollary 3.6.6 hold with* $\mathbb{F}_2$ *instead of* $\mathbb{F}_3, \mathbb{F}_4$*?*

By Corollary 3.6.5, it suffices to find some Γ and $\Lambda \geq \mathbb{F}_2$ such that there is a 2-generated group between $\Gamma \wr_L^\oplus \Lambda$ and $\Gamma \wr_L \Lambda$ (and Γ non-amenable if we want compressibility).

In view of Theorem 3.2.6 and Corollary 3.6.6 one can ask whether the following very strong realization result is true:

Problem 3.6.8. *Does every non-smooth aperiodic* CBER *have a realization as a subshift of* 2^Γ *for some group* Γ*? Also does it have a realization as a minimal subshift?*

(D) Minimal subshift universality

Let Γ be a countable group. We say that Γ is **minimal subshift universal** if there is a minimal subshift K of 2^Γ such that if E is the shift equivalence relation on 2^Γ, then $E|K$ is universal. We note the following equivalent formulation of this notion. Recall that a point $x \in 2^\Gamma$ is minimal if $\overline{\Gamma \cdot x}$ is a minimal Γ-flow; equivalently, for every finite $A \subseteq \Gamma$, the set $\{\gamma \in \Gamma : (\gamma \cdot x)\restriction_A = x\restriction_A\}$ is **left syndetic**, i.e. finitely many left translates of it cover Γ (see [dV93, IV(1.2)]).

Proposition 3.6.9. *Let Γ be a countable group. Then the following are equivalent:*

(i) Γ is minimal subshift universal;

(ii) There is a minimal Γ-flow which admits a clopen 2-generator and such that the induced equivalence relation is universal;

(iii) If M is the set of minimal points in 2^Γ and E is the shift equivalence relation, then $E|M$ is universal

Proof. Clearly (i) implies (ii) implies (iii). Assume now (iii). Consider the Borel map f that sends $x \in M$ to the closure of its orbit (which is an element of the space of compact subsets of 2^Γ). Then by [MSS16, Theorem 3.1] there is some K such that $E|f^{-1}(K)$ is universal. But clearly $f^{-1}(K) = K$, so K is a minimal subshift, thus (i) holds. $\square$

Clearly if Γ is minimal subshift universal and there is a surjective homomorphism of Δ on Γ, then Δ is minimal subshift universal. The existence of a minimal subshift universal group was first proved by Brandon Seward, who showed that $\mathbb{F}_\infty$ has this property. Corollary 3.6.5 shows that any wreath product $\Gamma \wr \Lambda$, where Γ is infinite and Λ contains $\mathbb{F}_2$, is minimal subshift universal and in particular by Corollary 3.6.6, $\mathbb{F}_3$ is minimal subshift universal. We include below Seward's proof for $\mathbb{F}_\infty$ (and somewhat more), with his permission, as it is based on a very different method.

Theorem 3.6.10 (Seward)**.** *Let Γ be a group. Then $E(\Gamma, 2) \sqsubseteq_B E(\Gamma * \mathbb{F}_\infty, 2)\restriction_M$, where $M \subseteq 2^{\Gamma * \mathbb{F}_\infty}$ denotes the set of minimal points.*

Proof. We start with the following lemma.

Lemma 3.6.11. *Let Γ be a group and let $A \subseteq \Gamma$ be a finite subset. There is a Γ-equivariant Borel embedding $x \mapsto x'$ from $E(\Gamma, 2)$ to $E(\Gamma * \mathbb{Z}, 2)$ such that for every $x \in 2^\Gamma$,*

(i) $x'\restriction_\Gamma = x$ (i.e. x' extends x),

(ii)$((\Gamma * \mathbb{Z}) \cdot x')\restriction_\Gamma \subseteq \Gamma \cdot x$,

(iii) the $\mathbb{Z}$-action on $(\Gamma * \mathbb{Z}) \cdot x'$ factors via the restriction map $2^{\Gamma * \mathbb{Z}} \to 2^A$ to a transitive action (on the image).

Proof. Fix an enumeration of Γ, and let t denote the generator of $\mathbb{Z}$. For every nonempty subset P of 2^A, fix a transitive permutation σ_P of P. Let $x \in 2^\Gamma$, and let $P_x = (\Gamma \cdot x)\restriction_A$ be the set of A-patterns appearing in x. We define x' inductively on left cosets of Γ, starting with $x'\restriction_\Gamma = x$.

Let tw be a reduced word in $2^{\Gamma * \mathbb{Z}}$ for which $(w \cdot x')\restriction_\Gamma$ is already defined. Then set $(tw \cdot x')\restriction_\Gamma = \gamma \cdot ((w \cdot x')\restriction_\Gamma)$, where γ is minimal with $(\gamma \cdot ((w \cdot x')\restriction_\Gamma))\restriction_A = \sigma_{P_x}((w \cdot x')\restriction_A)$.

Similarly, if $t^{-1}w$ is a reduced word for which $(w \cdot x')\restriction_\Gamma$ is defined, then set $(t^{-1}w \cdot x')\restriction_\Gamma = \gamma^{-1} \cdot ((w \cdot x')\restriction_\Gamma)$, where γ is minimal with $(\gamma^{-1} \cdot ((w \cdot x')\restriction_\Gamma))\restriction_A = \sigma_{P_x}^{-1}((w \cdot x')\restriction_A)$. □

Let now $t_0, t_1, t_2, \ldots$ be the free generators of $\mathbb{F}_\infty$, and let $\Gamma_n = \Gamma * \langle t_i \rangle_{i<n}$ (this includes the case $n = \infty$). Let $(A_n)_n$ be an exhaustive increasing sequence of finite subsets of Γ_∞ such that $A_n \subseteq \Gamma_n$. For every n, apply the lemma with Γ_n and A_n to obtain a Borel embedding $E(\Gamma_n, 2) \sqsubseteq_B E(\Gamma_{n+1}, 2)$. Given $x_0 \in 2^{\Gamma_0}$, let x_1 denote the extension to 2^{Γ_1} of x_0, let x_2 denote the extension to 2^{Γ_2} of x_1, and so on for $x_n \in 2^{\Gamma_n}$. Let $x_\infty = \bigcup_n x_n$. We claim that for every n and every $m > n$ (including $m = \infty$), the $\langle t_n \rangle$-action on $\Gamma_m \cdot x_m$ factors via the restriction map $2^{\Gamma_m} \to 2^{A_n}$ to a transitive $\langle t_n \rangle$-action on the image. It suffices to show this for finite m. We proceed by induction on m, for which the base case $m = n + 1$ holds by the lemma. Now suppose that this holds for m. Let $\gamma \in \Gamma_{m+1}$. Then by the lemma, $(\gamma \cdot x_{m+1})\restriction_{\Gamma_m} = h \cdot x_m$ for some $h \in \Gamma_m$, and thus

$$(t_m\gamma \cdot x_{m+1})\restriction_{\Gamma_m} = t_m \cdot ((\gamma \cdot x_{m+1})\restriction_{\Gamma_m}) = t_m \cdot (h \cdot x_m) = t_m h \cdot x_m$$

so

$$(t_m\gamma \cdot x_{m+1})\restriction_{A_n} = (t_m\gamma \cdot x_{m+1}\restriction_{\Gamma_m})\restriction_{A_n} = (t_m h \cdot x_m)\restriction_{A_n}$$

which only depends on $(h \cdot x_m)\restriction_{A_n} = (\gamma \cdot x_{m+1})\restriction_{A_n}$, so the $\langle t_n \rangle$-action factors through $2^{\Gamma_{m+1}} \to 2^{A_n}$, and the action is clearly still transitive.

We show that the map $x_0 \mapsto x_\infty$ is the desired embedding. It is clearly a Γ_0-invariant Borel injection. To see that it is a cohomomorphism, if $(x_\infty, y_\infty) \in E(\Gamma_\infty, 2)$, then $x_\infty = \gamma \cdot y_\infty$ for some $\gamma \in \Gamma_\infty$. Now $\gamma \in \Gamma_n$ for some n, so $x_n = \gamma \cdot y_n$, and thus $(x_n, y_n) \in E(\Gamma_n, 2)$. Since each extension map is a cohomomorphism, we have $(x_0, y_0) \in E(\Gamma_0, 2)$.

It remains to show that the image lies in M. Fix x_∞ and let $A \subseteq \Gamma_\infty$. We show that the set $\{\gamma \in \Gamma_\infty : (\gamma \cdot x_\infty)\restriction_A = x_\infty\restriction_A\}$ is left syndetic. By enlarging A, we can assume that $A = A_n$ for some n. Let $T = \{(t_n)^k : 0 \leq k < 2^{|A_n|}\}$. Now let $\gamma \in \Gamma_\infty$. Then by transitivity, there is some $0 \leq k < 2^{|A_n|}$ for which $((t_n)^k\gamma \cdot x)\restriction_{A_n} = x_\infty\restriction_{A_n}$, so we are done. □

We can now restate Problem 3.6.7, in a more general form, as follows:

Problem 3.6.12. *Is $\mathbb{F}_2$ minimal subshift universal? More generally, is every group that contains $\mathbb{F}_2$ minimal subshift universal?*

3.7 Subshifts as tests for amenability

It is well known that a group Γ is amenable iff every Γ-flow admits an invariant Borel probability measure. Call a class $\mathcal{F}$ of Γ-flows a **test for amenability** for Γ if Γ is amenable provided that every Γ-flow in $\mathcal{F}$ admits an invariant Borel probability measure. In [GdlH97] a compact metrizable space X is called a **test space** for the amenability of Γ if the class of all Γ-flows on X is a test for amenability for Γ. Giordano and de la Harpe show in [GdlH97] that the Cantor space $2^{\mathbb{N}}$ is a test space for amenability of any group. Equivalently this says that the class of all subshifts of $(2^{\mathbb{N}})^\Gamma$ is a test of amenability for Γ. We show next that the strongest result along these lines is actually true, namely that the class of all subshifts of 2^Γ is a test of amenability for Γ. This gives another characterization of amenability.

Theorem 3.7.1. *Let Γ be a group. Then Γ is amenable iff every subshift of 2^Γ admits an invariant Borel probability measure.*

Proof. We have to show that if Γ is not amenable then there is a compressible subshift of 2^Γ. We will first give a proof for the case that Γ contains $\mathbb{F}_2$, which is much simpler, and then give the full proof for arbitrary non-amenable Γ.

Proof when $\Gamma \geq \mathbb{F}_2$.

It suffices to find a compressible $\mathbb{F}_2$-flow with a 2-generator, since if $\mathbf{a}$ is such an $\mathbb{F}_2$-flow, then $\mathrm{CInd}_{\mathbb{F}_2}^{\Gamma}(\mathbf{a})$ is a compressible Γ-flow with a 2-generator by Corollary 3.6.2. For the existence of such a $\mathbb{F}_2$-flow, see the proof of Theorem 3.2.6.

Proof for all non-amenable Γ.

By nonamenability, there is a finite symmetric subset $S \subseteq \Gamma$ containing 1 such that:

(i) for every finite $F \subseteq \Gamma$, we have $|FS| \geq 2|F|$;
(ii) there is an integer n with

$$4 + 3\log_2(|S|) \leq n \leq \frac{|S| - 6}{3\log_2(|S|)}$$

(iii) there is some $r \in S$ with $r^2 \neq 1$.

Let $T = S^n$. Given a point $x \in (T \sqcup \{*\})^\Gamma$, let $\mathrm{Supp}(x)$ denote the set of $\gamma \in \Gamma$ such that $x_\gamma \neq *$. Let X be the subshift of $(T \sqcup \{*\})^\Gamma$ such that $x \in X$ iff the following hold:

(i) $\mathrm{Supp}(x)$ is maximal right S^3-disjoint (a subset $A \subseteq \Gamma$ is **right S^3-disjoint** if for any distinct $a, a' \in \mathrm{Supp}(A)$, we have $a' \notin aS^3$);

(ii) the function $\gamma \mapsto \gamma x_\gamma$ is a 2-to-1 surjection from $\mathrm{Supp}(x)$ onto $\mathrm{Supp}(x)$.

We claim that X is the desired subshift. We first recall a fact from graph theory.

Lemma 3.7.2. *Let G be a locally finite (not necessarily simple) graph with vertex set V, such that every finite $F \subseteq V$ satisfies $|N_G(F)| \geq k|F|$, where $N_G(F)$ denotes the set of neighbours of F. Then there is a k-to-1 surjection $p\colon V \to V$ such that for every $v \in V$, there is an edge from v to $p(v)$.*

Proof. Consider the bipartite graph B with bipartition (V_l, V_r), where $V_l = V_r = V$, and where there is an edge from $v \in V_l$ to $w \in V_r$ if vw is an edge in V. Then every finite $F \subseteq V_l$ satisfies $|N_B(F)| \geq |F|$, and every finite $F \subseteq V_r$ satisfies $|N_B(F)| \geq k|F|$, so by Hall's theorem [TW16, C.4(b)], there are matchings $(M_i)_{i<k}$ such that every vertex in V_l is covered by a unique M_i, and every vertex is V_r is covered by every M_i. Then $\bigcup_{i<k} M_i$ is (the graph of) the desired k-to-1 surjection. $\square$

We show that X is nonempty. Let $A \subseteq \Gamma$ be any maximal right S^3-disjoint subset, and consider A as a (non-simple) graph where a and a' are adjacent iff $a' \in aT$. Let $F \subseteq A$ be a finite subset. By maximality of A, every element of FS^{n-3} is within S^3 of some element of $FT \cap A$, and thus

$$|FT \cap A| \geq \frac{|FS^{n-3}|}{|S^3|} \geq \frac{2^{n-3}|F|}{|S|^3} \geq 2|F|$$

by our choice of n. Thus by Lemma 3.7.2, there is a 2-to-1 surjection $p\colon A \to A$ such that $p(a) \in aT$ for every $a \in A$. Define $x \in X^\Gamma$ by

$$x_\gamma = \begin{cases} \gamma^{-1}p(\gamma) & \gamma \in A, \\ * & \text{otherwise.} \end{cases}$$

Then $x \in X$.

Next, we show that X is a compressible subshift. Let $Y \subseteq X$ be the set of $x \in X$ with $1 \in \mathrm{Supp}(x)$. Consider the Borel map $Y \to Y$ defined by $y \mapsto y_1^{-1} \cdot y$. This is a 2-to-1 surjection, since the preimage of $y \in Y$ is the set $\{\gamma^{-1} \cdot y : \gamma y_\gamma = 1\}$. Thus Y is a compressible subset, so since Y is a complete section, X is also a compressible subset.

It remains to show that there is a clopen 2-generator.

Recall that if G is a finite graph with maximum degree d, then every independent set $I \subseteq G$ can be extended to an independent set of size at least $\frac{|G|}{d+1}$. To see this, note that for any set $X \subseteq G$, we have that $|N_G[X]| \leq (d+1)|X|$, where $N_G[X]$ is the set of vertices within distance 1 of X. If I is maximal independent, then clearly $N_G[I] = G$ and we are done.

Consider S as a graph where s and s' are adjacent iff $s' = sr^{\pm 1}$ and $\{s, s'\} \neq \{1, r\}$. Then by the above, there is an independent set $S' \supseteq \{1, r\}$ of size at least $\frac{|S|}{3}$. Fix an injection $\varphi : T \hookrightarrow 2^{S'}$ such that $\varphi(t)_1 = \varphi(t)_r = 1$ for every $t \in T$; this is possible since

$$\log_2(|T|) \leq n \log_2(|S|) \leq \frac{|S|}{3} - 2 \leq |S'| - 2$$

by our choice of n. Define the continuous map $f \colon X \to 2$ by

$$f(x) = \begin{cases} \varphi(x_{s^{-1}})_s & s^{-1} \in \mathrm{Supp}(x) \text{ for some } s \in S' \\ 0 & \text{otherwise} \end{cases}$$

This is well-defined, since if s_0^{-1} and s_1^{-1} are both in $\mathrm{Supp}(x)$, then since $\mathrm{Supp}(x)$ is right S^3-disjoint, we have $s_0 = s_1$.

We claim that

$$\gamma \in \mathrm{Supp}(x) \iff f(\gamma^{-1} \cdot x) = f((\gamma r)^{-1} \cdot x) = 1$$

For ($\implies$), since $x_\gamma \in T$, we have $\varphi(x_\gamma)_1 = \varphi(x_\gamma)_r = 1$, which is equivalent to what we need. For ($\impliedby$), we must have some $s_0, s_1 \in S'$ such that $s_0^{-1} \in \mathrm{Supp}(\gamma^{-1} \cdot x)$ and $s_1^{-1} \in \mathrm{Supp}((\gamma r)^{-1} \cdot x)$. Thus $\gamma s_0^{-1}, \gamma r s_1^{-1} \in \mathrm{Supp}(x)$, but since $\mathrm{Supp}(x)$ is right S^3-disjoint, we get that $\gamma s_0^{-1} = \gamma r s_1^{-1}$, i.e. $s_1 = s_0 r$. Thus by our choice of S', we have $\{s_0, s_1\} = \{1, r\}$, and since $r^2 \neq 1$, we have $s_0 = 1$. and thus $1 \in \mathrm{Supp}(\gamma^{-1} \cdot x)$, i.e. $\gamma \in \mathrm{Supp}(x)$.

We now show that f is a generator. Let $x, x' \in X$, and suppose that $f(\gamma \cdot x) = f(\gamma \cdot x')$ for every $\gamma \in \Gamma$. Then by above, we have that $\mathrm{Supp}(x) = \mathrm{Supp}(x')$. If $\gamma \notin \mathrm{Supp}(x)$, then $x_\gamma = * = x_{\gamma'}$. If $\gamma \in \mathrm{Supp}(x)$, then for any $s \in S'$, we have

$$\varphi(x_\gamma)_s = f((\gamma s)^{-1} \cdot x) = f((\gamma s)^{-1} \cdot x') = \varphi(x'_\gamma)_s$$

So since φ is injective, we have $x_\gamma = x'_\gamma$. Thus $x = x'$, and f is a generator. □

It turns out that if one is willing to replace 2^Γ by k^Γ, where k depends on Γ, it is easier to get compressible subshifts.

Fix a group Γ. For finite subsets S and T of Γ, denote by $X_{S,T}$ the space of (S, T)-paradoxical decompositions of Γ, that is, the subshift of $(S \sqcup T)^\Gamma$ such that $x \in X_{S,T}$ iff $\{x^{-1}(s)s\}_{s \in S}$ and $\{x^{-1}(t)t\}_{t \in T}$ are both partitions of Γ (we allow pieces of a partition to be empty).

For a finite subset T of Γ, denote by X_T be the space of 2-to-1 T-surjections of Γ, that is, the subshift of T^Γ such that $x \in X_T$ iff the map $\Gamma \to \Gamma$ defined by $\gamma \mapsto \gamma x_\gamma$ is a 2-to-1 surjection.

Note that $X_{S,T}$ is a subset of $X_{S\cup T}$. Also, Γ is non-amenable iff $X_{S,T}$ is nonempty for some S and T iff X_T is nonempty for some T.

The Tarski number k_Γ of Γ is minimum of $|S| + |T|$ over all S and T with $X_{S,T}$ nonempty (it's the smallest number of pieces in a paradoxical decomposition). There is a number l_Γ which is the minimum of $|T|$ over all T with X_T nonempty, or equivalently, the minimum of $|S \cup T|$ over all S and T with $X_{S,T}$ nonempty (it's the smallest number of group elements required in a paradoxical decomposition). Note that we have $l_\Gamma < k_\Gamma$ for any non-amenable Γ, since if $X_{S,T}$ is nonempty, then $X_{S,T\gamma}$ is nonempty for any γ, and thus we can assume that S and T have at least one element in common, i.e. $|S \cup T| < |S| + |T|$. Note that by [EGS15] there are groups Γ with arbitrarily large k_Γ.

Proposition 3.7.3. *$X_{S,T}$ and X_T are compressible. Thus if Γ is non-amenable, then there is a compressible subshift of $(l_\Gamma)^\Gamma$.*

So, for example, this easily gives a compressible subshift of $3^{\mathbb{F}_2}$.

Proof. For $X_{S,T}$, let P and Q be the set of $x \in X_{S,T}$ such that $x_1 \in S$ and $x_1 \in T$ respectively. Then the map defined by $x \mapsto x_1^{-1}x$ is a bijection from $P \to X_{S,T}$ and a bijection $Q \to X_{S,T}$, so $X_{S,T}$ is equidecomposable with two copies of itself, and thus it is compressible.

For X_T, let P be the set of $x \in X_T$ such that x_1 is the least of the two elements of $\{\gamma : \gamma x_\gamma = x_1\}$ (in some fixed ordering). Then proceed as above. $\square$

From Theorem 3.7.1 a group Γ is non-amenable iff there is a compressible subshift of 2^Γ. The following question asks whether an analogous characterization exists for groups that contain $\mathbb{F}_2$.

Problem 3.7.4. *Is it true that a group Γ contains $\mathbb{F}_2$ iff there is a compressible, orbit-universal subshift of 2^Γ?*

3.8 The space of subshifts

(A) We will first review the standard embedding of actions into the shift action. Consider a continuous action of a countable group Γ on a Polish space Y and let Y be a closed subspace of a Polish space X. Define $f\colon Y \to X^\Gamma$ by

$$f(y)_\gamma = \gamma^{-1} \cdot y.$$

Then it is easy to check that f is Γ-equivariant, $f(Y)$ is a closed subset of X^Γ and f is a homeomorphism of Y with $f(Y)$, i.e., the action of Γ on Y is (topologically) isomorphic to a subshift of X^Γ, where of course Γ acts on itself by left translation.

For any Polish space X, define the standard Borel space of subshifts of X^Γ as follows:

$$\mathrm{Sh}(\Gamma, X) = \{F \in F(X^\Gamma) : F \text{ is } \Gamma\text{-invariant}\}$$

If X is compact, we view this as a compact Polish space with the Vietoris topology.

Consider the Hilbert cube $\mathbb{I}^\mathbb{N}$. Every compact Polish space is (up to homeomorphism) a closed subspace of $\mathbb{I}^\mathbb{N}$, and thus every Γ-flow is (topologically) isomorphic to a subshift of $(\mathbb{I}^\mathbb{N})^\Gamma$. We can thus consider the compact Polish space $\mathrm{Sh}(\Gamma, \mathbb{I}^\mathbb{N})$ as the universal space of Γ-flows.

Similarly consider the product space $\mathbb{R}^\mathbb{N}$. Every Polish space is (up to homeomorphism) a closed subspace of $\mathbb{R}^\mathbb{N}$, and thus every continuous Γ-action is (topologically) isomorphic to a subshift of $(\mathbb{R}^\mathbb{N})^\Gamma$. We can thus consider the standard Borel space $\mathrm{Sh}(\Gamma, \mathbb{R}^\mathbb{N})$ as the universal space of continuous Γ-actions.

In particular taking $\Gamma = \mathbb{F}_\infty$, the free group with a countably infinite set of generators, we see that every CBER is Borel isomorphic to the equivalence relation E_F induced on some subshift F of $(\mathbb{R}^\mathbb{N})^{\mathbb{F}_\infty}$ and so we can view $\mathrm{Sh}(\mathbb{F}_\infty, \mathbb{R}^\mathbb{N})$ also as the universal space of CBER and study the complexity of various classes of CBER (like, e.g., smooth, aperiodic, hyperfinite, etc.) as subsets of this universal space. Similarly we can view $\mathrm{Sh}(\mathbb{F}_\infty, \mathbb{I}^\mathbb{N})$ as the universal space of CBER that admit a compact action realization. In this case we can also consider complexity questions as well as generic questions of various classes.

(B) Let Φ be a property of continuous Γ-actions on Polish spaces which is invariant under (topological) isomorphism. Let

$$\mathrm{Sh}_\Phi(\Gamma, X) = \{F \in \mathrm{Sh}(\Gamma, X) : F \models \Phi\},$$

where we write $F \models \Phi$ to mean that F has the property Φ.

Denote by $\mathrm{Prob}(\Gamma)$ the closed Γ-subspace of $[0,1]^\Gamma$ consisting of probability measures on Γ.

A Borel action $\Gamma \curvearrowright X$ on a standard Borel space is **Borel amenable** if there is a sequence of Borel maps $p_n : X \to \mathrm{Prob}(\Gamma)$ such that $\|p_n^{\gamma \cdot x} - \gamma \cdot p_n^x\|_1 \to 0$ for every $\gamma \in \Gamma$ and $x \in X$. If μ is a Borel probability measure on X, then $\Gamma \curvearrowright X$ is **$\boldsymbol{\mu}$-amenable** if there is a Γ-invariant μ-conull Borel subset of X where the action is Borel amenable. The action $\Gamma \curvearrowright X$ is **measure-amenable** if it is μ-amenable for every μ.

Let Γ be a countable group, and let X be a Polish space. A continuous action $\Gamma \curvearrowright X$ is **topologically amenable** if for every finite $S \subseteq \Gamma$, every compact $K \subseteq X$, and every $\varepsilon > 0$, there is some continuous $p\colon X \to \mathrm{Prob}(\Gamma)$, such that, letting $p^x = p(x)$, we have

$$\max_{\substack{\gamma \in S \\ x \in K}} \|p^{\gamma \cdot x} - \gamma \cdot p^x\|_1 < \varepsilon.$$

By Theorem A.3.1, measure-amenability is equivalent to **topological amenability**.

A countable discrete group Γ is **exact** if it admits a measure-amenable (equiv. topologically amenable) action on a compact Polish space, in which case there exists such an action on the Cantor space $2^{\mathbb{N}}$, since every compact Γ-flow extends to a Γ-flow on $2^{\mathbb{N}}$, see [GdlH97].

A CBER E on X is ***μ*-hyperfinite** if if there is an invariant μ-conull Borel subset of X where the action is hyperfinite and it is **measure-hyperfinite** if it is μ-hyperfinite with respect to every μ.

A CBER E on X is **amenable** if there is a sequence $p_n\colon E \to [0,1]$ of Borel functions such that $p_n^x \in \mathrm{Prob}([x]_E)$ for every $x \in X$, and for every $(x,y) \in E$, we have $\|p_n^x - p_n^y\|_1 \to 0$ in $\ell^1([x]_E)$. Analogously we define what it means to say that E is ***μ*-amenable** and **measure-amenable**. By the Connes-Feldman-Weiss theorem, see, e.g., [KM04, Theorem 10.1], E is μ-amenable iff E is μ-hyperfinite and thus E is measure-amenable iff it is measure-hyperfinite.

We will consider below the following Φ:

1) fin: finite equivalence relation;

2) sm: smooth equivalence relation;

3) free: free action;

4) aper: aperiodic equivalence relation;

5) comp: compressible equivalence relation;

6) hyp: hyperfinite equivalence relation;

7) amen: amenable equivalence relation;

8) measHyp: measure-hyperfinite equivalence relation

9) freeMeasHyp: free action + measure-hyperfinite equivalence relation;

10) measAmen: measure-amenable action.

We summarize in the following table what we can prove concerning the descriptive or generic properties of the Φ above:

<table>
<tr><th>Φ</th><th>$\mathrm{Sh}_\Phi(\Gamma, \mathbb{I}^\mathbb{N})$</th><th colspan="2">$\mathrm{Sh}_\Phi(\Gamma, \mathbb{R}^\mathbb{N})$</th></tr>
<tr><td>fin</td><td rowspan="2">meager</td><td colspan="2" rowspan="2">$\mathbf{\Pi}^1_1$-complete</td></tr>
<tr><td>sm</td></tr>
<tr><td>free</td><td rowspan="3">comeager</td><td rowspan="2">G_δ</td><td rowspan="3">$\mathbf{\Pi}^1_1$-complete</td></tr>
<tr><td>aper</td></tr>
<tr><td>comp</td><td>open</td></tr>
<tr><td>hyp</td><td>?</td><td colspan="2" rowspan="2">$\mathbf{\Sigma}^1_2$, $\mathbf{\Pi}^1_1$-hard</td></tr>
<tr><td>amen</td><td rowspan="4">comeager</td></tr>
<tr><td>measHyp</td><td colspan="2">$\mathbf{\Pi}^1_1$-complete</td></tr>
<tr><td>freeMeasHyp</td><td rowspan="2">G_δ</td><td rowspan="2">$\mathbf{\Pi}^1_1$-complete</td></tr>
<tr><td>measAmen</td></tr>
</table>

In this table, Γ is an infinite group, Γ is residually finite in the "$\mathbf{\Pi^1_1}$-complete" entry of the first two rows, Γ is non-amenable in the "comeager" entry of the fifth row, Γ is non-amenable and residually finite in the "$\mathbf{\Pi^1_1}$-hard" and "$\mathbf{\Pi^1_1}$-complete" entries of the last five rows, and Γ is exact in the "comeager" entry of the last four rows.

The following two problems are open:

Problem 3.8.1. *Let Γ be an infinite group. Is $\mathrm{Sh}_{\mathrm{hyp}}(\Gamma, \mathbb{I}^\mathbb{N})$ comeager in $\mathrm{Sh}(\Gamma, \mathbb{I}^\mathbb{N})$?*

Problem 3.8.2. *Let Γ be an infinite group. What is the exact descriptive complexity of $\mathrm{Sh}_{hyp}(\Gamma, \mathbb{I}^\mathbb{N})$ in $\mathrm{Sh}(\Gamma, \mathbb{I}^\mathbb{N})$?*

Note that from the results in the 5th row, it follows that a countable group Γ is amenable iff the generic subshift of $(\mathbb{I}^\mathbb{N})^\Gamma$ admits an invariant probability Borel measure.

We will now prove the results in the table in a series of propositions. A property Φ of continuous actions of Γ on Polish spaces, invariant under topological isomorphism is:

- **satisfiable** if some Polish Γ-space satisfies Φ;
- **compactly satisfiable** if some (compact) Γ-flow satisfies Φ;

- **product-stable** if for any (compact) Γ-flows $\mathbf{a}$ and $\mathbf{b}$, if $\mathbf{a}$ satisfies Φ, then $\mathbf{a} \times \mathbf{b}$ satisfies Φ.

Proposition 3.8.3. *Let Φ be a compactly satisfiable, product-stable property. Then the set*

$$\{K \in \mathrm{Sh}(\Gamma, \mathbb{I}^{\mathbb{N}}) : K \text{ satisfies } \Phi\}$$

is dense in $\mathrm{Sh}(\Gamma, \mathbb{I}^{\mathbb{N}})$.

Proof. Since $\mathbb{I}^{\mathbb{N}}$ is the inverse limit of the spaces $\mathbb{I}^n$, we have that $\mathrm{Sh}(\Gamma, \mathbb{I}^{\mathbb{N}})$ is the inverse limit of $(\mathrm{Sh}(\Gamma, \mathbb{I}^n))_n$. Thus it suffices to show, for every $n \in \mathbb{N}$ and every nonempty open $U \subseteq \mathrm{Sh}(\Gamma, \mathbb{I}^n)$, that some subshift in $\pi_n^{-1}(U)$ satisfies Φ, where $\pi_n : \mathrm{Sh}(\Gamma, \mathbb{I}^{\mathbb{N}}) \to \mathrm{Sh}(\Gamma, \mathbb{I}^n)$ is the projection map. Fix $K \in U$, and fix $L \in \mathrm{Sh}(\Gamma, \mathbb{I}^{\mathbb{N} \setminus n})$ satisfying Φ. Then $K \times L$ satisfies Φ by product stability, and is contained in $\pi_n^{-1}(U)$, so we are done. $\square$

For compact Polish X, a subset $\mathcal{I} \subseteq \mathrm{Sh}(\Gamma, X)$ is a ***σ*-ideal** if the following hold:

i) if $K \in \mathcal{I}$, $L \in \mathrm{Sh}(\Gamma, X)$ and $L \subseteq K$, then $L \in \mathcal{I}$;

ii) if $K \in \mathrm{Sh}(\Gamma, X)$ and $K = \bigcup_n K_n$ for some countable sequence $K_n \in \mathcal{I}$, then $K \in \mathcal{I}$.

Every $\mathrm{Sh}_\Phi(\Gamma, X)$ in the above table is a σ-ideal. We will need the following to show $\mathbf{\Pi}^1_1$-hardness. It is an analog of [KLW87, Section 1.4, Theorem 7] and can be proved by the same argument which we repeat here for the convenience of the reader.

Proposition 3.8.4. *Let X be a compact Polish space and let $\mathcal{I}$ be a σ-ideal in* $\mathrm{Sh}(\Gamma, X)$*. If $\mathcal{I}$ is F_σ-hard, then $\mathcal{I}$ is $\mathbf{\Pi}^1_1$-hard.*

Proof. There is a continuous map $2^{\mathbb{N}} \to \mathrm{Sh}(\Gamma, X)$ reducing $2^{<\mathbb{N}} \subseteq 2^{\mathbb{N}}$ to $\mathcal{I}$, which we will denote by $\alpha \mapsto K_\alpha$. Then the continuous map $K(2^{\mathbb{N}}) \to \mathrm{Sh}(\Gamma, X)$ defined by $A \mapsto \bigcup_{\alpha \in A} K_\alpha$ reduces $K(2^{<\mathbb{N}}) = \{K \in K(2^{\mathbb{N}}) : K \subseteq 2^{<\mathbb{N}}\}$ to $\mathcal{I}$, since for every $A \in K(2^{\mathbb{N}})$, we have

$$\begin{aligned} A \subseteq 2^{<\mathbb{N}} &\implies K_\alpha \in \mathcal{I} \text{ for all } \alpha \in A, \text{ and } A \text{ is countable} \\ &\implies \bigcup_{\alpha \in A} K_\alpha \in \mathcal{I} \\ &\implies K_\alpha \in \mathcal{I} \text{ for all } \alpha \in A \\ &\implies A \subseteq 2^{<\mathbb{N}}. \end{aligned}$$

So the result follows, since $K(2^{<\mathbb{N}})$ is $\mathbf{\Pi}^1_1$-hard (see [Kec95, 27.4(ii)]). $\square$

For a subset $(F_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ of $\mathrm{Sh}(\Gamma,\mathbb{R}^\mathbb{N})$, there is a closed Γ-invariant subspace of the product Γ-space $\mathbb{N}^\mathbb{N}\times((\mathbb{R}^\mathbb{N})^\Gamma)^\mathbb{N}$ (where Γ acts trivially on $\mathbb{N}^\mathbb{N}$), given by

$$\coprod_{\alpha\in\mathbb{N}^\mathbb{N}}\prod_n F_{\alpha\restriction n}=\{(\alpha,(x_n)_n)\in\mathbb{N}^\mathbb{N}\times((\mathbb{R}^\mathbb{N})^\Gamma)^\mathbb{N}:\forall n\,[x_n\in F_{\alpha\restriction n}]\}.$$

Fixing a closed embedding $\mathbb{N}^\mathbb{N}\times((\mathbb{R}^\mathbb{N})^\Gamma)^\mathbb{N}\hookrightarrow\mathbb{R}^\mathbb{N}$, we obtain an element of $\mathrm{Sh}(\Gamma,\mathbb{R}^\mathbb{N})$, which we denote by $\mathcal{A}_sF_s$.

Proposition 3.8.5. *Let Φ and Ψ be disjoint satisfiable properties of Polish Γ-spaces such that*

i) there is some $F_\Phi\in\mathrm{Sh}(\Gamma,\mathbb{R}^\mathbb{N})$ such that for every subset $(F_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ of $\mathrm{Sh}(\Gamma,\mathbb{R}^\mathbb{N})$ such that $\{s\in\mathbb{N}^{<\mathbb{N}}:F_s\neq F_\Phi\}$ is well-founded, we have that $\mathcal{A}_sF_s$ satisfies Φ;

ii) there is some $F_\Psi\in\mathrm{Sh}(\Gamma,\mathbb{R}^\mathbb{N})$ such that for every subset $(F_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ of $\mathrm{Sh}(\Gamma,\mathbb{R}^\mathbb{N})$ such that $\{s\in\mathbb{N}^{<\mathbb{N}}:F_s=F_\Psi\}$ is ill-founded, we have that $\mathcal{A}_sF_s$ satisfies Ψ.

Then $\mathrm{Sh}_\Phi(\Gamma,\mathbb{R}^\mathbb{N})$ is $\mathbf{\Pi}^1_1$-hard.

Proof. Let $\mathrm{Tr}\subseteq 2^{\mathbb{N}^{<\mathbb{N}}}$ denote the space of trees, and let $\mathrm{WF}\subseteq\mathrm{Tr}$ be the subset of well-founded trees, which is $\mathbf{\Pi}^1_1$-complete; see [Kec95, 33.A].

For every $T\in\mathrm{Tr}$ and $s\in\mathbb{N}^{<\mathbb{N}}$, define $F^T_s\in\mathrm{Sh}_\Phi(\Gamma,\mathbb{R}^\mathbb{N})$ by

$$F^T_s:=\begin{cases}F_\Phi & s\notin T\\ F_\Psi & s\in T\end{cases}$$

Then

$$\begin{aligned}T\in\mathrm{WF}&\implies\mathcal{A}_sF^T_s\models\Phi\\ T\notin\mathrm{WF}&\implies\mathcal{A}_sF^T_s\models\Psi\end{aligned}$$

so the Borel map $T\mapsto\mathcal{A}_sF^T_s$ is a reduction from WF to $\mathrm{Sh}_\Phi(\Gamma,\mathbb{R}^\mathbb{N})$, whence the latter is $\mathbf{\Pi}^1_1$-hard. $\square$

Proposition 3.8.6. *Let Γ be a countably infinite group, and let $\Phi\in\{\mathrm{free},\mathrm{aper}\}$. Then $\mathrm{Sh}_\Phi(\Gamma,\mathbb{I}^\mathbb{N})$ is dense G_δ, and $\mathrm{Sh}_\Phi(\Gamma,\mathbb{R}^\mathbb{N})$ is $\mathbf{\Pi}^1_1$-complete.*

Proof. For every $\gamma\in\Gamma$, the set of fixed points of γ in $(\mathbb{I}^\mathbb{N})^\Gamma$ (resp., $(\mathbb{R}^\mathbb{N})^\Gamma$) is closed (resp., Borel). Thus the set of points with free orbit is G_δ (resp., Borel), whence $\mathrm{Sh}_{\mathrm{free}}(\Gamma,\mathbb{I}^\mathbb{N})$ is G_δ (resp., $\mathrm{Sh}_{\mathrm{free}}(\Gamma,\mathbb{R}^\mathbb{N})$ is $\mathbf{\Pi}^1_1$). Similarly, the set of aperiodic points in $(\mathbb{I}^{\mathbb{N}})^\Gamma$ (resp., $(\mathbb{R}^{\mathbb{N}})^\Gamma$) is G_δ (resp., Borel), so $\mathrm{Sh}_{\mathrm{aper}}(\Gamma, I^{\mathbb{N}})$ is G_δ (resp., $\mathrm{Sh}_{\mathrm{free}}(\Gamma, \mathbb{R}^{\mathbb{N}})$ is $\mathbf{\Pi}^1_1$).

The property Φ is compactly satisfiable (see, e.g. [KPT05, 1(B)]) and product-stable, so density of $\mathrm{Sh}_\Phi(\Gamma, \mathbb{I}^{\mathbb{N}})$ follows from Proposition 3.8.3.

Finally, $\mathbf{\Pi}^1_1$-completeness follows from Proposition 3.8.5 by taking Ψ to be "has a fixed point". □

Proposition 3.8.7. *Let Γ be a countably infinite group. Then $\mathrm{Sh}_{\mathrm{comp}}(\Gamma, \mathbb{I}^{\mathbb{N}})$ is open, $\mathrm{Sh}_{\mathrm{comp}}(\Gamma, \mathbb{R}^{\mathbb{N}})$ is $\mathbf{\Pi}^1_1$-complete, and if Γ is non-amenable, then the former is dense.*

Proof. By Nadkarni's theorem, F is non-compressible iff

$$\exists \mu \in P(F)\, \forall \gamma\, [\gamma \cdot \mu = \mu],$$

where $P(F)$ is the set of Borel probability measures on F, which is a compact Polish space for $\mathbb{I}^{\mathbb{N}}$ and a standard Borel space for $\mathbb{R}^{\mathbb{N}}$. Thus the set of compressible subshifts is open for $\mathbb{I}^{\mathbb{N}}$, and $\mathbf{\Pi}^1_1$ for $\mathbb{R}^{\mathbb{N}}$. Moreover, $\mathbf{\Pi}^1_1$-completeness follows from Proposition 3.8.5 by taking Φ to be "compressible" and Ψ to be "has a fixed point", and taking any F_Φ satisfying Φ and any F_Ψ satisfying Ψ.

Now suppose Γ is non-amenable. Then compressibility is compactly satisfiable by non-amenability, and it is product-stable, so density follows from Proposition 3.8.3. □

Proposition 3.8.8. *Let Γ be a countably infinite group, let X be a Polish space, and let $\Phi \in \{\mathrm{fin}, \mathrm{sm}\}$. Then $\mathrm{Sh}_\Phi(\Gamma, X)$ is $\mathbf{\Pi}^1_1$, and if $X = \mathbb{I}^{\mathbb{N}}$, then it is meager.*

Proof. The set of periodic points in X^Γ is Borel. Also, a subshift is smooth iff every orbit is discrete (see, e.g., [Kec10, Corollary 22.3]). The set of points with discrete orbit is Borel. So in either case, $\mathrm{Sh}_\Phi(\Gamma, X)$ is $\mathbf{\Pi}^1_1$.

If $X = \mathbb{I}^{\mathbb{N}}$, then meagerness follows since $\mathrm{Sh}_\Phi(\Gamma, \mathbb{I}^{\mathbb{N}})$ is disjoint from $\mathrm{Sh}_{\mathrm{aper}}(\Gamma, \mathbb{I}^{\mathbb{N}})$, which is comeager by Proposition 3.8.6 (see also here Proposition 3.2.3). □

We now turn to the various notions of amenability.

Proposition 3.8.9. *Let Γ be a countably infinite group and let X be a Polish space. Then for Φ in $\{\mathrm{hyp}, \mathrm{amen}\}$ or $\{\mathrm{measHyp}, \mathrm{freeMeasHyp}, \mathrm{measAmen}\}$, $\mathrm{Sh}_\Phi(\Gamma, X)$ is $\mathbf{\Sigma}^1_2$ or $\mathbf{\Pi}^1_1$ respectively. If moreover Γ is non-amenable, then $\mathrm{Sh}_\Phi(\Gamma, \mathbb{R}^{\mathbb{N}})$ is $\mathbf{\Pi}^1_1$-hard.*

Proof. First, $\mathrm{Sh}_{\mathrm{hyp}}(\Gamma, X)$ is $\mathbf{\Sigma}^1_2$, since $F \in \mathrm{Sh}(\Gamma, X)$ is hyperfinite iff

$$\begin{aligned}&\exists \text{ sequence } (E_n)_n \text{ of Borel subsets of } (X^\Gamma)^2\\ &\quad [\forall n\, [E_n \text{ is a finite equivalence relation and } E_n \subseteq E_{n+1}]\\ &\quad \text{and } \forall x \in F\, \forall \gamma\, \exists n\, [(\gamma \cdot x, x) \in E_n]].\end{aligned}$$

Next, $\mathrm{Sh}_{\mathrm{amen}}(\Gamma, X)$ is $\mathbf{\Sigma}^1_2$, since $F \in \mathrm{Sh}(\Gamma, X)$ is amenable iff (letting R be the shift equivalence relation on F)

$$\exists \text{ sequence } (f_n)_n \text{ of Borel functions } f_n : F^2 \to [0,1]$$
$$\forall x \in F \left[\forall n \sum_{y \in [x]_R} f_n^x(y) = 1 \text{ and } \forall y \in [x]_R \, \|f_n^x - f_n^y\|_1 \to 0 \right],$$

where $\|\cdot\|_1$ is on $\ell^1([x]_R)$.

$\mathrm{Sh}_{\mathrm{measHyp}}(\Gamma, X)$ is $\mathbf{\Pi}^1_1$ by Miri Segal's effective witness to measure-hyperfiniteness (see [CM17, Theorem 1.7.8]).

Now $\mathrm{Sh}_{\mathrm{freeMeasHyp}}(\Gamma, X)$ is $\mathbf{\Pi}^1_1$, since $\mathrm{Sh}_{\mathrm{free}}(\Gamma, X)$ and $\mathrm{Sh}_{\mathrm{measHyp}}(\Gamma, X)$ are $\mathbf{\Pi}^1_1$.

Similarly, the set of points with amenable stabilizer is G_δ, since x has amenable stabilizer iff

$$\forall S \in \mathrm{Fin}(\Gamma) \left[S \subseteq \Gamma_x \implies \forall n \in \mathbb{N} \, \exists F \subseteq \langle S \rangle \, \frac{|SF \triangle F|}{|F|} < \frac{1}{n} \right].$$

Thus the set

$$\{F \in \mathrm{Sh}(\Gamma, X) : \forall x \in F \, [\Gamma_x \text{ is amenable}]\}$$

is $\mathbf{\Pi}^1_1$ (in fact G_δ when X is compact), and thus $\mathrm{Sh}_{\mathrm{measAmen}}(\Gamma, X)$ is $\mathbf{\Pi}^1_1$ by Corollary A.2.2.

If Γ is non-amenable, then $\mathbf{\Pi}^1_1$-hardness follows from Proposition 3.8.5 by taking F_Φ to be Γ with the Γ-action by left multiplication, by taking Ψ to be "has a free non-compressible Γ-invariant closed subspace", and by taking any F_Ψ satisfying Ψ. $\square$

Surprisingly, the free measure-hyperfinite subshifts of $(\mathbb{I}^{\mathbb{N}})^{\mathbb{F}_\infty}$ form a G_δ set:

Proposition 3.8.10. *Let Γ be an infinite exact group. Then $\mathrm{Sh}_{\mathrm{measAmen}}(\Gamma, \mathbb{I}^{\mathbb{N}})$ and $\mathrm{Sh}_{\mathrm{freeMeasHyp}}(\Gamma, \mathbb{I}^{\mathbb{N}})$ are dense G_δ. Moreover $\mathrm{Sh}_\Phi(\Gamma, \mathbb{I}^{\mathbb{N}})$ is comeager for $\Phi \in \{\mathrm{amen}, \mathrm{measHyp}\}$.*

Proof. Measure-amenability is compactly satisfiable (by exactness) and product-stable, so it is dense by Proposition 3.8.3. To show that $\mathrm{Sh}_{\mathrm{measAmen}}(\Gamma, X)$ is G_δ, by Proposition 3.8.4, it suffices to show that it is $\mathbf{\Sigma}^1_1$.

We use the characterization of measure-amenability as topological amenability, see Theorem A.3.1. By [Kec95, 12.13], there is Borel function $D : K(X) \to X^{\mathbb{N}}$ such that $D(K)$ is a dense subset of K for every nonempty $K \in K(X)$, and we can assume that $D(K)$ is Γ-invariant. Fix a compatible metric d on X. Then a subshift K is topologically amenable iff for every $\varepsilon > 0$ and any finite $S \subseteq \Gamma$, there is a function $p : \mathbb{N} \to \mathrm{Prob}(\Gamma)$ such that

i) (uniform continuity) for every ε_1, there is a ε_2 such that for every $n, m \in \mathbb{N}$, if $d(D(K)_n, D(K)_m) < \varepsilon_2$, then $\|p^n - p^m\|_1 < \varepsilon_1$;

ii) (invariance) for every $\gamma \in S$ and every $n, m \in \mathbb{N}$, if $D(K)_n = \gamma \cdot D(K)_m$, then $\|p^n - \gamma \cdot p^m\|_1 < \varepsilon$.

So it is $\mathbf{\Sigma}^1_1$.

Now $\mathrm{Sh}_{\mathrm{freeMeasHyp}}(\Gamma, \mathbb{I}^{\mathbb{N}})$ is G_δ, since by Corollary A.2.2, it is the intersection of $\mathrm{Sh}_{\mathrm{free}}(\Gamma, \mathbb{I}^{\mathbb{N}})$ and $\mathrm{Sh}_{\mathrm{measAmen}}(\Gamma, \mathbb{I}^{\mathbb{N}})$ which are both dense G_δ (the former by Proposition 3.8.6).

Finally, by the diagram of implications in the beginning of Appendix A, we have that $\mathrm{Sh}_{\mathrm{freeMeasHyp}}(\Gamma, \mathbb{I}^{\mathbb{N}}) \subseteq \mathrm{Sh}_{\mathrm{amen}}(\Gamma, \mathbb{I}^{\mathbb{N}}) \subseteq \mathrm{Sh}_{\mathrm{measHyp}}(\Gamma, \mathbb{I}^{\mathbb{N}})$, so the last two classes are also comeager. $\square$

We show $\mathbf{\Pi}^1_1$-hardness of $\mathrm{Sh}_\Phi(\mathbb{F}_\infty, \mathbb{I}^{\mathbb{N}})$ for various Φ.

Proposition 3.8.11. *Let Γ be an infinite residually finite group, and let X be $\mathbb{I}^{\mathbb{N}}$ or $\mathbb{R}^{\mathbb{N}}$. Then $\mathrm{Sh}_\Phi(\Gamma, X)$ is $\mathbf{\Pi}^1_1$-hard, where $\Phi \in \{\mathrm{fin}, \mathrm{sm}\}$. If moreover Γ is non-amenable, then $\mathrm{Sh}_\Phi(\Gamma, \mathbb{I}^{\mathbb{N}})$ is $\mathbf{\Pi}^1_1$-hard, where $\Phi \in \{\mathrm{hyp}, \mathrm{amen}, \mathrm{measHyp}\}$.*

Proof. Since $\mathrm{Sh}_\Phi(\Gamma, \mathbb{I}^{\mathbb{N}})$ reduces to $\mathrm{Sh}_\Phi(\Gamma, \mathbb{R}^{\mathbb{N}})$ via the inclusion map, it suffices to consider the case where $X = \mathbb{I}^{\mathbb{N}}$. By Proposition 3.8.4, it suffices to show F_σ-hardness. We will define a continuous map $2^{\mathbb{N}} \to \mathrm{Sh}(\Gamma, \mathbb{I}^{\mathbb{N}})$ which simultaneously reduces $2^{<\mathbb{N}}$ to $\mathrm{Sh}_{\mathrm{fin}}(\Gamma, \mathbb{I}^{\mathbb{N}})$ and to $\mathrm{Sh}_{\mathrm{sm}}(\Gamma, \mathbb{I}^{\mathbb{N}})$, and if moreover Γ is also non-amenable also to $\mathrm{Sh}_{\mathrm{measHyp}}(\Gamma, \mathbb{I}^{\mathbb{N}})$. Fix a descending sequence $(\Lambda_n)_n$ of finite index subgroups of Γ such that $\bigcap_n \Lambda_n = \{1\}$.

Let $K^n_i \in \mathrm{Sh}(\Gamma, \mathbb{I}^{\mathbb{N}}), n \in \mathbb{N}, i \in \{0, 1\}$, be defined as follows: K^n_0 is an invariant singleton and K^n_1 is a subshift isomorphic to the action of Γ on Γ/Λ_n. Consider now the space $\prod_n (\mathbb{I}^{\mathbb{N}})^\Gamma$ on which Γ acts diagonally and let $F \colon \prod_n (\mathbb{I}^{\mathbb{N}})^\Gamma \to (\mathbb{I}^{\mathbb{N}})^\Gamma$ be a Γ-equivariant continuous embedding. Finally for each $\alpha \in 2^{\mathbb{N}}$, let

$$f(\alpha) = F\left(\prod_n K^n_{\alpha_n}\right).$$

Then $f \colon 2^{\mathbb{N}} \to \mathrm{Sh}(\Gamma, \mathbb{I}^{\mathbb{N}})$ is continuous. If $\alpha \in 2^{<\mathbb{N}}$, clearly $f(\alpha)$ is finite. If $\alpha \notin 2^{<\mathbb{N}}$, then $f(\alpha)$ is a free subshift admitting an invariant probability Borel measure, so it is not smooth. If moreover Γ is non-amenable, it is also not measure-hyperfinite. $\square$

In connection to Problem 3.8.1, we can show density of hyperfinite shifts for certain groups. A countable group Γ is **Borel exact** if there is a free hyperfinite Γ-flow (note that replacing "hyperfinite" by "amenable" gives the definition of an exact group). An immediate consequence of Proposition 3.8.3 is that there are densely many of these:

Proposition 3.8.12. *Let* Γ *be Borel exact. Then* $\mathrm{Sh}_{\mathrm{hyp}}(\Gamma, \mathbb{I}^{\mathbb{N}})$ *is dense.*

We have a criterion for Borel exactness. A countable group is **hyperfinite** if all of its Borel actions have hyperfinite orbit equivalence relations.

Proposition 3.8.13. *Let* Γ *be a countable group with a* Γ*-flow* X *such that*

(i) X *is hyperfinite;*

(ii) the set $S = \{\mathrm{Stab}_\Gamma(x) : x \in X\}$ *is countable;*

(iii) for every infinite $\Delta \in S$*, the normalizer* $N_\Gamma(\Delta)$ *of* Δ *in* Γ *is hyperfinite.*

Then Γ *is Borel exact.*

Proof. We show that for any Borel Γ-space Y that $E = E_\Gamma^{X\times Y}$ is hyperfinite; the result follows from taking Y to be any free Γ-flow. By partitioning X, it suffices to show this in the case where every stabilizer of X is finite, and the case where every stabilizer of X is infinite.

First assume that every stabilizer of X is finite. We first show that $X \times Y$ is hyperfinite. Since X is hyperfinite, we can write $E_\Gamma^X = \bigcup_n F_n$, where each F_n is finite. Then each $E \cap (F_n \times I_Y)$ is a finite CBER (since each stabilizer is finite), so their increasing union E is hyperfinite.

Now assume that every stabilizer of X is infinite. Fix a transversal T for the conjugation action of Γ on S, and for every $x \in X$, choose $\gamma_x \in \Gamma$ such that $\mathrm{Stab}_\Gamma(\gamma_x \cdot x)$ is in T. For every Δ in T, let $Z_\Delta = \{(x,y) \in X \times Y : \mathrm{Stab}_\Gamma(x) = \Delta\}$. The map $(x,y) \mapsto \gamma_x \cdot (x,y)$ maps each E-class into itself, with image contained in $\bigsqcup_{\Delta\in T} Z_\Delta$, so it suffices to show that each $E{\restriction}Z_\Delta$ is hyperfinite, but $E{\restriction}Z_\Delta = E_{N_\Gamma(\Delta)}^{Z_\Delta}$, which is hyperfinite since $N_\Gamma(\Delta)$ is hyperfinite. ◻

We use this to show Borel exactness of hyperbolic groups.

Proposition 3.8.14. *Every hyperbolic group* Γ *is Borel exact.*

Proof. The Gromov boundary $\partial\Gamma$ is a Γ-flow. We show that it satisfies the conditions of Proposition 3.8.13. Hyperfiniteness follows from [MS20, Theorem A]. By [GdlH90, Chapitre 8, 30], every stabilizer is virtually cyclic, so there are only countably many stabilizers. Finally, let Δ be an infinite stabilizer. Since Δ is virtually cyclic, by [BH99, III.Γ, 3.10], it is a quasiconvex subgroup of Γ. So since Δ is infinite, by [BH99, Chapter III.Γ, 3.16], it has finite index in its normalizer, so its normalizer is also virtually cyclic, and thus hyperfinite. ◻

Since Borel exactness is closed under subgroups, every subgroup of a hyperbolic group is Borel exact. In particular, $\mathbb{F}_\infty$ is Borel exact, so $\mathrm{Sh}_{\mathrm{hyp}}(\mathbb{F}_\infty, \mathbb{I}^{\mathbb{N}})$ is dense.

The preceding constructions have some relevance to the question of whether every non-smooth, aperiodic CBER admits a compact action realization.

Proposition 3.8.15. *Suppose that Γ has an infinite Borel exact quotient. Then the set*

$$\{F \in \mathrm{Sh}(\Gamma, \mathbb{R}^{\mathbb{N}}) : E_F \text{ has a compact action realization}\}$$

is $\mathbf{\Sigma}^1_1$-hard.

Proof. For any quotient Γ' of Γ, the natural map $\mathrm{Sh}(\Gamma', \mathbb{R}^{\mathbb{N}}) \to \mathrm{Sh}(\Gamma, \mathbb{R}^{\mathbb{N}})$ reduces shifts with a compact action realization to shifts with a compact action realization. So without loss of generality, we assume that Γ itself is infinite and Borel exact.

Let $K, L \in \mathrm{Sh}(\Gamma, \mathbb{R}^{\mathbb{N}})$, where K is free compact hyperfinite, and L is free smooth. For every $T \in \mathrm{Tr}$ and $s \in \mathbb{N}^{<\mathbb{N}}$, define $F^T_s \in \mathrm{Sh}(\Gamma, \mathbb{R}^{\mathbb{N}})$ by

$$F^T_s := \begin{cases} K & s \in T \\ L & s \notin T \end{cases}$$

Let $f : \mathrm{Tr} \to \mathrm{Sh}(\Gamma, \mathbb{R}^{\mathbb{N}})$ be defined by $f(T) = \mathcal{A}_s F^T_s$. We claim that T is ill-founded iff $E_f(T)$ has a compact action realization.

Note that for every $T \in \mathrm{Tr}$, every branch of $(F^T_s)_s$ consists of free hyperfinite actions, so the product along every branch is free hyperfinite, and thus $f(T)$ is (free) hyperfinite.

If T has an infinite branch, then there is a branch of $(F^T_s)_s$ consisting only of K which is compact free, so the product along this branch is compact free, and thus non-smooth. Thus $f(T)$ is non-smooth hyperfinite, and thus has a compact action realization.

On the other hand, if T has no infinite branches, then every branch of $(F^T_s)_s$ contains L which is free smooth, so the product along every branch is free smooth, and thus $f(T)$ is free smooth, and thus does not have a compact action realization. $\square$

Proposition 3.8.16. *For every $x \in 2^{\mathbb{N}}$, there is some $F \in \mathrm{Sh}(\mathbb{F}_\infty, \mathbb{R}^{\mathbb{N}})$ such that E_F has a compact action realization, but also such that there is no $\Delta^1_1(F, x)$ isomorphism of E_F with some E_K, $K \in \mathrm{Sh}(\mathbb{F}_\infty, \mathbb{I}^{\mathbb{N}})$.*

Proof. Assume this fails toward a contradiction. Then there is a $\mathbf{\Pi}^1_1$ definition of the class of all $F \in \mathrm{Sh}(\mathbb{F}_\infty, \mathbb{R}^{\mathbb{N}})$ for which E_F admits a compact action realization, contradicting $\mathbf{\Sigma}^1_1$-hardness. $\square$

Informally this implies that there is no "uniform Borel method" that will construct a compact action realization for each CBER with a compact action realization, even if it is given as a subshift of $(\mathbb{R}^{\mathbb{N}})^{\mathbb{F}_\infty}$. An alternative form of such a result is proved in [KW22, Remark 4.3].

Another version is the following: For every Borel function $f\colon \mathrm{Sh}(\mathbb{F}_\infty, \mathbb{R}^{\mathbb{N}}) \to \mathrm{Sh}(\mathbb{F}_\infty, \mathbb{I}^{\mathbb{N}})$, there is some $F \in \mathrm{Sh}(\mathbb{F}_\infty, \mathbb{R}^{\mathbb{N}})$ such that E_F has a compact action realization but $E_F \not\cong_B E_{f(F)}$.

(C) For each $\kappa \in \{0, 1, 2, 3, \dots, \aleph_0, 2^{\aleph_0}\}$, we say that a continuous action of a countable group Γ on a Polish space is $\boldsymbol{\kappa}$**−ergodic** if the action or equivalently the corresponding equivalence relation admits exactly κ ergodic invariant probability measures. Thus 0-ergodicity is equivalent to compressibility and 1-ergodicity is what is usually called **unique ergodicity**. Let

$$\mathrm{Sh}_\kappa(\Gamma, X) = \{F \in \mathrm{Sh}(\Gamma, X) : F \text{ is } \kappa\text{−ergodic}\}.$$

and

$$\mathrm{Sh}_{\mathrm{free}\kappa}(\Gamma, X) = \{F \in \mathrm{Sh}(\Gamma, X) : F \text{ is free and } \kappa\text{−ergodic}\}.$$

We will classify next the complexity of $\mathrm{Sh}_\kappa(\Gamma, \mathbb{I}^{\mathbb{N}}), \mathrm{Sh}_\kappa(\Gamma, \mathbb{N}^{\mathbb{N}})$ as well as their free versions $\mathrm{Sh}_{\mathrm{free}\kappa}(\Gamma, \mathbb{I}^{\mathbb{N}}), \mathrm{Sh}_{\mathrm{free}\kappa}(\Gamma, \mathbb{N}^{\mathbb{N}})$, for $\kappa \geq 1$ and various Γ, in comparison with the results in (B) above for $\kappa = 0$.

We will need first some results concerning trees in descriptive set theory. As before we denote by Tr the space of all subtrees of $\mathbb{N}^{<\mathbb{N}}$ and by Tr_2 the space of all subtrees of $2^{<\mathbb{N}}$. For each such tree T, we denote the cardinality of its set of infinite branches by $\kappa(T)$. We now have:

Proposition 3.8.17. *(i)* $\{T \in \mathrm{Tr}_2 \colon \kappa(T) = \aleph_0\}$ *is* $\mathbf{\Pi^1_1}$*-complete and* $\{T \in \mathrm{Tr}_2 \colon \kappa(T) = 2^{\aleph_0}\}$ *is* $\mathbf{\Sigma^1_1}$*-complete.*

(ii) $\{T \in \mathrm{Tr} \colon \kappa(T) = \kappa\}$*, where* $1 \leq \kappa \leq \aleph_0$*, is* $\mathbf{\Pi^1_1}$*-complete and* $\{T \in \mathrm{Tr} \colon \kappa(T) = 2^{\aleph_0}\}$ *is* $\mathbf{\Sigma^1_1}$*-complete.*

Proof. (i) It is well known that $\{T \in \mathrm{Tr}_2 \colon \kappa(T) \leq \aleph_0\}$ is $\mathbf{\Pi^1_1}$-complete, so $\{T \in \mathrm{Tr}_2 \colon \kappa(T) = 2^{\aleph_0}\}$ is $\mathbf{\Sigma^1_1}$-complete. Fix now a tree T_0 in Tr_2 with $\aleph_0$ infinite branches and note that $T \mapsto T \cup T_0$ is a Borel reduction of $\{T \in \mathrm{Tr}_2 \colon \kappa(T) \leq \aleph_0\}$ to $\{T \in \mathrm{Tr}_2 \colon \kappa(T) = \aleph_0\}$, so the latter is also $\mathbf{\Pi^1_1}$-complete.

(ii) Let $A \subseteq \mathbb{N}^{\mathbb{N}}$ be $\mathbf{\Pi^1_1}$. Then there is a closed set $F \subseteq \mathbb{N}^{\mathbb{N}} \times (\mathbb{N}^{\mathbb{N}})_0$, where $(\mathbb{N}^{\mathbb{N}})_0 = \{x \in \mathbb{N}^{\mathbb{N}} \colon x(0) = 0\}$, such that

$$x \notin A \iff \exists y (x, y) \in F.$$

Let now for $1 \leq \kappa < \aleph_0$, C_κ be a closed subset of $(\mathbb{N}^\mathbb{N})_1$, where $(\mathbb{N}^\mathbb{N})_1 = \{x \in \mathbb{N}^\mathbb{N} \colon x(0) = 1\}$, such that C_κ has cardinality κ. Finally let $F_\kappa = F \cup (\mathbb{N}^\mathbb{N} \times C_\kappa)$, which is a closed subset of $\mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ and let T_κ be a subtree of $\mathbb{N}^{<\mathbb{N}} \times \mathbb{N}^{<\mathbb{N}}$ such that $F_\kappa = [T_\kappa]$, the set of infinite branches of T_κ. Then if $T_n(x) = \bigcup_k \{t \in \mathbb{N}^k \colon (x|k, t) \in T_n\}$, clearly

$$x \in A \iff \kappa(T_\kappa(x)) = \kappa,$$

so the Borel map $x \mapsto T_\kappa(x)$ reduces A to $\{T \in \mathrm{Tr} \colon \kappa(T) = \kappa\}$, so the latter is $\mathbf{\Pi^1_1}$-complete. The results about $\kappa = \aleph_0, 2^{\aleph_0}$ follow from the corresponding ones in (i) above. □

Using these results and the method of proof of Proposition 3.8.5, we can now show the following, where we call a countable infinite group **good** if it has a sequence of free, uniquely ergodic, continuous actions $(\boldsymbol{a_n})$ on compact, metrizable spaces such that the infinite (diagonal) product action $\prod_n \boldsymbol{a_n}$ is also uniquely ergodic. One can see, using rationally independent rotations of $\mathbb{T}$ and Kronecker's Theorem, that $\mathbb{Z}$ is good. Tsankov showed that every amenable group is good and Glasner pointed out that $\mathrm{SL}_n(\mathbb{Z})$ and the free groups are good. It is not clear what is the full extent of goodness among countable groups.

Theorem 3.8.18. *(i) For any infinite countable group* Γ*,*

(a) $\mathrm{Sh}_1(\Gamma, \mathbb{I}^\mathbb{N})$ *and* $\mathrm{Sh}_{free1}(\Gamma, \mathbb{I}^\mathbb{N})$ *are* G_δ *and* $\mathrm{Sh}_\kappa(\Gamma, \mathbb{I}^\mathbb{N})$ *and* $\mathrm{Sh}_{free\kappa}(\Gamma, \mathbb{I}^\mathbb{N})$*, are differences of two* G_δ *sets, for* $1 < \kappa < \aleph_0$*, while* $\mathrm{Sh}_\kappa(\Gamma, \mathbb{R}^\mathbb{N})$ *is* $\mathbf{\Pi^1_1}$*-complete, for* $1 \leq \kappa < \aleph_0$*.*

(b) $\mathrm{Sh}_{\aleph_0}(\Gamma, \mathbb{I}^\mathbb{N})$ *and* $\mathrm{Sh}_{\aleph_0}(\Gamma, \mathbb{R}^\mathbb{N})$ *are* $\mathbf{\Pi^1_1}$*-complete.*

(c) $\mathrm{Sh}_{2^{\aleph_0}}(\Gamma, \mathbb{I}^\mathbb{N})$ *and* $\mathrm{Sh}_{2^{\aleph_0}}(\Gamma, \mathbb{R}^\mathbb{N})$ *are* $\mathbf{\Sigma^1_1}$*-complete.*

(ii) For any good *infinite countable group* Γ*,*

(a) $\mathrm{Sh}_{free\kappa}(\Gamma, \mathbb{R}^\mathbb{N})$ *is* $\mathbf{\Pi^1_1}$*-complete, for* $1 \leq \kappa < \aleph_0$*.*

(b) $\mathrm{Sh}_{free\aleph_0}(\Gamma, \mathbb{I}^\mathbb{N})$ *and* $\mathrm{Sh}_{free\aleph_0}(\Gamma, \mathbb{R}^\mathbb{N})$ *are* $\mathbf{\Pi^1_1}$*-complete.*

(c) $\mathrm{Sh}_{free2^{\aleph_0}}(\Gamma, \mathbb{I}^\mathbb{N})$ *and* $\mathrm{Sh}_{free2^{\aleph_0}}(\Gamma, \mathbb{R}^\mathbb{N})$ *are* $\mathbf{\Sigma^1_1}$*-complete.*

Proof. (i)(a) Consider first $\mathrm{Sh}_\kappa(\Gamma, \mathbb{I}^\mathbb{N})$. It is enough to show that for $1 \leq \kappa < \aleph_0$, $\mathrm{Sh}_{\leq\kappa}(\Gamma, \mathbb{I}^\mathbb{N}) = \bigcup_{\lambda \leq \kappa} \mathrm{Sh}_\lambda(\Gamma, \mathbb{I}^\mathbb{N})$ is G_δ. Denoting by P the compact space of shift-invariant probability Borel measures on $(\mathbb{I}^\mathbb{N})^\Gamma$, we note that, using the ergodic decomposition, we have that $F \in \mathrm{Sh}_{\leq\kappa}(\Gamma, \mathbb{I}^\mathbb{N}) \iff \forall \mu_1, \dots, \mu_{\kappa+1} \in P\big([\forall i \leq (\kappa + 1)(\mu_i(F) = 1 \implies \exists a_1, \dots, a_{\kappa+1} \in [-1,1](\sum_{i\leq\kappa+1} |a_i| = 1 \;\&\; \sum_{i\leq\kappa+1}(a_i\mu_i) = 0)\big]$.

Since freeness is a G_δ condition, the results about $\mathrm{Sh}_{free\kappa}(\Gamma, \mathbb{I}^\mathbb{N})$ follow.

Finally the proof of the result about $\mathrm{Sh}_\kappa(\Gamma, \mathbb{R}^\mathbb{N})$ is similar to that in (i)(b) below, using Proposition 3.8.17 (ii), instead of Proposition 3.8.17 (i).

(i)(b) It is enough to consider $\mathrm{Sh}_{\aleph_0}(\Gamma, \mathbb{I}^{\mathbb{N}})$. For any $T \in \mathrm{Tr}_2$, $s \in 2^{<\mathbb{N}}$, let F_s^T be a 1-point subshift of $(\mathbb{I}^{\mathbb{N}})^{\Gamma}$, if $s \in T$, and a compressible subshift of $(\mathbb{I}^{\mathbb{N}})^{\Gamma}$, if $s \notin T$. Then, as is in the proof of Proposition 3.8.5, but defining $A_s F_s^T$ using the space $2^{\mathbb{N}} \times ((\mathbb{I}^{\mathbb{N}})^{\Gamma})^{\mathbb{N}}$ instead of $\mathbb{N}^{\mathbb{N}} \times ((\mathbb{R}^{\mathbb{N}})^{\Gamma})^{\mathbb{N}}$ in the paragraph preceding Proposition 3.8.5, the map $T \in \mathrm{Tr}_2 \mapsto A_s F_s^T$ is a Borel reduction of $\{T \in \mathrm{Tr}_2 \colon \kappa(T) = \aleph_0\}$ to $\mathrm{Sh}_{\aleph_0}(\Gamma, \mathbb{I}^{\mathbb{N}})$, so we are done by Proposition 3.8.17 (i).

(i)(c) The proof is similar to that in (i)(b).

(ii) Since Γ is good there is a family $(F_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$ of free subshifts of $(\mathbb{I}^{\mathbb{N}})^{\Gamma}$ which are uniquely ergodic and whose product is also uniquely ergodic. Then, again as in the proof of Proposition 3.8.5, define for any $T \in \mathrm{Tr}$, $s \in \mathbb{N}^{<\mathbb{N}}$, F_s^T to be F_s, if $s \in T$, and a free, compressible subshift of $(\mathbb{I}^{\mathbb{N}})^{\Gamma}$, if $s \notin T$. The the Borel map $T \in \mathrm{Tr} \mapsto A_s F_s^T$ is a Borel reduction of $\{T \in \mathrm{Tr} \colon \kappa(T) = \kappa\}$ to $\mathrm{Sh}_{\kappa}(\Gamma, \mathbb{R}^{\mathbb{N}})$, so we are done for (ii)(a) by Proposition 3.8.17 (ii). The case of $\mathbb{I}^{\mathbb{N}}$ in (ii)(b,c), is analogous to that in (i)(b,c) and the case of $\mathbb{R}^{\mathbb{N}}$ clearly follows from this. □

Remark 3.8.19. If Γ is a group that has a good infinite group as a factor, it is clear that all the results in Theorem 3.8.18 (ii) hold if "free" is replaced by "aperiodic".

In combination with results in Section 3.9, we also have the following dichotomy result, which also gives another characterization of amenability for countable groups (at least in the finitely generated case).

Theorem 3.8.20. *Let Γ be an infinite countable group.*

(i) If Γ is not amenable, then the generic subshift of $(\mathbb{I}^{N})^{\Gamma}$ is compressible, i.e., has no ergodic invariant measures.

(ii) If Γ is finitely generated and amenable, then the generic subshift of $(\mathbb{I}^{N})^{\Gamma}$ has $2^{\aleph_0}$ ergodic invariant measures.

Proof. (i) This follows from Proposition 3.8.7. (ii) This follows from the results in Section 3.9, see the third paragraph following Theorem 3.9.7. □

3.9 The space of actions on the Cantor space

(A) Let Γ be a countable group, and let X be a compact zero-dimensional Polish space. Denote by $\mathrm{Act}(\Gamma, X)$ the space of group homomorphisms $\Gamma \to \mathrm{Homeo}(X)$, i.e., Γ-flows on the space X. The group $\mathrm{Homeo}(X)$ is a Polish group with the uniform convergence topology and $\mathrm{Act}(\Gamma, X)$ is a closed subspace of $\mathrm{Homeo}(X)^{\Gamma}$ with the product topology, so it is a Polish space in this topology. The Polish group $\mathrm{Homeo}(X)$ acts continuously by conjugation on $\mathrm{Act}(\Gamma, X)$.

For $\mathbf{a} \in \mathrm{Act}(\Gamma, X)$, let $\mathcal{A}_{\mathbf{a}}$ denote the Boolean algebra of clopen $\mathbf{a}$-invariant subsets of X, and let $\mathrm{St}(\mathcal{A}_{\mathbf{a}})$ denote its Stone space. There is a continuous $\mathbf{a}$-equivariant surjection $X \twoheadrightarrow \mathrm{St}(\mathcal{A}_{\mathbf{a}})$ defined by sending x to the ultrafilter $\{A \in \mathcal{A}_{\mathbf{a}} : x \in A\}$. For every $\mathcal{U} \in \mathrm{St}(\mathcal{A}_{\mathbf{a}})$, the fiber $C^{\mathbf{a}}_{\mathcal{U}}$ above $\mathcal{U}$ is a closed $\mathbf{a}$-invariant subset of X, giving the decomposition

$$X = \bigsqcup_{\mathcal{U} \in \mathrm{St}(\mathcal{A}_{\mathbf{a}})} C^{\mathbf{a}}_{\mathcal{U}}.$$

Let $\mathrm{CEINV}(\mathbf{a})$ denote the subset of the space $\mathrm{INV}(\mathbf{a})$ of invariant probability Borel measures for $\mathbf{a}$, consisting of **clopen-ergodic** measures, that is, measures $\mu \in \mathrm{INV}(\mathbf{a})$ for which every $A \in \mathcal{A}_{\mathbf{a}}$ is μ-null or μ-conull. Note that $\mathrm{CEINV}(\mathbf{a})$ is closed by the Portmanteau Theorem [Kec95, 17.20(v)], and we have

$$\mathrm{EINV}(\mathbf{a}) \subseteq \mathrm{CEINV}(\mathbf{a}) \subseteq \mathrm{INV}(\mathbf{a}).$$

There is a map $\mathrm{CEINV}(\mathbf{a}) \twoheadrightarrow \mathrm{St}(\mathcal{A}_{\mathbf{a}})$ sending μ to the ultrafilter $\{A \in \mathcal{A}_{\mathbf{a}} : \mu(A) = 1\}$, which is surjective if Γ is amenable, and the fiber above $\mathcal{U}$ can be identified with $\mathrm{INV}(\mathbf{a} \restriction C^{\mathbf{a}}_{\mathcal{U}})$, giving a decomposition

$$\mathrm{CEINV}(\mathbf{a}) = \bigsqcup_{\mathcal{U} \in \mathrm{St}(\mathcal{A}_{\mathbf{a}})} \mathrm{INV}(\mathbf{a} \restriction C^{\mathbf{a}}_{\mathcal{U}}).$$

Proposition 3.9.1. *Suppose Γ is amenable. Let $\mathbf{a} \in \mathrm{Act}(\Gamma, X)$. If $|\mathcal{A}_{\mathbf{a}}| > 2$, then $\mathrm{CEINV}(\mathbf{a})$ is a proper subset of $\mathrm{INV}(\mathbf{a})$, so in particular, $\mathrm{INV}(\mathbf{a})$ is not Poulsen. If $\mathcal{A}_{\mathbf{a}}$ is atomless, then $\mathrm{EINV}(\mathbf{a})$ has size continuum.*

Proof. If $|\mathcal{A}_{\mathbf{a}}| > 2$, then $|\mathrm{St}(\mathcal{A}_{\mathbf{a}})| \geq 2$, so we see from the decomposition that $\mathrm{CEINV}(\mathbf{a})$ is not closed under convex combinations, and is thus a strict subset of $\mathrm{INV}(\mathbf{a})$. If $\mathcal{A}_{\mathbf{a}}$ is atomless, then $\mathrm{St}(\mathcal{A}_{\mathbf{a}})$ has size continuum, so $\mathrm{EINV}(\mathbf{a})$ has size continuum, since each $\mathrm{INV}(\mathbf{a} \restriction C^{\mathbf{a}}_{\mathcal{U}})$ is nonempty by amenability of Γ. □

The following fact appears in [Ele19, Remark 5], as was pointed out to us by J. Melleray:

Proposition 3.9.2. *Consider the action of $\mathrm{Homeo}(2^{\mathbb{N}})$ by conjugation on $\mathrm{Act}(\Gamma, 2^{\mathbb{N}})$. Then there is a dense conjugacy class.*

Proof. Let $(\mathbf{a}_n)$ be a dense sequence in $\mathrm{Act}(\Gamma, 2^{\mathbb{N}})$ and consider the product action $\prod_n \mathbf{a}_n$. Then an isomorphic copy of this action in $\mathrm{Act}(\Gamma, 2^{\mathbb{N}})$ has dense conjugacy class. □

Proposition 3.9.3. *Suppose* Γ *is finitely generated. Then for comeagerly many* $\mathbf{a} \in \mathrm{Act}(\Gamma, 2^{\mathbb{N}})$, $\mathcal{A}_{\mathbf{a}}$ *is atomless, so in particular if* Γ *is amenable, then* $\mathrm{EINV}(\mathbf{a})$ *has size continuum and* $\mathrm{INV}(\mathbf{a})$ *is not Poulsen.*

Proof. Let $\mathcal{A}$ be the set of nonempty clopen subsets of $2^{\mathbb{N}}$. Then $\mathcal{A}_{\mathbf{a}}$ is atomless iff for every $A \in \mathcal{A}$, if A is $\mathbf{a}$-invariant, then there is a partition $A = A_0 \sqcup A_1$ into $\mathbf{a}$-invariant $A_0, A_1 \in \mathcal{A}$. So it suffices to fix $A \in \mathcal{A}$, and show comeagerness of the set of $\mathbf{a}$ such that if A is $\mathbf{a}$-invariant, then there is a partition $A = A_0 \sqcup A_1$ into $\mathbf{a}$-invariant $A_0, A_1 \in \mathcal{A}$. This set is open, since Γ is finitely generated, so it suffices to show that it is dense. Let V be a nonempty open subset of $\mathrm{Act}(\Gamma, 2^{\mathbb{N}})$. We can assume that A is $\mathbf{a}$-invariant for every $\mathbf{a} \in V$, otherwise we are done. Then V gives by restriction an open subset of $\mathrm{Act}(\Gamma, A)$, so since the set of $\mathbf{a} \in \mathrm{Act}(\Gamma, A)$ with a partition $A = A_0 \sqcup A_1$ into $\mathbf{a}$-invariant sets is closed under conjugation, we are done, since $\mathrm{Act}(\Gamma, A)$ has a dense conjugacy class (because $A \cong 2^{\mathbb{N}}$). □

Remark 3.9.4. Proposition 3.9.3 may fail if Γ is not finitely generated. If $\Gamma = \mathbb{F}_\infty$, then the set of all minimal actions $\mathbf{a} \in \mathrm{Act}(\Gamma, 2^{\mathbb{N}})$ is dense G_δ but $|\mathcal{A}_{\mathbf{a}}| = 2$.

Problem 3.9.5. *If* Γ *is finitely generated and amenable, is it true that for comeager many* $\mathbf{a} \in \mathrm{Act}(\Gamma, 2^{\mathbb{N}})$, $\mathrm{INV}(\mathbf{a})$ *is a Bauer simplex, i.e.,* $\mathrm{EINV}(\mathbf{a})$ *is closed in* $\mathrm{INV}(\mathbf{a})$*?*

(B) In [Ele19] Elek studies a notion of weak containment in the space $\mathrm{Act}(\Gamma, 2^{\mathbb{N}})$. An equivalent form of his definition is the following: Given $\mathbf{a}, \mathbf{b} \in \mathrm{Act}(\Gamma, 2^{\mathbb{N}})$, we say that $\mathbf{a}$ is **weakly contained** in $\mathbf{b}$, in symbols

$$\mathbf{a} \preceq \mathbf{b},$$

if $\mathbf{a}$ is in the closure of the conjugacy class of $\mathbf{b}$. Weak containment is a quasi-order on $\mathrm{Act}(\Gamma, 2^{\mathbb{N}})$, which has a maximum element by Proposition 3.9.2. Note that the set of maximum elements, i.e., those with dense conjugacy class, is a dense G_δ in $\mathrm{Act}(\Gamma, 2^{\mathbb{N}})$. It is shown in [Ele19, Section 3.3] that there is also a minimum element among the free actions, for finitely generated Γ.

Certain properties of actions are upwards monotone in the quasi-order $\preceq$:

Theorem 3.9.6. *(i) If* $\mathbf{a}, \mathbf{b} \in \mathrm{Act}(\Gamma, 2^{\mathbb{N}})$, $\mathbf{a} \preceq \mathbf{b}$ *and* $\mathbf{a}$ *is free, so is* $\mathbf{b}$.

(ii) ([Ele19, Section 3.1]) *If* $\mathbf{a}, \mathbf{b} \in \mathrm{Act}(\Gamma, 2^{\mathbb{N}})$, $\mathbf{a} \preceq \mathbf{b}$ *and* $\mathbf{a}$ *is compressible, so is* $\mathbf{b}$.

(iii)([Ele19, Section 3.1]) *If* $\mathbf{a}, \mathbf{b} \in \mathrm{Act}(\Gamma, 2^{\mathbb{N}})$, $\mathbf{a} \preceq \mathbf{b}$ *and* $\mathbf{a}$ *is topologically amenable, so is* $\mathbf{b}$.

We include for the convenience of the reader proofs of these results, in the framework of the above definition of weak containment, in Appendix B.

(C) The Correspondence Theorem of Hochman [Hoc08, Theorem 1.3 and Section 10], shows the equivalence of genericity properties in the spaces $\mathrm{Act}(\Gamma, 2^{\mathbb{N}})$ and $\mathrm{Sh}(\Gamma, \mathbb{I}^{\mathbb{N}})$. Let Φ be a property of continuous Γ-actions on Polish spaces which is invariant under (topological) isomorphism. Recall that $\mathrm{Sh}_{\Phi}(\Gamma, \mathbb{I}^{\mathbb{N}})$ is the set of all subshifts of $(\mathbb{I}^{\mathbb{N}})^{\Gamma}$ that have the property Φ. Let also

$$\mathrm{Act}_{\Phi}(\Gamma, 2^{\mathbb{N}})$$

Be the set of all actions in $\mathrm{Act}(\Gamma, 2^{\mathbb{N}})$ that have the property Φ. Then we have:

Theorem 3.9.7 (Hochman [Hoc08, Theorem 1.3 and Section 10])**.** *Let Φ be a property of continuous Γ-actions on Polish spaces which is invariant under (topological) isomorphism. Then the following are equivalent:*
(i) $\mathrm{Act}_{\Phi}(\Gamma, 2^{\mathbb{N}})$ *is comeager;*
(ii) $\mathrm{Sh}_{\Phi}(\Gamma, \mathbb{I}^{\mathbb{N}})$ *is comeager.*

We include a somewhat simplified proof of this theorem in Appendix C.

In particular Theorem 3.9.6 and the Correspondence Theorem gives another proof that for $\Phi \in \{\text{free, compressible, topologically amenable}\}$, the set $\mathrm{Sh}_{\Phi}(\Gamma, \mathbb{I}^{\mathbb{N}})$ is comeager, for infinite Γ which are also non-amenable for the compressible case and exact in the topologically amenable case.

It also follows that for any amenable, finitely generated Γ the generic subshift of $(\mathbb{I}^{\mathbb{N}})^{\Gamma}$ admits continuum many ergodic invariant measures and thus if all Borel actions of Γ generate hyperfinite equivalence relations (e.g., if Γ is nilpotent), then the generic subshift of $(\mathbb{I}^{\mathbb{N}})^{\Gamma}$ gives an equivalence relation Borel isomorphic to $\mathbb{R}E_0$.

As we indicated before, we do not know if for nonamenable Γ the generic subshift in $\mathrm{Sh}(\Gamma, \mathbb{I}^{\mathbb{N}})$ is hyperfinite. If that was the case, since the generic subshift of $\mathrm{Sh}(\Gamma, \mathbb{I}^{\mathbb{N}})$ is compressible, it would follow that the (equivalence relation of the) generic subshift would be Borel isomorphic to E_t.

Of particular interest is the case $\Gamma = \mathbb{F}_2$. By the result of Kwiatkowska in [Kwi12] there is a generic action of $\mathbb{F}_2$ on $2^{\mathbb{N}}$, i.e., an action $\mathbf{a} \in \mathrm{Act}(\mathbb{F}_2, 2^{\mathbb{N}})$ with comeager conjugacy class. Then by the above Correspondence Theorem the question of genericity of hyperfiniteness in $\mathrm{Sh}(\mathbb{F}_2, \mathbb{I}^{\mathbb{N}})$ is equivalent to following question:

Problem 3.9.8. *Let $\mathbf{a} \in \mathrm{Act}(\mathbb{F}_2, 2^{\mathbb{N}})$ be the generic action of $\mathbb{F}_2$ on $2^{\mathbb{N}}$. Is the equivalence relation $E_{\mathbf{a}}$ hyperfinite?*

Note that by Proposition 3.8.10 and the Correspondence Theorem, this equivalence relation $E_{\mathbf{a}}$ is amenable.

3.10 K_σ and F_σ realizations

Clinton Conley raised the following question: Does every aperiodic CBER have a realization as a K_σ equivalence relation in a Polish space? We answer this question in the affirmative:

Theorem 3.10.1. *Every $E \in \mathcal{AE}$ has a transitive K_σ realization in the Cantor space $2^{\mathbb{N}}$.*

Proof. Let $Q = 2^{<\mathbb{N}} \subseteq 2^{\mathbb{N}}$, and let $N = 2^{\mathbb{N}} \setminus Q$. Then N is homeomorphic to Baire space, so by Proposition 3.2.1, we can assume that $E = E_\Gamma^N$, where $\Gamma \curvearrowright N$ is a continuous action of a countable group Γ on N. For each $\gamma \in \Gamma$, let R_γ be the relation on N defined by $x \mathrel{R_\gamma} y \iff y = \gamma \cdot x$. Let $\overline{R_\gamma}$ denote the closure of R_γ in $(2^{\mathbb{N}})^2$. We claim that $\overline{R_\gamma} \subseteq R_\gamma \oplus I_Q$. Let $(x, y) \in \overline{R_\gamma}$, and suppose that $x \in N$ (the case $y \in N$ is identical). Then there is a sequence $(x_n, y_n)_n$ in R_γ converging to (x, y). Since $x \in N$, we have that $y_n = \gamma \cdot x_n \to \gamma \cdot x$, so $y = \gamma \cdot x$, and thus $(x, y) \in R_\gamma$, proving the claim. Thus the relation $E \oplus I_Q$ on $2^{\mathbb{N}}$ (which is isomorphic to E) is equal to $I_Q \cup \bigcup_\gamma \overline{R_\gamma}$, so it is K_σ, and it has the dense class Q. □

We can ask about K_σ and F_σ realizations which are minimal. There is a known obstruction, due to Solecki:

Theorem 3.10.2 (Solecki, [Sol02, Corollary 3.2])**.** *Every minimal K_σ equivalence relation on a Polish space with at least two classes is not smooth.*

It is open whether this is the only obstruction:

Problem 3.10.3. *Let E be an aperiodic* CBER*. Does E have a minimal F_σ realization? If E is non-smooth, does E have a minimal K_σ realization?*

We do not even know if an aperiodic smooth CBER has a minimal F_σ realization. Theorem 3.2.6 shows that all non-smooth relations in $\mathcal{AH}$ have minimal K_σ realizations.

In contrast to Solecki's result, one can show the following:

Proposition 3.10.4. *Every aperiodic smooth* CBER *can be realized as a minimal equivalence relation which is a Boolean combination of K_σ relations in a compact Polish space.*

Proof. Here are two such realizations:

1. Consider the equivalence relation E_0 in $2^{\mathbb{N}}$. Let A be a Cantor set in $2^{\mathbb{N}}$ which is a partial transversal for E_0. Let B be the E_0-saturation of A and put $Y = 2^{\mathbb{N}} \setminus B$. Then Y is G_δ, so a zero-dimensional Polish space (in the relative topology). Every compact subset of Y has empty interior in Y, so Y is homeomorphic to the Baire space $\mathbb{N}^{\mathbb{N}}$ (see [Kec95, 7.7]). Therefore there is a continuous bijection $f\colon Y \to A$ (see [Kec95, 7.15]). Let F be the equivalence relation on $2^{\mathbb{N}}$ obtained by adding to each E_0 class $[a]_{E_0}$, with $a \in A$, the point $f^{-1}(a)$. Then F is smooth with all classes dense. Put

   $$S(x, y) \iff x \in B \;\&\; y \in Y \;\&\; \exists z \in A(xE_0 z \;\& f(y) = z)$$

   and

   $$T(x, y) \iff S(y, x).$$

   Then each of S, T is the intersection of two K_σ relations with a G_δ relation and

   $$xFy \iff (x, y \in B \;\&\; xE_0 y) \vee S(x, y) \vee T(x, y),$$

   so F is a Boolean combination of K_σ relations as well.

2. Let $X = \prod_{n \geq 1} 2^n$, where 2^n is the set of binary sequences of length n. Let $Y = \{(x_n) \in X : \exists m \forall n \geq m(x_n \prec x_{n+1})\}$, and define $f\colon X \to 2^{\mathbb{N}}$ as follows:

   $$f(x) = \begin{cases} \lim_n x_n & x \in Y \\ x_1 {}^\frown x_2 {}^\frown x_3 {}^\frown \cdots & x \notin Y \end{cases}$$

   Let $xEy \iff f(x) = f(y)$. Then E is a smooth CBER with all classes dense, and it is easy to check that $E = F_1 \cup F_2 \cup F_3 \cup F_4$, where F_1 is K_σ, F_2 and F_3 are intersections of a K_σ and a G_δ relation and F_4 is the equality relation on X.

$\square$

We next discuss a sharper notion of K_σ realization. Let X be a compact Polish space and E a CBER in X. Recall that we say that E is **compactly graphable** if there is a compact graphing of E, i.e., a compact graph (irreflexive, symmetric relation) $K \subseteq E$ so that the E-classes are the connected components of K. Note then that E is K_σ. A CBER E has a **compactly graphable realization** if it is Borel isomorphic to a compactly graphable CBER. Clearly every CBER that has a compact action realization implemented by a free continuous action of a *finitely generated* group has a compactly graphable realization. Also clearly a CBER that has a a compactly graphable realization admits a K_σ realization.

We now have the following result:

Theorem 3.10.5. *(a) Every aperiodic hyperfinite* CBER *has a compactly graphable realization.*

(B) Every compressible CBER *has a compactly graphable realization.*

Proof. (a) This follows from Theorem 3.2.6 for non-compressible hyperfinite CBER. The compressible case is covered in (b).

(b) The proof is a modification of the proof of Theorem 3.10.1. Let E be a compressible CBER. Then by [DJK94, Proposition 1.8], [Kec22, Proposition 3.27] and the arguments in the proof of Proposition 3.2.1, we can assume that E is of the form $E = E^N_{\mathbb{F}_2}$, where N is as in the proof of Theorem 3.10.1. Let α, β be free generators of $\mathbb{F}_2$ and let S consist of these generators and their inverses. Finally, as in the proof of Theorem 3.10.1, let $K = \bigcup_{\gamma\in S} \overline{R_\gamma}$, and note that if F is the equivalence relation generated by K (i.e., the smallest equivalence relation containing K), then F is of the form $E \oplus R$, where R is an equivalence relation on the space Q, Thus E is Borel bireducible to F. Now let $Y = \{1, 1/2, 1/3, \dots, 0\}$ and define on Y the compact, connected graph R, where yRy' iff

$$(y = 1, y' \leq 1/2) \text{ or } (y' = 1, y \leq 1/2) \text{ or } (y = \frac{1}{2}, y' \leq 1/3) \text{ or } (y' = \frac{1}{2}, y \leq \frac{1}{3}).$$

Consider now the equivalence relation $G = F \times I_Y$ on $2^{\mathbb{N}} \times Y$, where as usual $I_Y = Y^2$. Thus $(x, y)G(x', y') \iff xFx'$. Then the compact relation $\tilde{K}$ on $2^{\mathbb{N}} \times Y$ given by

$$(x, y)\tilde{K}(x', y') \iff xKx' \text{ and } yRy',$$

is a compact graphing of G. But G is Borel bireducible to F and thus to E. Since both E and G are compressible, they are Borel isomorphic by [Kec22, Proposition 3.27] and the proof is complete. □

The following is an open problem:

Problem 3.10.6. *Does every aperiodic* CBER *admit a compactly graphable realization?*

3.11 A σ-ideal associated to a K_σ countable Borel equivalence relation

Suppose that X is an (uncountable) Polish space and E a CBER on X. Denote by $K(X)$ the space of compact subsets of X with the usual Vietoris topology (see [Kec95, 4.F]). Let

$$I_E = \{K \in K(X) \colon [K]_E \neq X\}.$$

Recall that a σ ideal of compact sets is a nonempty subset $I \subseteq K(X)$ such that $K \subseteq L \in I \implies K \in I$ (i.e., it is hereditary) and $K \in K(X), K = \bigcup_n K_n, K_n \in I, \forall n \implies K \in I$ (i.e., it is closed under countable unions which are compact).

Proposition 3.11.1. *Let X be a Polish space and E a K_σ CBER on X with all E-classes dense. Then I_E is a G_δ σ-ideal of compact sets.*

Proof. Here and in the sequel, notice that since E is K_σ, $X = \{x \in X : (x, x) \in E\}$ (and X^2) is also K_σ and $F_\sigma = K_\sigma$ on X (and X^2).

Clearly I_E is hereditary. To check closure under countable unions, we will actually show that if $K_n \in I_E, \forall n$, then $[\bigcup_n K_n]_E \neq X$. Notice that because E is K_σ, for each compact K the set $[K]_E$ is also K_σ and thus if $K \in I_E$, then $X \setminus [K]_E$ is dense G_δ. So if $K_n \in I_E, \forall n$, and $[\bigcup_n K_n]_E = \bigcup_n [K_n]_E = X$ this contradicts the Baire Category Theorem. Since

$$K \in I_E \iff \exists x \forall y (y \in K \implies \neg x E y),$$

clearly I_E is $\mathbf{\Sigma}^1_1$, thus by [KLW87, Theorem 11] (see also [MZ07, Theorem 1.4]) it is G_δ. $\square$

Corollary 3.11.2. *If X, E are as in Proposition 3.11.1 and moreover E admits a meager complete section, then E admits a nowhere dense, compact complete section.*

Proof. We have a sequence K_n of nowhere dense compact sets with $[\bigcup_n K_n]_E = \bigcup_n [K_n]_E = X$. Thus for some n, $K_n \notin I_E$, so K_n is a nowhere dense, compact complete section. $\square$

Below denote by $K_{\aleph_0}(X)$ the σ-ideal of countable compact subsets of X and by $\mathrm{MGR}(X)$ the σ-ideal of nowhere dense (i.e., meager) compact subsets of X.

Corollary 3.11.3. *If X, E are as in Corollary 3.11.2, then*

$$K_{\aleph_0}(X) \subsetneqq I_E \subsetneqq \mathrm{MGR}(X).$$

Corollary 3.11.4. *If X, E are as in Proposition 3.11.1, then E does not admit a K_σ transversal.*

Proof. If F is a K_σ transversal, we can write $F = F_1 \sqcup F_2$, where each F_i is also K_σ and nonempty. Then each F_i is the union of countably many compact sets in I_E, a contradiction. $\square$

We say that a σ-ideal of compact sets I satisfies **Solecki's Property (*)** if for any sequence $K_n \in I, \forall n$, there is a G_δ set G such that $\bigcup_n K_n \subseteq G$ and $K(G) = \{K \in K(X) \colon K \subseteq G\} \subseteq I$; see [Sol11].

Proposition 3.11.5. *If X, E are as in Proposition 3.11.1, then I_E satisfies Solecki's Property (*).*

Proof. Let $K_n \in I_E, \forall n$. Then there is $x \in X$ such that $[x]_E \cap [\bigcup_n K_n]_E = \emptyset$ and thus if $G = X \setminus [x]_E$, G is G_δ and $K(G) \subseteq I_E$. $\square$

In particular I_E admits a representation as in [Sol11, Theorem 3.1].

A σ-ideal I of compact sets is **ccc** if there is no uncountable collection of pairwise disjoint compact sets which are not in I. Since for any CBER E every $K \notin I_E$ is a complete section, it follows that I_E is ccc.

On the other hand, let I_E^* be the σ-ideal of subsets of X generated by I_E, i.e, for $A \subset X$, $A \in I_E^* \iff \exists (K_n)(K_n \in I_E, \forall n,$ and $A \subseteq \bigcup_n K_n)$. Then I_E^* is not ccc, in fact we have the following:

Proposition 3.11.6. *Let X, E be as in Proposition 3.11.1 and moreover for every nonempty open set $U \subseteq X$ there is a meager complete section contained in U. Then there is a homeomorphic embedding $f \colon 2^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to X$ such that for every $\alpha \in 2^{\mathbb{N}}$, we have $f(\{\alpha\} \times \mathbb{N}^{\mathbb{N}}) \notin I_E^*$.*

Proof. By [KS95, Section 3, Lemma 9], it is enough to show that for every nonempty open $U \subseteq X$, there is a nowhere dense compact set $K \subseteq U$ with $K \notin I_E$. This follows as in the proof of Corollary 3.11.2. $\square$

A σ-ideal I of compact sets has the **covering property** if for every $\mathbf{\Sigma^1_1}$ set $A \subseteq X$, either $A \subseteq \bigcup_n K_n$, where $K_n \in I, \forall n$, or else $K(A) \nsubseteq I$. It is **calibrated** if whenever $K \in K(X)$ and $K_n \subseteq K$ are such that $K_n \in I, \forall n$, and $K(K \setminus \bigcup_n K_n) \subseteq I$, then $K \in I$.

Proposition 3.11.7. *Let X, E be as in Proposition 3.11.1. Then I_E does not have the covering property and is not calibrated.*

Proof. Fix $x \in X$ and let $G = X \setminus [x]_E$. This provides a counterexample to the covering property.

Let now K be a compact complete section, which exists by the proof of Proposition 3.11.1. Then $K = (K \cap [x]_E) \cup (K \setminus [x]_E)$ gives a counterexample to calibration. $\square$

We next provide an example of a pair X, E satisfying all the properties of Proposition 3.11.6, and which therefore satisfies all the preceding propositions. We take X to be the collection of all subsets A of $\mathbb{N}$ such that $0 \in A, 1 \notin A$, with the usual topology. We let then E be the restriction of many-one equivalence to X. It is easy to see that E is a K_σ CBER and every E-class is dense. Finally if U is an open subset of X, which we can assume that it has the form $U = \{A \in X \colon F_1 \subseteq A, F_2 \cap A = \emptyset\}$, for two disjoint finite subsets F_1, F_2 of $\mathbb{N}$, then for a large enough number n the set $K = \{A \in U \colon A \text{ contains only even numbers } > n\}$ is a meager complete section contained in U.

4 Generators and 2-adequate groups

For each infinite countable group Γ and standard Borel space X consider the shift action of Γ on X^Γ and let $E(\Gamma, X)$ be the associated equivalence relation and $E^{ap}(\Gamma, X)$ be its aperiodic part, i.e., the restriction of $E(\Gamma, X)$ to the set of points with infinite orbits. Consider now a Borel action of Γ on an uncountable standard Borel space, which we can assume is equal to $\mathbb{R}$. Then the map $f \colon X \to \mathbb{R}^\Gamma$ given by $x \mapsto p_x$, where $p_x(\gamma) = \gamma^{-1} \cdot x$, is an equivariant Borel embedding of this action to the shift action on $\mathbb{R}^\Gamma$. In particular for every aperiodic equivalence relation E induced by a Borel action of Γ we have that $E \sqsubseteq_B^i E(\Gamma, \mathbb{R})$, where for equivalence relations R, S on standard Borel spaces Y, Z, resp., we let $R \sqsubseteq_B^i S$ iff there is an injective Borel reduction $f \colon Y \to Z$ of R to S such that $f(Y)$ is S-invariant. Thus every aperiodic equivalence relation E induced by a Borel action of Γ can be realized as (i.e., is Borel isomorphic to) the restriction of $E^{ap}(\Gamma, \mathbb{R})$ to an invariant Borel set.

Now recall that for a Borel action of Γ on a standard Borel space X and $n \in \{2, 3, \dots, \dots, \mathbb{N}\}$ an ***n*-generator** is a Borel partition $X = \bigsqcup_{i<n} X_i$ such that $\{\gamma \cdot X_i \colon \gamma \in \Gamma, i < n\}$ generates the Borel sets in X. This is equivalent to having a Borel equivariant embedding of the action to the shift action on n^Γ.

It is shown in [JKL02] that for every such action with infinite orbits there exists an $\mathbb{N}$-generator. It follows that every aperiodic equivalence relation E induced by a Borel action of Γ can be realized as the restriction of $E^{ap}(\Gamma, \mathbb{N})$ to an invariant Borel set. In particular $E^{ap}(\Gamma, \mathbb{R}) \cong_B E^{ap}(\Gamma, \mathbb{N})$. However because of entropy considerations, even for the group $\Gamma = \mathbb{Z}$, it is not the case that every such action with invariant measure has a finite generator.

Weiss [Wei89] asked whether for $\Gamma = \mathbb{Z}$ any Borel action without invariant measure admits a finite generator. Tserunyan [Tse15] showed that answer is affirmative for *any* infinite countable group Γ if the action is Borel isomorphic to a continuous action on a σ-compact Polish space. Then Hochman [Hoc19] provided a positive answer to Weiss' question (for $\mathbb{Z}$). Finally this work culminated in the following complete answer:

Theorem 4.0.1 (Hochman-Seward)**.** *Every Borel action of a countable group on a standard Borel space without invariant measure admits a 2-generator.*

This however leaves open the possibility that every aperiodic CBER E induced by a Borel action of Γ can be realized as the restriction of $E^{ap}(\Gamma, 2)$ to an invariant Borel set. This is clearly equivalent to the statement that $E^{ap}(\Gamma, \mathbb{R}) \cong_B E^{ap}(\Gamma, 2)$ and it also equivalent to the statement that there is a Borel action of Γ that generates E and has a 2-generator. This leads to the following concept.

Definition 4.0.2. An infinite countable group Γ is called **2-adequate** if

$$E^{ap}(\Gamma, \mathbb{R}) \cong_B E^{ap}(\Gamma, 2).$$

Remark 4.0.3. Thomas [Tho12] studies the question of when $E(\Gamma, \mathbb{R}) \sim_B E(\Gamma, 2)$.

The first result here is the following:

Theorem 4.0.4. *Every infinite countable amenable group is* 2*-adequate.*

Proof. Let $X = \mathbb{R}^\Gamma, Y = 2^\Gamma, E = E^{ap}(\Gamma, \mathbb{R}), F = E^{ap}(\Gamma, 2)$. Note that $|\mathrm{EINV}_F| = |\mathrm{EINV}_E| = 2^{\aleph_0}$, so fix a Borel bijection $\pi \colon \mathrm{EINV}_E \to \mathrm{EINV}_F$. Fix also the ergodic decompositions $\{X_e\}_{e \in \mathrm{EINV_E}}$ of E and $\{Y_f\}_{f \in \mathrm{EINV_F}}$ of F, resp. By the Ornstein-Weiss Theorem, see. e.g., [Kec22, Theorem 8.31], let Z_e be an E-invariant Borel subset of X_e such that $E|Z_e$ is hyperfinite with unique invariant measure e. Again the construction of Z_e is effective enough, so that $Z = \bigcup_e Z_e$ is Borel. Put $X' = X \setminus Z$, so that $E|X'$ is compressible.

Then, by Theorem 4.0.1, there is a Borel F-invariant subset Y' of Y such that $E|X' \cong_B F|Y'$, say by the Borel isomorphism $g \colon X' \to Y'$. Put $W' = Y \setminus Y'$. Then let W_f be an F-invariant Borel subset of Y_f such that $W_f \subseteq W'$ and $F|W_f$ is hyperfinite with unique invariant measure f. Again the construction of W_f is effective enough, so that $W = \bigcup_f W_f$ is Borel and there is a Borel isomorphism h_e of $E|Z_e$ with $F|W_{\pi(e)}$ such that moreover $h = \bigcup_e h_e$ is Borel and thus a Borel isomorphism of $E|Z$ with $F|W$. Then $g \cup h$ shows that $E \sqsubseteq_B^i F$ and the proof is complete. $\square$

Thomas [Tho12, Page 391] asked the question of whether there is an infinite amenable Γ such that $E(\Gamma, \mathbb{R}) \not\leq_B E(\Gamma, 2)$. Theorem 4.0.4 provides a negative answer in a strong form.

To discuss other examples of 2-adequate groups, we will need the following strengthening of Theorem 2.2.2.

Proposition 4.0.5. *Let $E \in \mathcal{AE}$ and let $R \subseteq E$ be hyperfinite. Then there is $R \subseteq F \subseteq E$ with $F \in \mathcal{AH}$.*

Proof. Suppose E lives on the standard Borel space X and let

$$Y = \{x \colon [x]_E \text{ contains a finite nonempty set of finite } R\text{-classes}\}.$$

Then Y is E-invariant and $E|Y$ is smooth, thus we can let $F = E$ on Y. Let $W = \{x \colon [x]_E \text{ contains no finite } R\text{-classes}\}$. Then we can take $F = R$ on W.

So we can assume that each E-class contains infinitely many finite R-classes. Let $Z = \{x \colon [x]_R \text{ is finite}\}$. Then $R|Z$ is R-invariant and smooth, so let S be a Borel selector and T the associated Borel transversal $T = \{x \colon S(x) = x\}$. Then, by Theorem 2.2.2, let F' be a hyperfinite aperiodic Borel equivalence relation on T such that $F' \subseteq E|T$. Let then F'' be the equivalence relation on Z defined by $xF''y \iff S(x)F'S(y)$. It is clearly aperiodic, hyperfinite, and $R|Z \subseteq F'' \subseteq E|Z$. Finally put $F = F'' \cup R|(X \setminus Z)$. □

We also consider the following class of countable groups.

Definition 4.0.6. A countable group Γ is **hyperfinite generating** if for every $E \in \mathcal{AH}$ there is a Borel action of Γ that generates E.

We now have the next result that generalizes Proposition 4.0.5 from $\mathbb{Z}$ to any hyperfinite generating group. The proof is similar, noting that any smooth aperiodic CBER can be generated by a Borel action of any infinite countable group.

Proposition 4.0.7. *Let $E \in \mathcal{AE}$ and let $R \subseteq E$ be generated by a Borel action of Γ, where Γ is a hyperfinite generating group. Then there is $R \subseteq F \subseteq E$ with $F \in \mathcal{AE}$ generated by a Borel action of Γ.*

Proposition 4.0.8. *Let Γ be any countable group and Δ a hyperfinite generating, 2-adequate group. Then $\Gamma \star \Delta$ is 2-adequate.*

Proof. Fix a Borel action $\mathbf{a}$ of $\Gamma \star \Delta$ on an uncountable standard Borel space X generating an aperiodic equivalence relation that we denote by $E_{\mathbf{a}}$. Let $\mathbf{b} = \mathbf{a}\restriction\Delta, \mathbf{c} = \mathbf{a}\restriction\Gamma$ and denote by $E_{\mathbf{b}}, E_{\mathbf{c}}$ the associated equivalence relations, so that $E_{\mathbf{a}} = E_{\mathbf{b}} \vee E_{\mathbf{c}}$. By Proposition 4.0.7 find a Borel action $\mathbf{b}'$ of Δ such that $E_{\mathbf{b}'}$ is aperiodic and $E_{\mathbf{b}} \subseteq E_{\mathbf{b}'} \subseteq E_{\mathbf{a}}$, so that $E_{\mathbf{a}} = E_{\mathbf{b}'} \vee E_{\mathbf{c}}$. Let now $\mathbf{a}'$ be the action of $\Gamma \star \Delta$ such that $\mathbf{a}'\restriction\Delta = \mathbf{b}', \mathbf{a}'\restriction\Gamma = \mathbf{c}$, so that $E_{\mathbf{a}'} = E_{\mathbf{a}}$. Since $\mathbf{b}'$ has a 2-generator, so does $\mathbf{a}'$ and the proof is complete. □

It will be shown in Corollary 5.1.2 that all groups that have an infinite amenable factor are hyperfinite generating. Thus we have:

Corollary 4.0.9. *The free product of any countable group with a 2-adequate group that has an infinite amenable factor and thus, in particular, the free groups $\mathbb{F}_n, 1 \le n \le \infty$, are 2-adequate.*

The following is immediate:

Proposition 4.0.10. *If Γ, Δ are countable groups, every aperiodic equivalence relation induced by a Borel action of Γ can be also induced by a Borel action of Δ, Δ is a factor of Γ and Δ is 2-adequate, so is Γ. In particular, for any $1 \le n \le \infty$, every n-generated countable group that factors onto $\mathbb{F}_n$ is 2-adequate.*

The next two results owe a lot to some crucial observations by Brandon Seward.

Proposition 4.0.11. *Let Γ be n-generated, $1 \le n \le \infty$. Then $\Gamma \times \mathbb{F}_n$ is 2-adequate. In particular, all products $\mathbb{F}_m \times \mathbb{F}_n$, $1 \le m, n \le \infty$, are 2-adequate.*

Proof. Let $\{\gamma_i\}_{i<n}$ be generators for Γ and let $\{\alpha_i\}_{i<n}$ be free generators for $\mathbb{F}_n$. Consider a Borel action $\mathbf{a}$ of $\Gamma \times \mathbb{F}_n$ with E_a aperiodic. Then the equivalence relation E_i generated by $\mathbf{a}\restriction\langle\gamma_i, \alpha_i\rangle$ is generated by a Borel action of $\mathbb{Z}^2$ thus is hyperfinite, see, e.g., [Kec22, Theorem 8.33], and so is given by a Borel action $\mathbf{a}_i$ of $\mathbb{Z}$. Let $\mathbf{b}$ the Borel action of $\mathbb{F}_n$ in which the generator α_i acts like $\mathbf{a}_i$. Then $E_\mathbf{b} = \bigvee_i E_i = E$ and the proof is complete by Proposition 4.0.10. □

Finally not every infinite countable group is 2-amenable. The argument below follows the pattern of the proofs in [Tho12, Section 6].

Theorem 4.0.12. *The group $\mathrm{SL}_3(\mathbb{Z})$ is not 2-adequate.*

Proof. Assume that $\Gamma = \mathrm{SL}_3(\mathbb{Z})$ is 2-adequate, towards a contradiction. Then in particular $E^{ap}(\Gamma, 3) \cong_B E^{ap}(\Gamma, 2)$, say via the Borel isomorphism f. Let μ be the usual product of the uniform measure on 3^Γ. Then $\nu = f_*\mu$ is an ergodic, invariant measure for the shift action of Γ on 2^Γ, thus by Stuck-Zimmer [SZ94] this shift action is free ν-a.e. This gives a contradiction by the arguments in [Tho12, Section 6]. □

We conclude this section with the following problem:

Problem 4.0.13. *Characterize the 2-adequate groups.*

5 Additional results

5.1 Hyperfinite generating groups

We introduced in Section 4 the concept of hyperfinite generating groups. We will establish here some equivalent formulations of this concept and in particular prove the fact mentioned in the paragraph after Proposition 4.0.8. Below we let μ be the product of the uniform measure on $2^{\mathbb{N}}$ and by $[E_0] \leq \mathrm{Aut}(2^{\mathbb{N}}, \mu)$ the usual measure theoretic full group of the pmp equivalence relation E_0. For a countable group $\Delta \leq [E_0]$, we denote by E_Δ the subequivalence relation of E_0 induced by the action of Δ on $2^{\mathbb{N}}$. This is again understood to be defined only μ-a.e.

Below an **IRS** on a countable group Γ is a measure on the space of subgroups of Γ invariant under conjugation. We say that an IRS μ has some property P if μ-almost all $\Delta \leq \Gamma$ have property P. Finally a subgroup $\Delta \leq \Gamma$ is **co-amenable** if the action of Γ on Γ/Δ is amenable, i.e., admits a finitely additive probability measure.

Proposition 5.1.1. *Let Γ be an infinite countable group. Then the following are equivalent:*

(i) Γ is hyperfinite generating;

(ii) There is a Borel action of Γ that generates E_0;

(iii) Γ admits a Borel action which generates a non-compressible, aperiodic hyperfinite equivalence relation;

(iv) Γ admits a factor $\Delta \leq [E_0]$ such that E_Δ has a μ-positive set of infinite orbits.

Moreover, if Γ is hyperfinite generating, Γ admits a co-amenable IRS *with infinite index.*

Proof. Clearly (i) $\implies$ (ii) $\implies$ (iii). We next prove that (iii) $\implies$ (iv). Indeed (iii) implies that there is a Borel action of Γ on a standard Borel space X generating an aperiodic, hyperfinite equivalence relation E that has an ergodic, invariant measure μ. This action induces a homomorphism $\pi\colon \Gamma \to [E]$, the measure theoretic full group of E, with respect to μ. If $\Delta = \pi(\Gamma) \leq [E]$, then Δ generates E (again μ-a.e). But by Dye's Theorem, see, e.g., [Kec22, Theorem 8.8], E and E_0 are measure theoretically isomorphic, which proves (iv).

We now show that (iv) $\implies$ (i). Fix $E \in \mathcal{AH}$ which lives on a space X. If E is compressible, then it is generated by a Borel action of Γ, by [DJK94, 11.2]. Otherwise consider the ergodic decomposition $\{X_e\}_{e \in \mathrm{EINV}_E}$ of E. Now (iv) implies (iii) and it follows that Γ has a Borel action on a standard Borel space Z which generates an aperiodic hyperfinite equivalence relation F, which has an ergodic, invariant measure

μ. Find then, using Dye's Theorem, see, e.g., [Kec22, Theorem 8.8], invariant Borel sets $Y_e \subseteq X_e$ with $e(Y_e) = 1$ and $Z_e \subseteq Z$ with $\mu(Z_e) = 1$ such that $E|Y_e$ and $F|Z_e$ are Borel isomorphic. Then $E|Y_e$ can be generated by a Borel action of Γ, and, by the effectivity of this construction, we also have that $Y = \bigcup_e Y_e$ is Borel and putting together the acton of Γ on each Y_e, we get a Borel action of Γ on Y which generates $E|Y$. Since $E|(X \setminus Y)$ is compressible, this shows that E is generated by a Borel action of Γ.

Finally the last statement follows as in the proof of (vii) $\implies$ (x) in the last paragraph of [BK20, Appendix D] (finite generation is not required there). □

Corollary 5.1.2. *Every countable group that has an infinite amenable factor is hyperfinite generating.*

Proof. If Γ is infinite amenable, consider its shift action on 2^Γ, equipped with the product of the uniform measure, with associated equivalence relation $E = E(\Gamma, 2)$. Then E and E_0 are measure theoretically isomorphic, so the measure theoretic full group of E is isomorphic to $[E_0]$. Since $\Gamma \leq [E]$ we have an embedding $\pi \colon \Gamma \to [E_0]$ such that if $\Delta = \pi(\Gamma)$, then $E_\Delta = E_0$, which completes the proof. □

It also immediately follows from [Mil06, Theorem 13] that every countable group that has a factor of the form $\Gamma \star \Delta$, where Γ, Δ are non-trivial subgroups of $[E_0]$, is hyperfinite generating.

On the other hand, not every infinite countable group is hyperfinite generating.

Proposition 5.1.3. *No infinite countable group with property* (T) *is hyperfinite generating.*

Proof. See, for example, the proof of [Kec10, Proposition 4.14]. □

It is also shown in [Kec10, page 29] that there are groups that do not have property (T) and are not hyperfinite generating.

The following is an open problem.

Problem 5.1.4. *Characterize the hyperfinite generating groups.*

5.2 Dynamically compressible groups

In the course of the previous investigations the following property of countable groups came up. As usual we employ the notation E^X_Γ for the equivalence relation induced by a Borel action of a countable group Γ on a standard Borel space X.

Definition 5.2.1. An infinite countable group Γ is called **dynamically compressible** if for every aperiodic E_Γ^X, there is a compressible E_Γ^Y with $E_\Gamma^X \leq_B E_\Gamma^Y$.

Here is an equivalent formulation of this notion.

Proposition 5.2.2. *A countable group Γ is dynamically compressible iff for every aperiodic E_Γ^X, $E_\Gamma^X \times I_\mathbb{N}$ is induced by a Borel action of Γ*

Proof. Since $E_\Gamma^X \times I_\mathbb{N} \leq_B E_\Gamma^X$, if $E_\Gamma^X \leq_B E_\Gamma^Y$, with E_Γ^Y compressible, then $E_\Gamma^X \times I_\mathbb{N} \leq_B E_\Gamma^Y$, therefore $E_\Gamma^X \times I_\mathbb{N} \sqsubseteq_B^i E_\Gamma^Y$ by [Kec22, Proposition 3.27]. □

We now have:

Proposition 5.2.3. *Every infinite countable amenable group is dynamically compressible.*

Proof. Consider any aperiodic $E = E_\Gamma^X$, which we can clearly assume is not compressible, so admits an invariant measure. Then let $\{X_e\}_{e\in \mathrm{EINV}_E}$ be its ergodic decomposition. Then there is a Borel set $Y_e \subseteq X_e$ with $e(Y_e) = 1$ such that $E|Y_e$ is hyperfinite, thus $E|Y_e \leq_B E_t$. As usual $Y = \bigcup_e Y_e$ is Borel and $E|Y \leq_B \mathbb{R}E_t \leq_B E_t$. Now $E|(X \setminus Y)$ is compressible and E_t is induced by a Borel action of Γ by [DJK94, 11.2], so the proof is complete. □

Proposition 5.2.4. *If $\mathbb{F}_2 \leq \Gamma$, then Γ is dynamically compressible.*

Proof. Let E_Γ^X be aperiodic. Then $E_\Gamma^X = E_{\mathbb{F}_\infty} \leq_B E_{\mathbb{F}_\infty} \times I_\mathbb{N} = E_{\mathbb{F}_\infty}^Y$, for $Y = X \times \mathbb{N}$. Now $\mathbb{F}_\infty \leq \Gamma$, so by using the inducing construction from the action of $\mathbb{F}_\infty$ on Y, see [BK96, 2.3.5], we have $E_{\mathbb{F}_\infty}^Y \leq_B E_Z^\Gamma$ for some compressible E_Z^Γ. □

Therefore only the groups that are not amenable but do not contain $\mathbb{F}_2$ can possibly fail to be dynamically compressible. But even among those there exist dynamically compressible groups.

Proposition 5.2.5. *Let Γ be a countable group for which there is an infinite group Δ such that $\Gamma \times \Delta \leq \Gamma$. Then Γ is dynamically compressible.*

Proof. Let E_Γ^X be aperiodic. Then for $Y = X \times \mathbb{N}$, $E_\Gamma^X \leq_B E_\Gamma^X \times I_\mathbb{N} = E_{\Gamma\times\Delta}^Y \leq_B E_\Gamma^Z$, where E_Γ^Z is obtained by inducing from the action of $\Gamma \times \Delta$ on Y. □

As a result any countable group of the form $\Gamma \times \Delta^{<\mathbb{N}}$, for an infinite Δ, is dynamically compressible. Take now Γ to be any group that is not amenable and does not contain $\mathbb{F}_2$ and consider $G = \Gamma \times \mathbb{Z}^{<\mathbb{N}}$. Then G is dynamically compressible and clearly is not amenable. Moreover it does not contain $\mathbb{F}_2$ because of the following standard fact.

Proposition 5.2.6. *Let G, H be two groups such that $\mathbb{F}_2 \leq G \times H$. Then $\mathbb{F}_2 \leq G$ or $\mathbb{F}_2 \leq H$.*

Proof. Let $\pi\colon \mathbb{F}_2 \to H$ be the second projection, If it has trivial kernel, then $\mathbb{F}_2 \leq H$. Else either $\mathbb{F}_2 \leq \ker(\pi) \leq G$ or $\ker(\pi) \cong \mathbb{Z}$. In the latter case, by [LS01, 3.110], $[\mathbb{F}_2 : \ker(\pi)]$ is finite, so by [LS01, 3.9],

$$[\mathbb{F}_2 : \ker(\pi)] = \frac{\operatorname{rank}(\ker(\pi)) - 1}{\operatorname{rank}(\mathbb{F}_2) - 1} = 0,$$

a contradiction. □

We now have the following open problem:

Problem 5.2.7. *Is every infinite countable group dynamically compressible?*

We note that Γ fails to be dynamically compressible iff there is some aperiodic E_Γ^X such that every $E_\Gamma^X \leq_B E_\Gamma^Y$ admits an invariant measure.

We conclude with the following interesting consequence of Proposition 5.2.4. Let $\Gamma = \mathrm{SL}_3(\mathbb{Z})$ and consider the shift action of Γ on $\mathbb{R}^\Gamma$ and denote by $E = F(\Gamma, \mathbb{R})$ the restriction of $E(\Gamma, \mathbb{R})$ to the free part of the action. Then, by Proposition 5.2.4, $E \times I_\mathbb{N}$ is induced by a Borel action of Γ. On the other hand, $E \times I_\mathbb{N}$ cannot be induced by a *free* Borel action of Γ, since if that was the case then $E \times I_\mathbb{N} \sqsubseteq_B^i E$, contradicting the Addendum following [CK18, 5.28].

6 Open problems

For the convenience of the reader, we collect here some of the main open problems discussed in this paper.

Problem 6.0.1. *(Problem 2.3.3) Let $\kappa \geq 1$. Does $\mathcal{AT}_\kappa$ have a $\subseteq_B$-maximum element?*

Problem 6.0.2. *(Problem 3.2.10) Does* every *non-smooth $E \in \mathcal{AE}$ have any of the topological realizations stated in Definition 3.2.2? In particular, does* every *non-smooth $E \in \mathcal{AE}$ admit a compact action realization?*

Problem 6.0.3. *(Problem 3.2.11) If a* CBER *admits a compact action realization, does it admit one in which the underlying space is $2^\mathbb{N}$?*

Problem 6.0.4. *([Problem 3.2.12]) Is* every *non-smooth $E \in \mathcal{AE}$ Borel bireducible to some $F \in \mathcal{AE}$ which has any of the topological realizations stated in [Definition 3.2.2]? In particular, can one find such an F that admits a compact action realization?*

Note that by Theorem 3.3.11, every non-smooth $E \in \mathcal{AE}$ is Borel bireducible to some $F \in \mathcal{AE}$ which has a compact action realization iff every non-smooth compressible $E \in \mathcal{AE}$ has a compact action realization.

Problem 6.0.5. *(Problem 3.3.3) Is it true that for an arbitrary infinite countable group Γ, $E^{ap}(\Gamma, 2^{\mathbb{N}})$ admits a compact action realization?*

Problem 6.0.6. *(Problem 3.3.6) Let $E \in \mathcal{AE}$ be on a standard Borel space X and let μ be a measure on X such that the restriction of E to any invariant Borel set of measure 1 is not smooth. Is there is an invariant Borel set $Y \subseteq X$ with $\mu(Y) = 1$ such that $E|Y$ admits a compact action realization?*

Problem 6.0.7. *(Problem 3.3.15) Does an arbitrary (not necessarily compressible) aperiodic, universal* CBER *admit a compact action realization?*

Problem 6.0.8. *(Problem 3.4.2) Does Turing equivalence $\equiv_T$ on $2^{\mathbb{N}}$ admit a compact action realization?*

Problem 6.0.9. *(Problem 3.4.11) Is there a Baire class 1 map that is an isomorphism between $\equiv_T$ and an equivalence relation given by a continuous group action on $\mathbb{N}^{\mathbb{N}}$?*

Problem 6.0.10. *(Problem 3.4.12) Is there a Borel map $\Phi : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ that is an isomorphism between $\equiv_T$ and an equivalence relation given by a continuous group action on $\mathbb{N}^{\mathbb{N}}$ such that $\Phi(x) \equiv_T x'$ on a cone?*

Problem 6.0.11. *(Problem 3.5.7) In Theorem 3.5.3, can one replace (iii) by "***a** *is finitely compressible" and similarly for (iv).*

Problem 6.0.12. *(Problem 3.6.7) Does Corollary 3.6.6 hold with $\mathbb{F}_2$ instead of $\mathbb{F}_3, \mathbb{F}_4$?*

Problem 6.0.13. *(Problem 3.6.8) Does every non-smooth aperiodic* CBER *have a realization as a subshift of 2^{Γ} for some group Γ? Also does it have a realization as a minimal subshift?*

Problem 6.0.14. *(Problem 3.6.12) Is $\mathbb{F}_2$ minimal subshift universal? More generally, is every group that contains $\mathbb{F}_2$ minimal subshift universal?*

Problem 6.0.15. *(Problem 3.7.4) Is it true that a group* Γ *contains* $\mathbb{F}_2$ *iff there is a compressible, orbit-universal subshift of* 2^Γ*?*

Problem 6.0.16. *(Problem 3.8.1) Let* Γ *be an infinite group. Is* $\mathrm{Sh}_{\mathrm{hyp}}(\Gamma, \mathbb{I}^{\mathbb{N}})$ *comeager in* $\mathrm{Sh}(\Gamma, \mathbb{I}^{\mathbb{N}})$*?*

Problem 6.0.17. *(Problem 3.8.2) Let* Γ *be an infinite group. What is the exact descriptive complexity of* $\mathrm{Sh}_{hyp}(\Gamma, \mathbb{I}^{\mathbb{N}})$ *in* $\mathrm{Sh}(\Gamma, \mathbb{I}^{\mathbb{N}})$*?*

Problem 6.0.18. *(Problem 3.9.5) If* Γ *is finitely generated and amenable, is it true that for comeager many* $\mathbf{a} \in \mathrm{Act}(\Gamma, 2^{\mathbb{N}})$*,* $\mathrm{INV}(\mathbf{a})$ *is a Bauer simplex, i.e.,* $\mathrm{EINV}(\mathbf{a})$ *is closed in* $\mathrm{INV}(\mathbf{a})$*?*

Problem 6.0.19. *(Problem 3.9.8) Let* $\mathbf{a} \in \mathrm{Act}(\mathbb{F}_2, 2^{\mathbb{N}})$ *be the generic action of* $\mathbb{F}_2$ *on* $2^{\mathbb{N}}$*. Is the equivalence relation* $E_{\mathbf{a}}$ *hyperfinite?*

Problem 6.0.20. *(Problem 3.10.3) Let* E *be an aperiodic* CBER*. Does* E *have a minimal* F_σ *realization? If* E *is non-smooth, does* E *have a minimal* K_σ *realization?*

Problem 6.0.21. *(Problem 3.10.6) Does every aperiodic* CBER *admit a compactly generated realization?*

Problem 6.0.22. *(Problem 4.0.13) Characterize the* 2*-adequate groups.*

Problem 6.0.23. *(Problem 5.1.4) Characterize the hyperfinite generating groups.*

Problem 6.0.24. *(Problem 5.2.7) Is every infinite countable group dynamically compressible?*

Appendices

A Amenable actions

The purpose of this appendix is to explain the following implications for a continuous action $\Gamma \curvearrowright X$ of a countable group on a Polish space. Recall that E_Γ^X is the induced orbit equivalence relation and all the concepts in the diagram below are defined in Section 3.8, **(B)**, except for topological amenability which is defined below in Appendix A.3.

$$
\begin{tikzcd}
\begin{array}{c} E_\Gamma^X \text{ hyperfinite} \\ + \text{ amenable stabilizers} \end{array} \arrow[dr, Rightarrow] & & \\
& \Gamma \curvearrowright X \text{ Borel amenable} \arrow[dl, Rightarrow] & \\
\begin{array}{c} E_\Gamma^X \text{ amenable} \\ + \text{ amenable stabilizers} \end{array} \arrow[d, Rightarrow] & & \\
\begin{array}{c} E_\Gamma^X \text{ measure-amenable} \\ + \text{ amenable stabilizers} \end{array} \arrow[r, Leftrightarrow] & \Gamma \curvearrowright X \text{ measure-amenable} \arrow[r, Leftrightarrow] & \Gamma \curvearrowright X \text{ topologically amenable} \arrow[uul, Rightarrow, "\sigma-compact"']
\end{tikzcd}
$$

Denote by $\mathrm{Prob}(\Gamma)$ the closed Γ-subspace of $[0,1]^\Gamma$ consisting of probability measures on Γ.

A.1 Borel amenability

We first have the following result:

Theorem A.1.1. *Let $\Gamma \curvearrowright X$ be a Borel action of a countable group on a standard Borel space, and consider the following statements:*

(1) *E_Γ^X is hyperfinite and every stabilizer is amenable.*

(2) *$\Gamma \curvearrowright X$ is Borel amenable.*

(3) *E_Γ^X is amenable and every stabilizer is amenable.*

Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3).

Proof. Let $E := E_\Gamma^X$.

(1) $\implies$ (2): Since E is hyperfinite, it is amenable in a strong sense: there is a sequence $p_n\colon E \to [0,1]$ of Borel functions, such that p_n^x is a probability measure supported on $[x]_E$, for every $(x,y) \in E$, we have $\|p_n^x - p_n^y\|_1 \to 0$, and additionally, for every y, there are only finitely many x with $p_n^x(y) > 0$.

Let $\alpha\colon E \to \Gamma$ be a Borel function such that for every $(x,y) \in E$, we have $y = \alpha_x^y \cdot x$ and $\alpha_y^x \alpha_x^y = 1$. Write $\Gamma = \bigcup_n S_n$ as an increasing union of finite subsets.

We claim that there is a sequence $q_n\colon X \to \mathrm{Prob}(\Gamma)$ of Borel functions with q_n^x supported on Γ_x, such that for every $(x,y) \in E$ with $p_n^x(y) > 0$ and every $\gamma \in S_n$, we have $\|q_n^y - \alpha_{\gamma\cdot x}^y \gamma \alpha_y^x \cdot q_n^y\|_1 < \frac{1}{n}$. To see this, for every $y \in X$, by amenability of Γ_y, let A_n^y be the least (in some enumeration) finite subset of Γ such that $A_n^y \subseteq \Gamma_y$ and

$$\frac{|A_n^y \triangle \alpha_{\gamma\cdot x}^y \gamma \alpha_y^x A_n^y|}{|A_n^y|} < \frac{1}{n}$$

for every $x \in [y]_E$ with $p_n^x(y) > 0$ and every $\gamma \in S_n$. Then let $q_n^y := \frac{1}{|A_n^y|}\mathbf{1}_{A_n^y}$ be the uniform distribution on A_n^y. Then

$$\|q_n^y - \alpha_{\gamma\cdot x}^y \gamma \alpha_y^x \cdot q_n^y\|_1 = \frac{\|\mathbf{1}_{A_n^y} - \mathbf{1}_{\alpha_{\gamma\cdot x}^y \gamma \alpha_y^x A_n^y}\|_1}{|A_n^y|} = \frac{|A_n^y \triangle \alpha_{\gamma\cdot x}^y \gamma \alpha_y^x A_n^y|}{|A_n^y|} < \frac{1}{n}.$$

Let $r_n\colon X \to \mathrm{Prob}(\Gamma)$ be defined by

$$r_n^x(\gamma) = p_n^x(\gamma \cdot x) q_n^{\gamma\cdot x}(\gamma \alpha_{\gamma\cdot x}^x).$$

Let $x \in X$ and $\gamma \in \Gamma$. Then

$$\begin{aligned}
\|r_n^{\gamma\cdot x} - \gamma \cdot r_n^x\|_1 &= \sum_{\delta\in\Gamma} |r_n^{\gamma\cdot x}(\delta) - r_n^x(\gamma^{-1}\delta)| \\
&= \sum_{\delta\in\Gamma} \left| p_n^{\gamma\cdot x}(\delta^{-1}\gamma\cdot x) q_n^{\delta^{-1}\gamma\cdot x}(\alpha_{\gamma\cdot x}^{\delta^{-1}\gamma\cdot x}\delta) - p_n^x(\delta^{-1}\gamma\cdot x) q_n^{\delta^{-1}\gamma\cdot x}(\alpha_x^{\delta^{-1}\gamma\cdot x}\gamma^{-1}\delta)\right| \\
&= \sum_{y\in[x]_E}\sum_{\lambda\in\Gamma_y} \left| p_n^{\gamma\cdot x}(y) q_n^y(\lambda) - p_n^x(y) q_n^y(\alpha_x^y \gamma^{-1}\alpha_y^{\gamma\cdot x}\lambda)\right| \\
&\leq \sum_{y\in[x]_E} |p_n^{\gamma\cdot x}(y) - p_n^x(y)| \sum_{\lambda\in\Gamma_y} q_n^y(\lambda) \\
&\quad + \sum_{y\in[x]_E} p_n^x(y) \sum_{\lambda\in\Gamma_y} |q_n^y(\lambda) - q_n^y(\alpha_x^y\gamma^{-1}\alpha_y^{\gamma\cdot x}\lambda)| \\
&= \|p_n^{\gamma\cdot x} - p_n^x\|_1 + \sum_{y\in[x]_E} p_n^x(y) \|q_n^y - \alpha_{\gamma\cdot x}^y \gamma \alpha_y^x \cdot q_n^y\|_1.
\end{aligned}$$

If $\gamma \in S_n$, then the second term is bounded by

$$\sum_{\substack{y\in[x]_E \\ p_n^x(y)>0}} p_n^x(y)\frac{1}{n} = \frac{1}{n} \to 0$$

so the whole expression converges to 0.

(2) $\Longrightarrow$ (3): Let $p_n \colon X \to \mathrm{Prob}(\Gamma)$ witness the Borel amenability of the action $\Gamma \curvearrowright X$.

To show that E is amenable, define $q_n \colon E \to [0,1]$ by

$$q_n^x(y) := \sum_{\substack{\gamma\in\Gamma \\ \gamma\cdot y = x}} p_n^x(\gamma).$$

Now if $x \in X$ and $\gamma \in \Gamma$, then we have

$$\begin{aligned}
\|q_n^{\gamma\cdot x} - q_n^x\|_1 &= \sum_{y\in[x]_E} \left| \sum_{\substack{\delta\in\Gamma \\ \delta\cdot y=\gamma\cdot x}} p_n^{\gamma\cdot x}(\delta) - \sum_{\substack{\lambda\in\Gamma \\ \lambda\cdot y=x}} p_n^x(\lambda) \right| \\
&= \sum_{y\in[x]_E} \left| \sum_{\substack{\delta\in\Gamma \\ \delta\cdot y=\gamma\cdot x}} p_n^{\gamma\cdot x}(\delta) - \sum_{\substack{\delta\in\Gamma \\ \delta\cdot y=\gamma x}} p_n^x(\gamma^{-1}\delta) \right| \\
&\leq \sum_{y\in[x]_E} \sum_{\substack{\delta\in\Gamma \\ \delta\cdot y=\gamma\cdot x}} |p_n^{\gamma\cdot x}(\delta) - p_n^x(\gamma^{-1}\delta)| \\
&= \sum_{\delta\in\Gamma} |p_n^{\gamma\cdot x}(\delta) - p_n^x(\gamma^{-1}\delta)| \\
&= \|p_n^{\gamma\cdot x} - \gamma\cdot p_n^x\|_1 \\
&\to 0.
\end{aligned}$$

Thus E is amenable.

Now let $x \in X$. To see that Γ_x is amenable, let T be a transversal for the right cosets of Γ_x in Γ, and define $q_n \in \mathrm{Prob}(\Gamma_x)$ by

$$q_n(\gamma) := \sum_{t\in T} p_n^x(\gamma t).$$

Then for every $\gamma \in \Gamma_x$, we have

$$\begin{aligned}
\|q_n - \gamma\cdot q_n\|_1 &= \sum_{\delta\in\Gamma_x} |q_n(\delta) - q_n(\gamma^{-1}\delta)| \\
&= \sum_{\delta\in\Gamma_x} \left| \sum_{t\in T} p_n^x(\delta t) - \sum_{t\in T} p_n^x(\gamma^{-1}\delta t) \right| \\
&\leq \sum_{\delta\in\Gamma_x} \sum_{t\in T} |p_n^x(\delta t) - p_n^x(\gamma^{-1}\delta t)| \\
&= \sum_{\lambda\in\Gamma} |p_n^x(\lambda) - p_n^x(\gamma^{-1}\lambda)| \\
&= \|p_n^x - \gamma\cdot p_n^x\|_1 \\
&= \|p_n^{\gamma\cdot x} - \gamma\cdot p_n^x\|_1 \\
&\to 0.
\end{aligned}$$

Thus Γ_x is amenable.

$\square$

A.2 Measure amenability

By Theorem A.1.1 and the Connes-Feldman-Weiss theorem, see, e.g., [KM04, Theorem 10.1], we have the following analogue of [AEG94] (see also [ADR00, Corollary 5.3.33]):

Theorem A.2.1. *Let $\Gamma \curvearrowright X$ be a Borel action of a countable group on a standard Borel space, and let μ be a Borel probability measure on X. Then the following are equivalent:*

(1) *$\Gamma \curvearrowright X$ is μ-amenable.*

(2) *E_Γ^X is μ-amenable and μ-a.e. stabilizer is amenable.*

Corollary A.2.2. *Let $\Gamma \curvearrowright X$ be a Borel action of a countable group on a standard Borel space. Then the following are equivalent:*

(1) *$\Gamma \curvearrowright X$ is measure-amenable.*

(2) *E_Γ^X is measure-amenable and every stabilizer is amenable.*

A.3 Topological amenability

Let Γ be a countable group, and let X be a Polish space. A continuous action $\Gamma \curvearrowright X$ is **topologically amenable** if for every finite $S \subseteq \Gamma$, every compact $K \subseteq X$, and every $\varepsilon > 0$, there is some continuous $p\colon X \to \mathrm{Prob}(\Gamma)$, such that, letting $p^x = p(x)$, we have

$$\max_{\substack{\gamma \in S \\ x \in K}} \|p^{\gamma \cdot x} - \gamma \cdot p^x\|_1 < \varepsilon.$$

Topological amenability is equivalent to measure amenability (see [ADR00, Theorem 3.3.7] for the locally compact case, also the proof of [BO08, Proposition 5.2.1]):

Theorem A.3.1. *Let $\Gamma \curvearrowright X$ be a continuous action of a countable group on a Polish space. Then the following are equivalent:*

(1) *$\Gamma \curvearrowright X$ is topologically amenable.*

(2) *$\Gamma \curvearrowright X$ is measure-amenable.*

Moreover, if X is σ-compact, then these are also equivalent to

(3) $\Gamma \curvearrowright X$ *is Borel amenable.*

The following lemma says that in the definition of μ-amenability, we can upgrade the Borel functions to continuous ones:

Lemma A.3.2. *Let $\Gamma \curvearrowright X$ be a continuous action of a countable group on a Polish space, and let μ be a Borel probability measure on X. Then the following are equivalent:*

1. $\Gamma \curvearrowright X$ *is μ-amenable.*
2. *For every finite $S \subseteq \Gamma$ and every $\varepsilon > 0$, there is some continuous $p\colon X \to \mathrm{Prob}(\Gamma)$ such that for every $\gamma \in S$, we have*

$$\int_X \|p^{\gamma \cdot x} - \gamma \cdot p^x\|_1 \, \mathrm{d}\mu(x) < \varepsilon.$$

Proof of Lemma A.3.2. It suffices to show that for every Borel $p\colon X \to \mathrm{Prob}(\Gamma)$ and every $\varepsilon > 0$, there is some continuous $q\colon X \to \mathrm{Prob}(\Gamma)$ such that

$$\int_X \|p - q\|_1 \, \mathrm{d}\mu < \varepsilon.$$

By Lusin's theorem [Kec95, 17.12], there is a closed $F \subseteq X$ with $\mu(F) > 1 - \frac{\varepsilon}{2}$ such that $p\restriction F$ is continuous. By Dugundji's extension theorem [Dug51, 4.1], there is some continuous extension $q\colon X \to \mathrm{Prob}(\Gamma)$ of $p\restriction F$. Then p and q agree on F, so we are done. □

Proof of Theorem A.3.1.

(1) $\implies$ (2): This follows from tightness of Borel probability measures, see [Kec95, 17.11].

(2) $\implies$ (1): Let $S \subseteq \Gamma$ be finite and let $K \subseteq X$ be compact. Denote below by $C(X, \mathrm{Prob}(\Gamma))$ the set of continuous functions $X \to \mathrm{Prob}(\Gamma)$, and define $\Psi\colon C(X, \mathrm{Prob}(\Gamma)) \to C(K)$ by

$$\Psi_p(x) = \sum_{\gamma \in S} \|p^{\gamma \cdot x} - \gamma \cdot p^x\|_1.$$

By measure-amenability and Lemma A.3.2, for every Borel probability measure μ on K, we have

$$\inf_{f \in \operatorname{im} \Psi} \int_K f \, \mathrm{d}\mu = 0.$$

So by the Riesz representation theorem for $C(K)$, for every functional $\varphi \in C(K)^*$, we have

$$\inf_{f \in \operatorname{im} \Psi} |\varphi(f)| = 0.$$

Thus by the Hahn-Banach separation theorem, we have

$$\inf_{f \in \operatorname{Conv}(\operatorname{im} \Psi)} \|f\|_\infty = 0,$$

where $\operatorname{Conv}(\operatorname{im} \Psi)$ denotes the convex hull of $\operatorname{im} \Psi$. Since

$$\Psi_{\sum_{i<k} \alpha_i p_i} \leq \sum_{i<k} \alpha_i \Psi_{p_i},$$

we have

$$\inf_{f \in \operatorname{im} \Psi} \|f\|_\infty = 0,$$

so we are done.

Now suppose that X is σ-compact. It suffices to show (1) $\implies$ (3). Write $\Gamma = \bigcup_n S_n$ as an increasing union of finite subsets, and write $X = \bigcup_n K_n$ as an increasing union of compact subsets. Then for each n, by topological amenability, there is some continuous $p_n \colon X \to \operatorname{Prob}(\Gamma)$ such that

$$\max_{\substack{\gamma \in S_n \\ x \in K_n}} \|p^{\gamma \cdot x} - \gamma \cdot p^x\|_1 < \frac{1}{n}.$$

Then $(p_n)_n$ witnesses Borel amenability of $\Gamma \curvearrowright X$. $\square$

B Weak containment

We give here the proof of Theorem 3.9.6.

(i) Note that if $h \in \operatorname{Homeo}(2^{\mathbb{N}})$, then h is free (i.e., $h(x) \neq x, \forall x$) iff there is a finite sequence $(K_i)_{i=1}^n$ of clopen sets such that $2^{\mathbb{N}} = \bigcup_{i=1}^n K_i$ and $h(K_i) \cap K_i = \emptyset, \forall i$. From this is follows easily that if $\mathbf{a} \preceq \mathbf{b}$ and $\mathbf{a}$ is free, then so is $\mathbf{b}$, recalling that if

$h_n, h \in \mathrm{Homeo}(2^{\mathbb{N}})$ and $h_n \to h$, then for any clopen set K, for all large enough n, $h_n(K) = h(K)$.

(ii) By Nadkarni's Theorem it is enough to show that if $\mathbf{a} \preceq \mathbf{b}$ and $\mathbf{b}$ admits an invariant (probability Borel) measure, then so does $\mathbf{a}$. We have that $\mathbf{a}$ is the limit of a sequence of actions $\mathbf{a}_n$ that admit invariant measures μ_n, which by compactness in the space of measures we can assume that they converge to a measure μ. We will check that μ is invariant for $\mathbf{a}$. Let for $\gamma \in \Gamma$, $\gamma^{\mathbf{a}}(x) = \mathbf{a}(\gamma, x)$, and similarly for $\mathbf{a}_n$, so that $\gamma^{\mathbf{a}_n} \to \gamma^{\mathbf{a}}$. Fix a clopen set K. Then for all large enough n, we have that $\gamma^{\mathbf{a}}(K) = \gamma^{\mathbf{a}_n}(K)$. Then $\mu(\gamma^{\mathbf{a}}(K)) = \lim_n \mu_n(\gamma^{\mathbf{a}}(K)) = \lim_n \mu_n(\gamma^{\mathbf{a}_n}(K))) = \lim_n \mu_n(K) = \mu(K)$.

(iii) We will use the following lemma (see [Ele19, Lemma 3.3]).

Lemma B.0.1. *Let $p\colon 2^{\mathbb{N}} \to \mathrm{Prob}(\Gamma)$ be continuous. Then for each $\varepsilon > 0$, there is finite $F \subseteq \Gamma$ and continuous $q\colon 2^{\mathbb{N}} \to \mathrm{Prob}(\Gamma)$ such that for all $x \in 2^{\mathbb{N}}$ and $\gamma \notin F$, we have that $q^x(\gamma) = 0$, and moreover $||p^x - q^x||_1 < \varepsilon$.*

Proof. Let (F_n) be an increasing sequence of finite subsets of Γ with $1 \in F_0$ that covers Γ. Then there must exist n such that for all $x \in 2^{\mathbb{N}}$, $p^x(F_n) > 1 - \frac{\varepsilon}{3}$. Indeed otherwise there is a sequence of points x_n such that $p^{x_n}(F_n) \le 1 - \frac{\varepsilon}{3}$. Then there is a subsequence of x_n converging to some x for which $p^x(\Gamma) \le 1 - \frac{\varepsilon}{3}$, a contradiction. Take $F = F_n$.

Define now q as follows: If $\gamma \in F \setminus \{1\}$, then $q^x(\gamma) = p^x(\gamma)$; $q^x(\gamma) = 0$, if $x \notin F$; $q^x(1) = p^x(1) + p^x(\Gamma \setminus F)$. □

Let now $\mathbf{a} \preceq \mathbf{b}$ and assume that $\mathbf{a}$ is topologically amenable. Fix a finite subset $S \subseteq \Gamma$ and $\varepsilon > 0$. Then there is a continuous $p\colon 2^{\mathbb{N}} \to \mathrm{Prob}(\Gamma)$ such that

$$\max_{\substack{\gamma \in S \\ x \in 2^{\mathbb{N}}}} \|p^{\mathbf{a}(\gamma,x)} - \gamma \cdot p^x\|_1 < \varepsilon.$$

By Lemma B.0.1 we can also assume that there is finite $F \subseteq \Gamma$ such that $p^x(\gamma) = 0, \forall \gamma \notin F$. Since $\mathbf{a}$ is a limit of conjugates of $\mathbf{b}$, it follows that there is a conjugate $\mathbf{c}$ of $\mathbf{b}$, say via the homeomorphism h of $2^{\mathbb{N}}$, such that

$$\max_{\substack{\gamma \in S \\ x \in 2^{\mathbb{N}}}} \|p^{\mathbf{c}(\gamma,x)} - p^{\mathbf{a}(\gamma,x)}\|_1$$

can be made as small as we want, so that

$$\max_{\substack{\gamma \in S \\ x \in 2^{\mathbb{N}}}} \|p^{\mathbf{c}(\gamma,x)} - \gamma \cdot p^x\|_1 < \varepsilon.$$

Then if $q^x = p^{h(x)}$, we have that q is continuous and

$$\max_{\substack{\gamma\in S\\ x\in 2^{\mathbb{N}}}} \|q^{\mathbf{b}(\gamma,x)} - \gamma\cdot q^x\|_1 < \varepsilon.$$

and the proof is complete.

C The Correspondence Theorem of Hochman

We give here the proof of Theorem 3.9.7. Below for each Polish space X, we denote by $K(X)$ the Polish space of all compact subsets of X with the Vietoris topology, which is compact if X is compact. If $d \le 1$ is a compatible metric on X, let d_H be the corresponding Hausdorff metric for $K(X)$, which induces the Vietoris topology (see [Kec95, 4.F]). Fix a compatible metric $d \le 1$ for $\mathbb{I}^{\mathbb{N}}$ with corresponding Hausdorff metric d_H for $K(\mathbb{I}^{\mathbb{N}})$, the product metric d_1 for $(\mathbb{I}^{\mathbb{N}})^\Gamma$ given by $d_1((x_\gamma)_\gamma,(y_\gamma)_\gamma) = \sum_{i=0}^\infty 2^{-i} d(x_{\gamma_i}, y_{\gamma_i})$, where $\Gamma = \{\gamma_i : i \in \mathbb{N}\}$, with $\gamma_0 = 1$, and the corresponding Hausdorff metric $d_{1,H}$ on $K(\mathbb{I}^{\mathbb{N}})^\Gamma$.

The idea is to construct a Polish space E and a topological embedding

$$\pi\colon \mathrm{Act}(\Gamma, 2^{\mathbb{N}}) \times E \to \mathrm{Sh}(\Gamma, \mathbb{I}^{\mathbb{N}}),$$

so that for all $\mathbf{a} \in \mathrm{Act}(\Gamma, 2^{\mathbb{N}})$ and $e \in E$, the subshift $\pi(\mathbf{a}, e)$ is topologically isomorphic to $\mathbf{a}$ and moreover the range of π is dense (and therefore dense G_δ) in $\mathrm{Sh}(\Gamma, \mathbb{I}^{\mathbb{N}})$. The Correspondence Theorem follows immediately.

To construct π we will need a series of lemmas. Below denote by D the set of topological embeddings of $2^{\mathbb{N}}$ to $\mathbb{I}^{\mathbb{N}}$. This is a G_δ subset of the Polish space of all continuous maps from $2^{\mathbb{N}}$ to $\mathbb{I}^{\mathbb{N}}$ with the sup metric, so it is a Polish space.

Lemma C.0.1. *There is a G_δ subset $E \subseteq D$ so that the map $e \in E \mapsto e(2^{\mathbb{N}}) \in K(\mathbb{I}^{\mathbb{N}})$ is a topological embedding with dense (and therefore dense G_δ) image in $K(\mathbb{I}^{\mathbb{N}})$.*

Proof. The set $\mathcal{C}$ of all Cantor sets in $K(\mathbb{I}^{\mathbb{N}})$ is a dense G_δ subset of $K(\mathbb{I}^{\mathbb{N}})$ (see [Kec95, 8.8]). Moreover by the Jankov-von Neumann Uniformization Theorem (see, [Kec95, 18.1]) there is a Baire measurable map $f\colon \mathcal{C} \to D$ such that if $f(C) = e$, then $e(2^{\mathbb{N}}) = C$. Then by [Kec95, 8.38], there is a dense G_δ subset $\mathcal{C}'$ of $\mathcal{C}$ such that f is continuous on $\mathcal{C}'$. Clearly $f|\mathcal{C}'$ is a topological embedding and we can take $E = f(\mathcal{C}')$. $\square$

For each action $\mathbf{a} \in \mathrm{Act}(\Gamma, C)$, where C is a Cantor space, and topological embedding e of C into $\mathbb{I}^{\mathbb{N}}$, there is a canonical subshift in $\mathrm{Sh}(\Gamma, \mathbb{I}^{\mathbb{N}})$, denoted by $e^*_{\mathbf{a}}$, defined as follows: Let $f_{\mathbf{a}}\colon C \to (\mathbb{I}^{\mathbb{N}})^{\Gamma}$ be defined by:

$$f_{\mathbf{a}}(x)_{\gamma} = e(\mathbf{a}(\gamma^{-1}, x)).$$

We put $e^*_{\mathbf{a}} = f_{\mathbf{a}}(C)$. This is a subshift in $\mathrm{Sh}(\Gamma, \mathbb{I}^{\mathbb{N}})$ topologically isomorphic to $\mathbf{a}$. Finally we define for $\mathbf{a} \in \mathrm{Act}(\Gamma, 2^{\mathbb{N}}), e \in E$,

$$\pi(\mathbf{a}, e) = e^*_{\mathbf{a}}.$$

Clearly $\pi(\mathbf{a}, e)$ is isomorphic to $\mathbf{a}$, so it only remains to show that π is a topological embedding with dense range.

Lemma C.0.2. *The map π is a topological embedding.*

Proof. It is clear form Lemma C.0.1 that π is injective and continuous. We will verify next that its inverse is continuous.

Assume that $\pi(\mathbf{a}_n, e_n) \to \pi(\mathbf{a}, e)$, in order to show that $\mathbf{a}_n \to \mathbf{a}$ and $e_n \to e$.

(1) $e_n \to e$: Let π_1 be the projection map $\pi_1\colon (\mathbb{I}^{\mathbb{N}})^{\Gamma} \to \mathbb{I}^{\mathbb{N}}$ given by $\pi_1(x) = x(1)$. Then $\pi_1(e^*_{\mathbf{a}}) = e(2^{\mathbb{N}})$. Since $\pi(\mathbf{a}_n, e_n) \to \pi(\mathbf{a}, e)$, by applying π_1 we have that $e_n(2^{\mathbb{N}}) \to e(2^{\mathbb{N}})$, therefore $e_n \to e$ by Lemma C.0.1.

(2) $\mathbf{a}_n \to \mathbf{a}$: Since $e_n \to e$ uniformly, it is clear that $\pi(\mathbf{a}_n, e) \to \pi(\mathbf{a}, e)$. Replacing $2^{\mathbb{N}}$ with $C = e(2^{\mathbb{N}})$ and each action on $2^{\mathbb{N}}$ by its copy on C via e, we can assume that actually $\mathbf{a}_n, \mathbf{a} \in \mathrm{Act}(\Gamma, C)$ and if i_C is the identity on C, then we have that $(i_C)^*_{\mathbf{a}_n} \to (i_C)^*_{\mathbf{a}}$. To show that $\mathbf{a}_n \to \mathbf{a}$ we need to show that for each $\gamma \in \Gamma$, $\mathbf{a}_n(\gamma, c) \to \mathbf{a}(\gamma, c)$, uniformly for $c \in C$.

Fix $\varepsilon > 0$ and then find $\delta < \frac{\varepsilon}{k_0}$, where k_0 is some positive constant, defined below, depending only on γ, such that

$$d(c, d) < \delta \implies d(\mathbf{a}(\gamma, c), \mathbf{a}(\gamma, d)) < \varepsilon.$$

Next fix N such that for all $n \geq N$, we have that $d_{1,H}((i_C)^*_{\mathbf{a}_n}, (i_C)^*_{\mathbf{a}}) < \delta$. Let now $c \in C$. Then there is $c' \in C$ such that

$$d_1((\mathbf{a}_n(\gamma, c))_{\gamma}, (\mathbf{a}(\gamma, c'))_{\gamma}) < \delta$$

and therefore $d(c, c') < \delta$ and $d(\mathbf{a}_n(\gamma, c), d(\mathbf{a}(\gamma, c')) < k_0\delta$, for some constant $k_0 > 0$ depending only on γ. Thus $d(\mathbf{a}(\gamma, c), \mathbf{a}(\gamma, c')) < \varepsilon$ and so $d(\mathbf{a}_n(\gamma, c), \mathbf{a}(\gamma, c)) < \varepsilon + k_0\delta < 2\varepsilon$ and the proof is complete. $\square$

We next need the following general topological lemma. Below C will be a Cantor space with compatible metric d_C, X a perfect Polish space with metric d and $\mathcal{K}'$ is a dense subset of $K(X)$ consisting of Cantor sets.

Lemma C.0.3. *If $f\colon C \to X$ is such that*

$$d(f(c), f(d)) \leq d_C(c, d),$$

for all $c.d \in C$, then for every $\varepsilon > 0$, there is $K \in \mathcal{K}'$ and a homeomorphism $g\colon C \to K$ such that

$$d(f(c), g(c)) < \varepsilon, \forall c \in C.$$

Proof. Let $C = \bigsqcup_{i=1}^n C_i$ be a partition of C into nonempty clopen sets with $d_C(C_i) < \frac{\varepsilon}{3}, 1 \leq i \leq n$. Let $c_i \in C_i, 1 \leq i \leq n$. Recursively define distinct $x_i \in X, 1 \leq i \leq n$, such that $d(f(c_i), x_i) < \frac{\varepsilon}{3}$. Let δ be the minimum of the distances $d(x_i, x_j)$, for $1 \leq i \neq j \leq n$. Then find $K \in \mathcal{K}'$ such that

$$d_H(K, \{x_1, \dots, x_n\}) < \rho \leq \min\{\frac{\delta}{2}, \frac{\varepsilon}{3}\}.$$

Then for each $k \in K$, there is *unique* $1 \leq i \leq n$ such that $d(k, x_i) < \rho$. Put $K_i = K \cap B_{x_i}(\rho)$, where $B_{x_i}(\rho)$ is the open ball of radius ρ with center x_i in the metric d. Then

(1) $K = \bigsqcup_{i=1}^n K_i$,
(2) For each $1 \leq i \leq n$, $K_i \neq \emptyset$, since there is some $k \in K$ with $d(k, x_i) < \rho$,
(3) For each $1 \leq i \leq n$, K_i is clopen in K.

Thus $K = \bigsqcup_i^n K_i$ is a clopen partition of K into Cantor sets. Therefore there is a homeomorphism $g_i\colon C_i \to K_i$, $1 \leq i \leq n$. Put $g = \bigsqcup_{i=1}^n g_i$ so that $g\colon C \to K$ is a homeomorphism.

Let now $c \in C$. Then for some (unique) $1 \leq i \leq n$, $c \in C_i$, so $d_C(c, c_i) < \frac{\varepsilon}{3}$. Thus $d(f(c), f(c_i)) < \frac{\varepsilon}{3}$. Also $d(f(c_i), x_i) < \frac{\varepsilon}{3}$ and $g(c) = g_i(c) \in K_i$, so $d_X(g(c), x_i) < \rho < \frac{\varepsilon}{3}$. Thus $d(f(c), g(c)) < \varepsilon$. □

Lemma C.0.4. *Let $\mathcal{K}_0$ be a dense subset of $K(\mathbb{I}^{\mathbb{N}})$ consisting of Cantor sets. Then for each Cantor subshift K in $\mathrm{Sh}(\Gamma, \mathbb{I}^{\mathbb{N}})$, K is the limit of a sequence $K_n = (i_{C_n})^*_{\mathbf{a}_n}$, with $C_n \in \mathcal{K}_0$, $\mathbf{a}_n \in \mathrm{Act}(\Gamma, C_n)$.*

Proof. Let $\varepsilon > 0$. Let $K_1 = \pi_1(K)$. Then by Lemma C.0.3 (with $C = K, X = \mathbb{I}^{\mathbb{N}}, f = \pi_1, \mathcal{K}' = \mathcal{K}_0$), there is a homeomorphism $g\colon K \to C_0$, where $C_0 \in \mathcal{K}_0$ and $d(x(1), g(x)) < \varepsilon$, for all $x \in K$. Push the shift action on K via g to the action $\mathbf{a} \in \mathrm{Act}(\Gamma, C_0)$. Let $L = (i_{C_0})_{\mathbf{a}}(C_0)$ and put $j = (i_{C_0})_{\mathbf{a}} \circ g\colon K \to L$. Then for all $x \in K$,

$$d(x(1), j(x)(1)) = d(x(1), g(x)) < \varepsilon.$$

and so for every $\gamma \in \Gamma, x \in K$,

$$d(x(\gamma), j(x)(\gamma)) = d(\mathbf{s}(\gamma^{-1}, x)(1), (i_{C_0})_{\mathbf{a}}(g(x))(\gamma)) = d(\mathbf{s}(\gamma^{-1}, x)(1), g(\mathbf{s}(\gamma^{-1}, x)) < \varepsilon,$$

where $\mathbf{s}$ is the shift action. Thus $d_{1,H}(K, L) < \varepsilon$ and the proof is complete. □

Next note the following:

Lemma C.0.5. *The Cantor subshifts are dense in* $\mathrm{Sh}(\Gamma, \mathbb{I}^{\mathbb{N}})$.

Proof. Let $K \in \mathrm{Sh}(\Gamma, \mathbb{I}^{\mathbb{N}})$ and let $D = \pi_1(K)$, which is a compact subset of $\mathbb{I}^{\mathbb{N}}$. Let then C be a Cantor set such that the Hausdorff distance of C from D is less than $\frac{\varepsilon}{2}$. Put

$$K_0' = \{x' \in C^\Gamma : \exists x \in K(d(x(\gamma), x'(\gamma)) < \frac{\varepsilon}{2}, \forall \gamma \in \Gamma)\}.$$

and let K' be the closure of K_0'. Then K' is a Cantor subshift with $d_{1,H}(K, K') < \varepsilon$. □

We finally complete the proof of the Correspondence Theorem by showing that π has dense range.

Let K be a Cantor subshift. Let $\mathcal{K}_0 = \{e(2^{\mathbb{N}}) : e \in E\}$, which is dense in $K(\mathbb{I}^{\mathbb{N}})$ by Lemma C.0.1. Then by Lemma C.0.4, we can find $K_n \to K$ with $K_n = (i_{C_n})^*_{\mathbf{a}_n}$, where $C_n \in \mathcal{K}_0$ and $\mathbf{a}_n \in \mathrm{Act}(\Gamma, C_n)$. Let $e_n \in E$ be a homeomorphism of $2^{\mathbb{N}}$ with C_n and pull back the action $\mathbf{a}_n$ via e_n to the action $\mathbf{b}_n \in \mathrm{Act}(\Gamma, 2^{\mathbb{N}})$. Then clearly $\pi(\mathbf{b}_n, e_n) = K_n$ and we are done.

References

[ADR00] C. Anantharaman-Delaroche and J. Renault. *Amenable groupoids*, volume 36 of *Monographies de L'Enseignement Mathématique [Monographs of L'Enseignement Mathématique]*. L'Enseignement Mathématique, Geneva, 2000. With a foreword by Georges Skandalis and Appendix B by E. Germain.

[AEG94] Scot Adams, George A. Elliott, and Thierry Giordano. Amenable actions of groups. *Trans. Amer. Math. Soc.*, 344(2):803–822, 1994.

[Ber17] Anton Bernshteyn. Free subshifts with invariant measures from the Lovász Local Lemma. *arXiv:1702.02792*, 2017.

[BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[BK96] Howard Becker and Alexander S. Kechris. *The descriptive set theory of Polish group actions*, volume 232 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1996.

[BK20] Peter J. Burton and Alexander S. Kechris. Weak containment of measure-preserving group actions. *Ergodic Theory Dynam. Systems*, 40(10):2681–2733, 2020.

[BO08] Nathanial P. Brown and Narutaka Ozawa. *C^*-algebras and finite-dimensional approximations*, volume 88 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

[CC20] John D Clemens and Samuel Coskey. New jump operators on equivalence relations. *arXiv:2008.06613*, 2020.

[Che21] Ruiyuan Chen. Decompositions and measures on countable Borel equivalence relations. *Ergodic Theory Dynam. Systems*, 41(12):3671–3703, 2021.

[CK18] Ruiyuan Chen and Alexander S. Kechris. Structurable equivalence relations. *Fund. Math.*, 242(2):109–185, 2018.

[CKM13] Clinton T. Conley, Alexander S. Kechris, and Benjamin D. Miller. Stationary probability measures and topological realizations. *Israel J. Math.*, 198(1):333–345, 2013.

[CM14] Clinton T. Conley and Benjamin D. Miller. An antibasis result for graphs of infinite Borel chromatic number. *Proc. Amer. Math. Soc.*, 142(6):2123–2133, 2014.

[CM17] Clinton T. Conley and Benjamin D. Miller. Measure reducibility of countable Borel equivalence relations. *Ann. of Math. (2)*, 185(2):347–402, 2017.

[Dan01] Alexandre I. Danilenko. Strong orbit equivalence of locally compact Cantor minimal systems. *Internat. J. Math.*, 12(1):113–123, 2001.

[DJK94] R. Dougherty, S. Jackson, and A. S. Kechris. The structure of hyperfinite Borel equivalence relations. *Trans. Amer. Math. Soc.*, 341(1):193–225, 1994.

[DK00] Randall Dougherty and Alexander S. Kechris. How many Turing degrees are there? In *Computability theory and its applications (Boulder, CO, 1999)*, volume 257 of *Contemp. Math.*, pages 83–94. Amer. Math. Soc., Providence, RI, 2000.

[Dow91] Tomasz Downarowicz. The Choquet simplex of invariant measures for minimal flows. *Israel J. Math.*, 74(2-3):241–256, 1991.

[Dug51] J. Dugundji. An extension of Tietze's theorem. *Pacific J. Math.*, 1:353–367, 1951.

[dV93] J. de Vries. *Elements of topological dynamics*, volume 257 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1993.

[EGS15] Mikhail Ershov, Gili Golan, and Mark Sapir. The Tarski numbers of groups. *Adv. Math.*, 284:21–53, 2015.

[Ele18] Gábor Elek. On universal continuous actions on the Cantor set. *arXiv:1803.05461*, 2018.

[Ele19] Gábor Elek. Qualitative graph limit theory. Cantor dynamical systems and constant-time distributed algorithms. *1812.07511*, 2019.

[GdlH90] É. Ghys and P. de la Harpe, editors. *Sur les groupes hyperboliques d'après Mikhael Gromov*, volume 83 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1990. Papers from the Swiss Seminar on Hyperbolic Groups held in Bern, 1988.

[GdlH97] Thierry Giordano and Pierre de la Harpe. Moyennabilité des groupes dénombrables et actions sur les espaces de Cantor. *C. R. Acad. Sci. Paris Sér. I Math.*, 324(11):1255–1258, 1997.

[GJS09] Su Gao, Steve Jackson, and Brandon Seward. A coloring property for countable groups. *Math. Proc. Cambridge Philos. Soc.*, 147(3):579–592, 2009.

[Hoc08] Michael Hochman. Genericity in topological dynamics. *Ergodic Theory Dynam. Systems*, 28(1):125–165, 2008.

[Hoc19] Michael Hochman. Every Borel automorphism without finite invariant measures admits a two-set generator. *J. Eur. Math. Soc. (JEMS)*, 21(1):271–317, 2019.

[HR94] John M. Howie and N. Ruškuc. Constructions and presentations for monoids. *Comm. Algebra*, 22(15):6209–6224, 1994.

[JKL02] S. Jackson, A. S. Kechris, and A. Louveau. Countable Borel equivalence relations. *J. Math. Log.*, 2(1):1–80, 2002.

[Kec88] Alexander S. Kechris. "$\mathsf{AD} + \mathsf{UNIFORMIZATION}$" is equivalent to "half $\mathsf{AD}_{\mathbb{R}}$". In *Cabal Seminar 81–85*, volume 1333 of *Lecture Notes in Math.*, pages 98–102. Springer, Berlin, 1988.

[Kec94] A. S. Kechris. Lectures on definable group actions and equivalence relations. *preprint*, 1994.

[Kec95] Alexander S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[Kec10] Alexander S. Kechris. *Global aspects of ergodic group actions*, volume 160 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2010.

[Kec21] A. S. Kechris. The spaces of measure preserving equivalence relations and graphs. *preprint*, 2021.

[Kec22] A. S. Kechris. The theory of countable Borel equivalence relations. *preprint*, 2022.

[KLW87] A. S. Kechris, A. Louveau, and W. H. Woodin. The structure of σ-ideals of compact sets. *Trans. Amer. Math. Soc.*, 301(1):263–288, 1987.

[KM04] Alexander S. Kechris and Benjamin D. Miller. *Topics in orbit equivalence*, volume 1852 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2004.

[KPT05] A. S. Kechris, V. G. Pestov, and S. Todorcevic. Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups. *Geom. Funct. Anal.*, 15(1):106–189, 2005.

[KS95] A. S. Kechris and S. Solecki. Approximation of analytic by Borel sets and definable countable chain conditions. *Israel J. Math.*, 89(1-3):343–356, 1995.

[KST99] A. S. Kechris, S. Solecki, and S. Todorcevic. Borel chromatic numbers. *Adv. Math.*, 141(1):1–44, 1999.

[KW22] A. S. Kechris and M.S. Wolman. An effective version of Nadkarni's theorem. *preprint*, 2022.

[Kwi12] Aleksandra Kwiatkowska. The group of homeomorphisms of the Cantor set has ample generics. *Bull. Lond. Math. Soc.*, 44(6):1132–1146, 2012.

[LS01] Roger C. Lyndon and Paul E. Schupp. *Combinatorial group theory*. Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[Mil06] B. D. Miller. Borel equivalence relations and everywhere faithful actions of free products. *preprint*, 2006.

[MS20] Timothée Marquis and Marcin Sabok. Hyperfiniteness of boundary actions of hyperbolic groups. *Math. Ann.*, 377(3-4):1129–1153, 2020.

[MSS16] Andrew Marks, Theodore A. Slaman, and John R. Steel. Martin's conjecture, arithmetic equivalence, and countable Borel equivalence relations. In *Ordinal definability and recursion theory: The Cabal Seminar. Vol. III*, volume 43 of *Lect. Notes Log.*, pages 493–519. Assoc. Symbol. Logic, Ithaca, NY, 2016.

[MZ07] Étienne Matheron and Miroslav Zelený. Descriptive set theory of families of small sets. *Bull. Symbolic Logic*, 13(4):482–537, 2007.

[Sho58] J. R. Shoenfield. The class of recursive functions. *Proc. Amer. Math. Soc.*, 9:690–692, 1958.

[Sol02] Sławomir Solecki. The space of composants of an indecomposable continuum. *Adv. Math.*, 166(2):149–192, 2002.

[Sol11] Sławomir Solecki. G_δ ideals of compact sets. *J. Eur. Math. Soc. (JEMS)*, 13(4):853–882, 2011.

[SZ94] Garrett Stuck and Robert J. Zimmer. Stabilizers for ergodic actions of higher rank semisimple groups. *Ann. of Math. (2)*, 139(3):723–747, 1994.

[Tho12] Simon Thomas. Universal Borel actions of countable groups. *Groups Geom. Dyn.*, 6(2):389–407, 2012.

[Tse15] Anush Tserunyan. Finite generators for countable group actions in the Borel and Baire category settings. *Adv. Math.*, 269:585–646, 2015.

[TW16] Grzegorz Tomkowicz and Stan Wagon. *The Banach-Tarski paradox*, volume 163 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, New York, second edition, 2016. With a foreword by Jan Mycielski.

[Wei89] Benjamin Weiss. Countable generators in dynamics—universal minimal models. In *Measure and measurable dynamics (Rochester, NY, 1987)*, volume 94 of *Contemp. Math.*, pages 321–326. Amer. Math. Soc., Providence, RI, 1989.

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