Abstract. The class of ergodic, invariant probability Borel measure for the shift action of a countable group is a $G_δ$ set in the compact, metrizable space of probability Borel measures. We study in this paper the descriptive complexity of the class of ergodic, quasi-invariant probability Borel measures and show that for any infinite countable group $Γ$ it is $Π^0_3$-hard, for the group $Z$ it is $Π^0_3$-complete, while for the free group $F_∞$ with infinite, countably many generators it is $Π^0_α$-complete, for some ordinal $α$ with $3 ≤ α ≤ ω + 2$. The exact value of this ordinal is unknown.

1. Introduction

For any Polish space $X$, let $P(X)$ be the Polish space of probability Borel measures on $X$ with the usual topology (see, e.g., [K, 17.E]). It is compact, metrizable, if $X$ is compact, metrizable. Any $f: X → Y$ induces the map $f_*: P(X) → P(Y)$, defined by $f_*(µ)(B) = µ(f^{-1}(B))$.

If $E$ is an equivalence relation on $X$, a measure $µ ∈ P(X)$ is ergodic for $E$ if for any Borel $E$-invariant set $A ⊆ X$, $µ(A) ∈ \{0, 1\}$. We denote by $ERG_E$ the set of such measures. Similarly if $a: Γ × X → X$ is an action of a group $Γ$ on $X$, a measure $µ$ is ergodic for $a$ if for any invariant under $a$ Borel set $A$, we have $µ(A) ∈ \{0, 1\}$. We denote again...
by ERG\(_a\) the set of such measures. Clearly ERG\(_a\) = ERG\(_{E_a}\), where \(E_a\) is the equivalence relation induced by (the orbits of) the action \(a\).

Consider now a continuous action \(a\) of a countable (discrete) group \(\Gamma\) on a compact, metrizable space \(K\). If \(a\) is understood from the context, we write \(\gamma \cdot x\) instead of \(a(\gamma, x)\). We also let \(\gamma^a(x) = a(\gamma, x)\).

It is a standard fact that the set INV\(_a\) of invariant measures for \(a\) is closed in \(P(K)\) and the set EINV\(_a\) of invariant, ergodic measures for \(a\) is \(G_\delta\) in \(P(K)\) (see, e.g., [G, Theorem 4.2]).

Recall that \(\mu \in P(K)\) is called quasi-invariant for the action \(a\) if for any \(\gamma \in \Gamma\), \(\gamma \cdot \mu \sim \mu\), where \(\sim\) denotes measure equivalence and \(\gamma \cdot \mu = (\gamma^a)\_\bullet (\mu)\). Denote by QINV\(_a\) the set of quasi-invariant measures for \(a\) and by EQINV\(_a\) the subset of ergodic, quasi-invariant measures for \(a\). Since the relation \(\sim\) of measure equivalence is \(\Pi^0_3\) in \(P(K)\), it follows that QINV\(_a\) is \(\Pi^0_3\) in \(P(K)\). From a (more general) result of Ditzen in [D], it follows that ERG\(_a\) is Borel and, again from a (more general) result in Louveau-Mokobodzki [LM, page 4823], this can be improved to ERG\(_a\) \(\in \Pi^0_{\omega+2}\). Thus EQINV\(_a\) = ERG\(_a\) \(\cap\) QINV\(_a\) is also \(\Pi^0_{\omega+2}\) in \(P(K)\).

In this paper we are interested in the Borel complexity of the sets QINV\(_a\) and EQINV\(_a\). To avoid technical complications involving the topology of \(K\), we will consider here the case where \(K\) is 0-dimensional and thus can be viewed as a closed subspace of the Cantor space \(C = 2^\mathbb{N}\).

Under these circumstances, the action \(a\) of \(\Gamma\) on \(K\) can be topologically embedded, via the map \(f(x) = (\gamma^{-1} \cdot x)\_\gamma\), into the shift action \(s\_\Gamma\) of \(\Gamma\) on \(C\). Therefore QINV\(_a\) and EQINV\(_a\) are Wadge reducible, via the continuous map \(\mu \mapsto f\_s(\mu)\), to QINV\(_{s\_\Gamma}\) and EQINV\(_{s\_\Gamma}\), resp. Recall that if \(A \subseteq X\), \(B \subseteq Y\), then \(A \leq_W B\). We will thus focus our attention on the study of the Borel complexity of the quasi-invariant and ergodic, quasi-invariant measures for the shift action. For convenience we write

\[
\text{QINV}\_\Gamma = \text{QINV}\_s\_\Gamma, \quad \text{ERG}\_\Gamma = \text{ERG}_s\_\Gamma, \quad \text{EQINV}\_\Gamma = \text{EQINV}_s\_\Gamma.
\]

We prove below the following results, where for a class \(\Phi\) of sets in Polish spaces, a set \(A \subseteq X\), \(X\) a Polish space, is called \(\Phi\)-hard if for any \(B \subseteq Y\), \(Y\) a 0-dimensional Polish space, with \(B \in \Phi\), we have \(B \leq_W A\). If in addition \(A \in \Phi\), then \(A\) is called \(\Phi\)-complete.

**Theorem 1.** For any infinite, countable group \(\Gamma\), QINV\(_\Gamma\) is \(\Pi^0_3\)-complete and ERG\(_\Gamma\), EQINV\(_\Gamma\) are \(\Pi^0_3\)-hard.

**Theorem 2.** The set EQINV\(_Z\) is \(\Pi^0_3\)-complete.
Theorem 3. Let $F_\infty$ be the group with infinite, countably many generators. Then there is a countable ordinal $\alpha_\infty \geq 3$ such that the set $\text{EQINV}_{F_\infty}$ is $\Pi^0_{\alpha_\infty}$-complete.

Thus $3 \leq \alpha_\infty \leq \omega + 2$.

Problem 4. Calculate $\alpha_\infty$.

We note that from Theorem 3 it follows that $\text{EQINV}_\Gamma \in \Pi^0_{\alpha_\infty}$, for any countable group $\Gamma$.

Remark 5. The proof of Theorem 3 in Section 4 below also shows that for any countable group $\Gamma$ that can be mapped onto the direct sum of infinite, countably many copies of itself, there is a countable ordinal $\alpha_\Gamma$ (thus $3 \leq \alpha_\Gamma \leq \omega + 2$) such that $\text{EQINV}_\Gamma$ is $\Pi^0_{\alpha_\Gamma}$-complete.

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2. Proof of Theorem 1

We first note the following standard fact.

Lemma 2.1. For any continuous action $a$ of a countable group $\Gamma$ on a compact, metrizable space $K$, $\text{ERG}_a \leq_w \text{EQINV}_a$.

Proof. Let $\Gamma = \{ \gamma_n : n \in \mathbb{N} \}$. The map $\mu \in P(K) \mapsto \sum_n 2^{-(n+1)} \gamma_n \cdot \mu \in P(K)$ is a continuous reduction of $\text{ERG}_a$ to $\text{EQINV}_a$. $\square$

Thus to complete the proof of Theorem 1, it is enough to show that $\text{QINV}_\Gamma$ is $\Pi^3_3$-complete and that $\text{ERG}_\Gamma$ is $\Pi^3_3$-hard.

(A) $\text{QINV}_\Gamma$ is $\Pi^3_3$-complete.

Let $X$ be a perfect Polish space and $\Gamma$ an infinite, countable group, which acts freely and continuously on $X$. Put

$$S = \{(x_n) \in X^\mathbb{N} : \{x_n : n \in \mathbb{N}\} \text{ is } \Gamma\text{-invariant}\}$$

$$= \{(x_n) \in X^\mathbb{N} : \forall n \forall \gamma \exists m (\gamma \cdot x_n = x_m)\}$$

Proposition 2.2. $S$ is not $G_\delta$.

Proof. First notice that $S$ is dense: Given $U_0, \ldots, U_{k-1}$ non-$\emptyset$ open in $X$ consider $U_0 \times \cdots \times U_{k-1} \times X^\mathbb{N}$. We will show that it intersects $S$. Pick $x^0_i \in V_i, i < k$. Then clearly there are $x^0_k, x^0_{k+1}, \ldots$ such that $(x^0_n) \in S$.

So if $S$ is $G_\delta$, it is comeager. We will show that there is a dense $G_\delta$ set $G$ such that $G \cap S = \emptyset$, a contradiction.
Let $\gamma \neq 1, \gamma \in \Gamma$ and put
\[ G = \{(x_n) : \forall m (\gamma \cdot x_0 \neq x_m)\}. \]
Clearly $G \cap S = \emptyset$. Now
\[ G = \bigcap_m G_m, \text{ where} \]
\[ G_m = \{(x_n) : \gamma \cdot x_0 \neq x_m\}. \]
Clearly $G_m$ is dense, open, so $G$ is comeager. \hfill \Box

Let now $K$ be perfect, compact, metrizable and let $a$ be a free, continuous action of $\Gamma$ on $K$.

**Proposition 2.3.** $\text{QINV}_a$ is not $G_\delta$ in $P(K)$.

**Proof.** It is enough to find a continuous function
\[ F : K^\mathbb{N} \to P(K) \]
such that $F^{-1}(\text{QINV}_a) = S$, where $S$ is as above for $(K, \Gamma)$.

Put
\[ F((x_n)) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \delta_{x_n}, \]
where $\delta_x$ is the Dirac measure at $x \in K$.

*Claim. $F$ is continuous.*

*Proof. We need to check that if $f \in C(K)$, and $(x_n^i) \to (x_n)$ in $K^\mathbb{N}$,\]
then $F((x_n^i))(f) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f(x_n^i) \to F((x_n))(f) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} f(x_n)$,\]
which is clear as $f(x_n^i) \to f(x_n), \forall n$.*

*Claim. $F^{-1}(\text{QINV}_\Gamma) = S$.*

*Proof. If $(x_n) \in S$, then clearly $\gamma \cdot (F(x_n)) \sim F((x_n)), \forall \gamma \in \Gamma$. Conversely assume $(x_n) \notin S$. Let then $n, \gamma$ be such that $\forall m (\gamma \cdot x_n \neq x_m)$. Then $\gamma \cdot F((x_n)) \not\sim F((x_n))$.* \hfill \Box

Thus we have shown:

**Proposition 2.4.** Let $a$ be a continuous and free action of an infinite countable group $\Gamma$ on the perfect, compact metrizable space $K$. Then $\text{QINV}_a$ is not $G_\delta$ in $P(K)$.

Let now $Q = \{x \in C : x(n) = 0 \text{ for all but finitely many } n\}$. Then $Q$ is $F_a$ in the Cantor space $C$ and for any Polish space $X$ and Borel set $A \subseteq X$, if $A$ is not $G_\delta$, then $Q \leq_W A$ (see [K, 24.20 and 22.13]). Thus we have:
Corollary 2.5. Let \( a \) be a continuous and free action of an infinite, countable group \( \Gamma \) on the perfect, compact metrizable space \( K \). Then there is a continuous function \( f : \mathcal{C} \to P(K) \) with \( f^{-1}(\text{QINV}_a) = Q \).

Consider now the set \( Q^N \subseteq C^N \). It is known that \( Q^N \) is a \( \Pi^0_3 \)-complete set (see [K, page 179] where this set is denoted by \( P_3 \)). Let now \( K = C^F \) with the shift action. By a result of Gao-Jackson-Seward [GJS, 3.7], there are infinitely many, pairwise disjoint, invariant compact subsets \( K_n \) of \( K \) on which \( \Gamma \) acts minimally and freely. Note that each \( K_n \) is perfect. By the preceding corollary, there is a continuous function \( f_n : \mathcal{C} \to P(K_n) \) such that \( f_n^{-1}(\text{QINV}_{a_n}) = Q \), where \( a_n \) is the restriction of the shift action to \( K_n \). Define now \( f : C^N \to P(K) \) by \( f((x_n)) = \sum_n \frac{1}{2^n} f_n(x_n) \). Then \( f \) is continuous and \( f^{-1}(\text{QINV}_\Gamma) = Q^N \), so \( \text{QINV}_\Gamma \) is \( \Pi^0_3 \)-complete.

(B) \( \text{ERG}_\Gamma \) is \( \Pi^0_3 \)-hard.

This follows from the following more general result, where a Borel equivalence relation \( E \) on a Polish space \( X \) is smooth if there is a Borel map \( f : X \to Y, Y \) a Polish space, such that \( xEy \iff f(x) = f(y) \).

Theorem 2.6. Let \( E \) be a non-smooth, Borel equivalence relation on a Polish space \( X \). Then \( \text{ERG}_E \) is \( \Pi^0_3 \)-hard.

Proof. Let \( E_0^k \) be the equivalence relation of \( k^N \) given by
\[
(x_n)E_0^k(y_n) \iff \exists n \forall m \geq n(x_m = y_m).
\]

Then by [HKL], \( E_0^3 \) can be continuously embedded, say by the function \( f : 3^N \to X \), into \( E \). The function \( f_x \) from \( P(3^N) \) to \( P(X) \) is continuous and \( \mu \) is ergodic for \( E_0^3 \) iff \( f_x(\mu) \) is ergodic for \( E \). It is thus enough to prove this result for \( E = E_0^3 \).

Consider the \( \Pi^0_3 \)-complete set \( P_3 = Q^N \subseteq C^N \), as in the paragraph following Corollary 2.5. We will define a continuous function \( f : C^N \times C \to 3^N \) as follows:

Fix a bijection \( \langle \cdot, \cdot \rangle : \mathbb{N}^2 \to \mathbb{N} \). Define first a function \( \bar{f} \) by:
\[
\bar{f}((a_k), x)(\langle n, m \rangle) = x(n + m), \text{ if } a_n(m) = 0; x|n, n + m, \text{ if } a_n(m) = 1.
\]
Let then \( f((a_k), x) = y \), where letting \( y_n(m) = y(\langle n, m \rangle) \), \( y_n \) is equal to:
\[
2 \bar{f}((a_k), x)(\langle n, 0 \rangle) 2 \bar{f}((a_k), x)(\langle n, 1 \rangle) 2 \cdots 2 \bar{f}((a_k), x)(\langle n, m \rangle) 2 \cdots,
\]
which is the concatenation of 2 followed by \( \bar{f}((a_k), x)(\langle n, 0 \rangle) \) followed by 2 followed by \( \bar{f}((a_k), x)(\langle n, 1 \rangle) \)...

Since \( y(\langle n, m \rangle) \) depends only on \( a_n(l) \), for \( l \leq m \), and \( x(n), \ldots, x(n + m) \), it is clear that \( f \) is continuous.
It is also clear that for each fixed \((a_k)\), the section \(f(a_k)(x) = f((a_k), x)\) is injective. For each \((a_k) \in \mathcal{C}^N\) let now 

\[ \mu((a_k)) = (f(a_k))_*(\lambda), \]

where \(\lambda\) is the usual product measure on \(\mathcal{C}\). The function \(\mu: \mathcal{C}^N \rightarrow P(3^N)\) is continuous, so its is enough to show that 

\[ (a_k) \in P_3 \iff \mu((a_k)) \in \operatorname{ERG}_{E^3_0}. \]

(A) Let \((a_k) \in P_3\). We claim then that \(xE_0^2 y \Rightarrow f(a_k)(x)E_0^3 f(a_k)(y)\). Indeed, if \(xE_0 y\), say \(x(k) = y(k)\) for all \(k \geq n_0\), then for \(n \geq n_0\), clearly \(f(a_k)(x)((n, m)) = f(a_k)(y)((n, m)), \forall m\). Let also \(m\) be large enough so that \(a_n(m) = 0\), for all \(n < n_0\) and all \(m \geq m_0\). Then for some \(k_0\) and all \(m \geq k_0, n < n_0\), we have \(f(a_k)(x)((n, m)) = f(a_k)(y)((n, m))\), so \(f(a_k)(x)E_0^3 f(a_k)(y)\).

Thus if \(A \subseteq 3^N\) is Borel, \(E_0^3\)-invariant, then \(f^{-1}_{(a_k)}(A)\) is Borel \(E_0^2\)-invariant, so, since \(\lambda\) is ergodic for \(E_0^2\), it has \(\lambda\)-measure 0 or 1, and thus \(A\) has \(\mu((a_k))\)-measure 0 or 1. So \(\mu((a_k)) \in \operatorname{ERG}_{E^3_0}\).

(B) Let \((a_k) \notin P_3\). Fix then \(n_0 = 1 < m_0 < m_1 < m_2 \ldots\) be such that \(a_{n_0}(m_i) = 1, \forall i\). Fix also a tree \(T \subseteq 2^{<N}\) such that \(0 < \lambda([T]) < 1\).

Put 

\[ B = \bigcup_{s \in 2^{n_0}} N_s \ast [T], \]

where for \(s \in 2^{n_0}\):

\[ N_s \ast [T] = \{ a \in \mathcal{C}: s \subseteq a \& (a_{n_0}, a_{n_0+1}, \ldots) \in [T] \}. \]

Then \(\lambda(B) = \lambda([T]) \in (0, 1)\). Put \(f(a_k)(B) = C\) and \(A = [C]_{E_0^3}\). Then \(A\) is Borel, \(E_0^3\)-invariant and we will show that \(f^{-1}_{(a_k)}(A) = B\), so that \(\mu((a_k))\) is not ergodic for \(E^3_0\), completing the proof.

Let \(f(a_k)(x) \in A\) and choose \(y \in B\) such that \(f(a_k)(x)E_0^3 f(a_k)(y)\). Then, in particular, if \(f(a_k)(x) = (x_n), f(a_k)(y) = y_n\), we have \(x_n E_0^3 y_n\). Now \(x_{n_0} = 2 s_0 2 s_1 \ldots, y_{n_0} = 2 t_0 2 t_1 \ldots\), where for each \(i, s_i, t_i\) are binary sequences of the same length. Let then \(k\) be such that for all \(i \geq k, s_i = t_i\). If \(m_j \geq k\), then \(t_{m_j} = (y_{n_0}, \ldots, y_{n_0+m_j})\) and so \(s_{m_j} = t_{m_j} \in T\). Since also \(s_{m_j} = (x_{n_0}, \ldots, x_{n_0+m_j})\), we have that \((x_{n_0}, x_{n_0+1}, \ldots) \in [T], \text{i.e., } x \in B\). \(\square\)

3. Proof of Theorem 2

Ditzen [D, page 47] shows that \(\operatorname{EQINV}_2\) is \(\Pi^0_3\) and thus by Theorem 1 it is \(\Pi^0_3\)-complete.
4. Proof of Theorem 3

Theorem 3 will follow from the next proposition:

**Proposition 4.1.** Let $X$ be a Polish space and let $A \subseteq X$. If $A \leq_{w} \text{ERG}_{\mathbb{F}_{\infty}}$, then $A^{\infty}(\subseteq X^{\infty}) \leq_{w} \text{ERG}_{\mathbb{F}_{\infty}}$.

**Proof.** Recall that for any countable Borel equivalence relation $E$, we denote by $\text{ERG}_{E}$ the set of probability Borel measures that are ergodic for $E$.

**Lemma 4.2.** Let $E_{n}$ be a countable Borel equivalence relation in the Polish space $X_{n}$ and let $\mu_{n}$ be a probability Borel measure on $X_{n}$. Let $E_{\infty}$ be the following equivalence relation on $X^{\infty}$:

\[(x_{n})E_{\infty}(y_{n}) \iff \forall n(x_{n}E_{n}y_{n}) \& \exists m \forall n \geq m(x_{n} = y_{n}).\]

Then

\[\prod_{n} \mu_{n} \in \text{ERG}_{E_{\infty}} \iff \forall n(\mu_{n} \in \text{ERG}_{E_{n}}).\]

**Proof.** $\implies$ : Put $\mu = \prod_{n} \mu_{n}$. Let $A \subseteq X_{n}$ be Borel and $E_{n}$-invariant. Let $B = X_{0} \times \cdots X_{n-1} \times A \times X_{n+1} \times \cdots$. Then $B$ is Borel and $E_{\infty}$-invariant, so $\mu_{n}(A) = \mu(B) \in \{0,1\}$.

$\impliedby$: Assume that each $\mu_{n}$ is ergodic for $E_{n}$. Let $A \subseteq \prod_{n} X_{n}$ be Borel and $E_{\infty}$-invariant. For each Borel set $B \subseteq \prod_{n} X_{n}$, let $\nu(B) = \mu(A \cap B)$. If we can show that for each Borel cylinder $B \subseteq \prod_{n} X_{n}$, $\nu(B) = \mu(A)\mu(B)$, then since the class of all Borel sets $B$ with the property that $\nu(B) = \mu(A)\mu(B)$ is closed under complements and countable disjoint unions, by the $\pi - \lambda$ Theorem (see, e.g., [K, 10.1, iii]) it contains all Borel sets, and in particular $A$, so $\nu(A) = \mu(A) = \mu(A)^{2}$, thus $\mu(A) \in \{0,1\}$.

Let then $B = D \times \prod_{i \geq n} X_{i}$ be a Borel cylinder, where $D \subseteq \prod_{i < n} X_{i}$. For $y \in \prod_{i \geq n} X_{i}$, let $A^{y} = \{(x_{i})_{i < n} \in \prod_{i < n} X_{i}: ((x_{i})_{i < n}, y)) \in A\}$. Then $A^{y}$ is $\prod_{i < n} E_{i}$-invariant.

**Claim.** $\prod_{i < n} \mu_{i} \in \text{ERG}_{\prod_{i < n} E_{i}}$.

**Proof.** It is enough to consider the case $n = 2$, so let $A \subseteq X_{0} \times X_{1}$ be Borel and $(E_{0} \times E_{1})$-invariant. Note that for $x_{0} \in X_{0}$ the section $A_{x_{0}} \subseteq X_{1}$ is $E_{1}$ invariant, so $\mu_{1}(A_{x_{0}}) \in \{0,1\}$. Let $P_{i} = \{x_{0}: \mu_{1}(A_{x_{0}}) = i\}$, for $i \in \{0,1\}$. Then each $P_{i}$ is $E_{0}$-invariant. If $\mu_{0}(P_{0}) = 0$, then $\mu_{0}(P_{1}) = 1$, so $(\mu_{0} \times \mu_{1})(A) = 1$. If $\mu_{0}(P_{0}) = 1$, then $(\mu_{0} \times \mu_{1})(A) = 1$.

Thus $A^{y}$ has $\prod_{i < n} \mu_{i}$-measure 0 or 1. Let

\[C = \{y \in \prod_{i \geq n} X_{i}: (\prod_{i < n} \mu_{i})(A^{y}) = 1\}.\]
Then $\mu(A) = (\prod_{i \geq n} \mu_i)(C)$. Now for $y \in C$, $(\prod_{i < n} \mu_i)(A_y \cap D) = (\prod_{i < n} \mu_i)(D)$ and for $y \notin C$, $(\prod_{i < n} \mu_i)(A_y \cap D) = 0$, so

$$
\mu(A \cap B) = \mu(A \cap (D \times \prod_{i \geq n} X_i)) \\
= \int (\prod_{i < n} \mu_i)(A_y \cap D) d(\prod_{i \geq n} \mu_i)(y) \\
= (\prod_{i < n} \mu_i)(D) \cdot (\prod_{i \geq n} \mu_i)(C) \\
= \mu(B)\mu(A).
$$

\[\square\]

Let now $E$ be the equivalence relation on $C^{\mathbb{F}_\infty}$ induced by the shift action of $\mathbb{F}_\infty$. We have to show that if $A \leq_W \text{ERG}_E$, then $A^N \leq_W \text{ERG}_E$. Let $f : X \to P(C^{\mathbb{F}_\infty})$ be a continuous function witnessing that $A \leq_W \text{ERG}_E$. Define $f_\infty : X^N \to P((C^{\mathbb{F}_\infty})^N)$ by $f_\infty((x_n)) = \prod_n f(x_n)$. Then $f_\infty$ is continuous and if $E^\infty$ is as in Lemma 4.2 with $E_n = E$ for each $n$, then

$$
f_\infty((x_n)) \in \text{ERG}_{E^\infty} \iff \forall n (f(x_n) \in \text{ERG}_E) \iff (x_n) \in A^N.
$$

So $A^N \leq_W \text{ERG}_{E^\infty}$.

Now consider the continuous action of $\bigoplus_n \mathbb{F}_\infty$ on $(C^{\mathbb{F}_\infty})^N$ given by $(\gamma_n) \cdot (x_n) = (\gamma_n \cdot x_n)$. The equivalence relation it induces is exactly $E^\infty$. Mapping $\mathbb{F}_\infty$ onto $\bigoplus_n \mathbb{F}_\infty$, this gives a continuous action $a$ of $\mathbb{F}_\infty$ on $(C^{\mathbb{F}_\infty})^N$ for which $\text{ERG}_a = \text{ERG}_{E^\infty}$ and thus $A^N \leq_W \text{ERG}_a$. Noting that $(C^{\mathbb{F}_\infty})^N$ is homeomorphic to $\mathcal{C}$, we can embed this action to the shift action of $\mathbb{F}_\infty$ on $C^{\mathbb{F}_\infty}$ and thus $A^N \leq_W \text{ERG}_{\mathbb{F}_\infty}$. \[\square\]

Using Proposition 4.1, we now complete the proof of Theorem 3 as follows. Let $\alpha$ be least such that $\text{ERG}_{\mathbb{F}_\infty} \in \Sigma^0_\alpha$. By Theorem 1, $\alpha \geq 3$.

**Claim.** $\text{ERG}_{\mathbb{F}_\infty}$ is $\Pi^0_\alpha$-complete.

**Proof.** Let $A = \{B \subseteq Y : Y \text{ Polish, 0-dimensional, } B \leq_W \text{ERG}_{\mathbb{F}_\infty}\}$. Then $A$ is closed under countable intersections, since if $B_n \in A, B_n \subseteq Y$, there is a continuous function $f_n : Y \to P(C^{\mathbb{F}_\infty})$ such that $B_n = f_n^{-1}(\text{ERG}_{\mathbb{F}_\infty})$. Put $X = P(C^{\mathbb{F}_\infty}), A = \text{ERG}_{\mathbb{F}_\infty}$ and let $f : Y \to X^N$ be given by $f(y) = f_n(y)$. Then $f$ witnesses that $\bigcap_n B_n \leq_W A^N \leq_W A = \text{ERG}_{\mathbb{F}_\infty}$, so $\bigcap_n B_n \in A$.

Let now $B \in \Pi^0_\alpha, B \subseteq Y, Y \text{ Polish and 0-dimensional}$. Then $B = \bigcap_n B_n$, where $B_n \in \Sigma^0_{\alpha_n}$, for some $\alpha_n < \alpha$. By a result of Saint-Raymond (see [K, 24.20 and 22.13]) $B_n \leq_W \text{ERG}_{\mathbb{F}_\infty}$, so $B \leq_W \text{ERG}_{\mathbb{F}_\infty}$.\[\square\]
Now, as $\alpha \geq 3$, $\text{EQINV}_{F_\infty}$ is in $\Pi^0_\alpha$. Also by Lemma 2.1, if $B \in \Pi^0_\alpha$, then $B \leq_W \text{ERG}_{F_\infty} \leq_W \text{EQINV}_{F_\infty}$, so $\text{EQINV}_{F_\infty}$ is $\Pi^0_\alpha$-complete.

**Remark 4.3.** Note that the only property of $F_\infty$ that we used in the preceding proof is that it can be mapped onto the direct sum of countably many copies of itself. It follows that Theorem 3 is valid as well for any countable group $\Gamma$ that has this property.

**References**


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