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The theory of countable Borel equivalence relations

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1. Introduction

The theory of definable equivalence relations has been a very active area of research in descriptive set theory during the last three decades. It serves as a foundation of a theory of complexity of classification problems in mathematics. Such problems can be often represented by a definable (usually Borel or analytic) equivalence relation on a standard Borel space. To compare the difficulty of a classification problem with respect to another, one introduces a basic order among equivalence relations, called **Borel reducibility**, which is defined as follows. An equivalence relation E on a standard Borel space X is Borel reducible to an equivalence relation F on a standard Borel space Y, in symbols $E \leq_B F$, if there is a Borel map $f \colon X \to Y$ such that $xEy \iff f(x)Ff(y)$. In this case one views E as less complex than F. The study of this hierarchical order and the discovery of various canonical benchmarks in this hierarchy occupies a major part of this theory.

Another source of motivation for the theory of definable equivalence relations comes from the study of group actions, in a descriptive, topological or measure theoretic context, where one naturally studies the structure of the equivalence relation whose classes are the orbits of an action and the associated orbit space.

An important part of this theory is concerned with the structure of **countable Borel equivalence relations**, i.e., those Borel equivalence relations all of whose classes are countable. It turns out that these are exactly the equivalence relations generated by Borel actions of countable discrete groups (Feldman-Moore) and this brings into this subject important connections with group theory, dynamical systems and operator algebras.

Our goal here is to provide a survey of the state of the art in the theory of countable Borel equivalence relations. Although this subject has a long history in the context of ergodic theory and operator algebras, the system-

atic study of countable Borel equivalence relations in the purely Borel context dates back to the mid-1990's and originates in the papers [DJK] and [JKL]. Since that time there has been extensive work in this area leading to major progress on many of the fundamental problems.

The paper is organized as follows. Chapter 2 reviews some basic general concepts concerning equivalence relations and morphisms between them. In Chapter 3 we introduce countable Borel equivalence relations, discuss some of their properties and mention several examples. The scope of the theory of countable Borel equivalence relations is actually much wider as it encompasses a great variety of other equivalence relations up to Borel bireducibility. Chapter 4 deals with such equivalence relations, called **essentially countable**. In Chapter 5, we consider invariant and quasi-invariant measures for equivalence relations.

In Chapter 6, we start studying the hierarchical order of Borel reducibility, introducing the important benchmarks of the simplest (non-trivial) and the most complex countable Borel equivalence relations. The next Chapter 7, demonstrates the complexity and richness of the structure of this hierarchical order and discusses the role of rigidity phenomena in both the set theoretic and ergodic theoretic contexts. We next consider various important classes of countable Borel equivalence relations such as hyperfinite (Chapter 8), amenable (Section 9), treeable (Chapter 10), freely generated (Chapter 11) and finally universal ones (Chapter 12). The last four chapters deal with the algebraic structure of the Borel reducibility order (Chapter 13), the concept of structurability of countable Borel equivalence relations (Chapter 14), topological realizations of countable Borel equivalence relations (Chapter 15) and a universal space for actions and equivalence relations (Chapter 16).

With a few minor exceptions, this survey contains no proofs but it includes detailed references to the literature where these can be found. The emphasis here is primarily on the descriptive aspects of the theory of countable Borel equivalence relations and ergodic theoretic aspects are brought in when relevant. It is not our intention though to survey the research in ergodic theory related to this subject, including in particular the theory of orbit equivalence and its relations with operator algebras. These can be found, for example, in [Z2], [AP], and [I3], which also contains a detailed bibliography of the extensive work in this area over the last two decades. Exposition of other related subjects, like the Levitt-Gaboriau theory of cost, can be found in [KM1]. Finally there are important connec-

tions with descriptive aspects of graph combinatorics, for which we refer the reader to [KM].

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2. Equivalence relations and reductions

2.1 Generalities on equivalence relations

Let E be an equivalence relation on a set X. If $A \subseteq X$, we let $E \upharpoonright A = E \cap A^2$ be its **restriction** to A. We also let $[A]_E = \{x \in X \colon \exists y \in A(xEy)\}$ be its **E-saturation**. The set A is **E-invariant** if $A = [A]_E$. In particular, for each $x \in X$, $[x]_E$ is the **equivalence class**, or **E-class**, of x. A function $f \colon X \to Y$ is **E-invariant** if $xEy \implies f(x) = f(y)$. Finally $X/E = \{[x]_E \colon x \in X\}$ is the **quotient space** of X modulo E.

Suppose E, F are equivalence relations on sets X, Y, respectively, and $f \colon X/E \to Y/F$ is a function. A **lifting** of f is a function $\tilde{f} \colon X \to Y$ such that $f([x]_E) = [\tilde{f}(x)]_F, \forall x \in X$. Similarly if $R \subseteq (X/E)^n$ its lifting is $\tilde{R} \subseteq X^n$, where $(x_i)_{i<n} \, \tilde{R} \, (y_i)_{i<n} \iff ([x_i]_E)_{i<n} \, R \, ([y_i]_E)_{i<n}$.

If $E_i, i \in I$, is a family of equivalence relations, with E_i living on X_i, we define the **direct sum** $\bigoplus_i E_i$ to be the equivalence relation on $\bigoplus_i X_i = \{(x,i) \colon x \in X_i\}$ defined by

$$(x,j) \bigoplus_i E_i \ (y,k) \iff j = k \ \& \ xE_jy.$$

In particular, we let for $n \geq 1$, $nE = \bigoplus_{i<n} E$. Also let $\mathbb{N}E = \bigoplus_{i \in \mathbb{N}} E$.

We define the **direct product** $\prod_i E_i$ to be the equivalence relation on the space $\prod_i X_i$ defined by

$$(x_j) \prod_i E_i \ (y_j) \iff \forall j(x_j E_j y_j).$$

In particular, we let for $n \geq 1$, $E^n = \prod_{i<n} E$. Also let $E^{\mathbb{N}} = \prod_{i \in \mathbb{N}} E$.

If E, F are equivalence relations on X, then $E \subseteq F$ means that E is a subset of F, when these are viewed as subsets of X^2, i.e., E is finer than F or equivalently F is coarser than E. The **index** of F over E, in symbols $[F : E]$, is the supremum of the cardinality of the set of E-classes contained in an F-class. Thus $[F : E] \leq \aleph_0$ means that every F-class contains only countably many E-classes.

We denote by $\Delta_X = \{(x, y) : x = y\}$ the equality relation on a set X and we also let $I_X = X^2$. Note that if $E_y = E, y \in Y$, where E is an equivalence relation on a set X, then $\bigoplus_y E_y = E \times \Delta_Y$.

If $E_i, i \in I$, are equivalence relations on X, we denote by $\bigwedge_i E_i = \bigcap_i E_i$ the largest (under inclusion) equivalence relation contained in all E_i and by $\bigvee_i E_i$ the smallest (under inclusion) equivalence relation containing each E_i. We call $\bigwedge_i E_i$ the **meet** and $\bigvee_i E_i$ the **join** of (E_i).

If E is an equivalence relation on X, a set $S \subseteq X$ is a **complete section** of E if S intersects every E-class. If moreover S intersects every E-class in exactly one point, then S is a **transversal** of E.

Consider now an action $a : G \times X \to X$ of a group G on a set X. We often write $g \cdot x = a(g, x)$, if there is no danger of confusion. Let $G \cdot x = \{g \cdot x : g \in G\}$ be the **orbit** of $x \in X$. The action a induces an equivalence relation E_a on X whose classes are the orbits, i.e., $x E_a y \iff \exists g (g \cdot x = y)$. When a is understood, sometimes the equivalence relation E_a is also denoted by E_G^X. The action a is **free** if $g \cdot x \neq x$ for every $x \in X, g \in G, g \neq 1_G$.

2.2 Morphisms

Let E, F be equivalence relations on spaces X, Y, resp. A map $f : X \to Y$ is a **homomorphism** from E to F if $x E y \implies f(x) F f(y)$. In this case we write $f : (X, E) \to (Y, F)$ or just $f : E \to F$, if there is no danger of confusion. A homomorphism f is a **reduction** if moreover $x E y \iff f(x) F f(y)$. We denote this by $f : (X, E) \leq (Y, F)$ or just $f : E \leq F$. Note that a homomorphism as above induces a map form X/E to Y/F, which is an injection if f is a reduction. In other words, a homomorphism is a lifting of a map from X/E to Y/F and a reduction is a lifting of an injection of X/E into Y/F. An **embedding** is an injective reduction. This is denoted by $f : (X, E) \sqsubseteq (Y, F)$ or just $f : E \sqsubseteq F$. An **invariant embedding** is an injective reduction whose range is an F-invariant subset of Y. This is

denoted by $f\colon (X, E) \sqsubseteq^i (Y, F)$ or just $f\colon E \sqsubseteq^i F$. Finally an **isomorphism** is a surjective embedding. This is denoted by $f\colon (X, E) \cong (Y, F)$ or just $f\colon E \cong F$.

If a, b are actions of a group G on spaces X, Y, resp., a **homomorphism** from a to b is a map $f\colon X \to Y$ such that $f(g \cdot x) = g \cdot f(x), \forall g \in G, x \in X$. If f is injective, we call it an **embedding** of a to b.

2.3   The Borel category

We are interested here in studying (classes of) Borel equivalence relations on **standard Borel spaces** (i.e., Polish spaces with the associated Borel structure). If X is a standard Borel space space and E is an equivalence relation on X, then E is Borel if E is a Borel subset of X^2.

Given a class of functions Φ between standard Borel spaces, we can restrict the above notions of morphism to functions in Φ in which case we use the subscript Φ in the above notation (e.g., $f\colon E \to_\Phi F$, $f\colon E \leq_\Phi F$, etc.). In particular if Φ is the class of Borel functions, we write $f\colon E \to_B F, f\colon E \leq_B F, f\colon E \sqsubseteq_B F, f\colon E \sqsubseteq_B^i F, f\colon E \cong_B F$ to denote that f is a Borel morphism of the appropriate type. Similarly when we consider the underlying topology, we use the subscript c in the case Φ is the class of continuous functions between Polish spaces and write $f\colon E \to_c F, f\colon E \leq_c F, f\colon E \sqsubseteq_c F, f\colon E \sqsubseteq_c^i F, f\colon E \cong_c F$ in this case.

We say that E is **Borel reducible** to F if there is a Borel reduction from E to F. In this case we write $E \leq_B F$. If $E \leq_B F$ and $F \leq_B E$, then E, F are **Borel bireducible**, in symbols $E \sim_B F$. Finally we let $E <_B F$ if $E \leq_B F$ but $F \not\leq_B E$. Similarly we define the notions of E being **Borel embeddable** to F and E being **Borel invariantly embeddable** to F, for which we use the notations $E \sqsubseteq_B F$ and $E \sqsubseteq_B^i F$. Also we use $E \simeq_B F, E \simeq_B^i F$ for the corresponding notions of being **Borel biembeddable** and **Borel invariantly biembeddable**. We also use $E \sqsubset_B F$ and $E \sqsubset_B^i F$ for the corresponding strict notions. More generally, if Φ is as above, we analogously define $E \leq_\Phi F, E \sqsubseteq_\Phi F$, etc.

Finally E, F are **Borel isomorphic**, in symbols $E \cong_B F$, if there is a Borel isomorphism from E to F. Note that by the usual (Borel) Schroeder-Bernstein argument, E, F are Borel isomorphic iff they are Borel invariantly biembeddable, i.e., $\simeq_B^i = \cong_B$.

3. Countable Borel equivalence relations

Definition 3.1. *An equivalence relation E is* **countable** *if every E-class is countable. It is* **finite** *if every E-class is finite.*

3.1 Some examples

We discuss first some examples of countable Borel equivalence relations.

Examples 3.2. 1) Let $X = 2^{\mathbb{N}}$. Then the **eventual equality** relation

$$xE_0y \iff \exists m \forall n \geq m(x_n = y_n)$$

and the **tail** equivalence relation

$$xE_ty \iff \exists m \exists k \forall n(x_{m+n} = y_{k+n})$$

are countable Borel.

More generally, let $(S, \times, 1)$ be a monoid. We usually write st for $s \times t$. An action a of S on a set X is a map $a \colon S \times X \to X$ such that, letting as usual, $s \cdot x = a(s, x)$, we have $1 \cdot x = x$, $s \cdot (t \cdot x) = st \cdot x$. If now S is abelian, this action gives rise to two equivalence relations $E_{0,a}, E_{t,a}$ on X, defined by $xE_{0,a}y \iff \exists s(s \cdot x = s \cdot y), xE_{t,a}y \iff \exists s \exists t(s \cdot x = t \cdot y)$. If we take $S = (\mathbb{N}, +, 0)$, $X = 2^{\mathbb{N}}$, $1 \cdot (x_n) = (x_{n+1})$ (the shift map), we obtain E_0, E_t. If S is countable (discrete), abelian, and a is a Borel action such that for each $s \in S$ the map $x \mapsto s \cdot x$ is countable-to 1, then these two equivalence relations are countable Borel.

2) Take again $X = 2^{\mathbb{N}}$ and consider $\equiv_T$ and $\equiv_A$, the **Turing** and **arithmetical equivalence** relations, resp. These are countable Borel.

3) Let now $X = \mathbb{R}$. Then the **Vitali equivalence relation** defined by $x E_v y \iff x - y \in \mathbb{Q}$ is countable Borel.

4) Let $X = \mathbb{R}^+$. The **commensurability relation** is given by $x E_c y \iff \frac{x}{y} \in \mathbb{Q}$. This is countable Borel (and one of the earliest equivalence relations in the history of mathematics).

5) Let $k \geq 2$ and let X be the space of subshifts of $k^{\mathbb{Z}}$, where a **subshift** is a closed subset of $k^{\mathbb{Z}}$ invariant under the shift map $S(x)_i = x_{i-1}$. This is a compact subspace of the hyperspace of all compact subsets of $k^{\mathbb{Z}}$, thus compact, metrizable. Let E be the equivalence relation of isomorphism of subshifts, where two subshifts are isomorphic if there is a homeomorphism between the closed sets that commutes with the shift. Then E is a countable Borel equivalence relation, see [Cl2].

6) Let now a be Borel action of a countable (discrete) group G on a standard Borel space X. Then E_a is a countable Borel equivalence relation.

3.2   The Feldman-Moore Theorem

It turns out that Example 6) in the list of Examples 3.2 includes all countable Borel equivalence relations.

Theorem 3.3 ([FM]). *If E is a countable Borel equivalence relation on a standard Borel space X, then there is a countable group G and a Borel action a of G on X such that $E = E_a$.*

This is an immediate consequence of the following result that can be proved using the classical Luzin-Novikov Theorem in descriptive set theory, see [Ke6, 18.10]

Theorem 3.4 ([FM]). *If E is a countable Borel equivalence relation on a standard Borel space X, then there is a sequence of Borel involutions (T_n) on X such that $x E y \iff \exists n (T_n(x) = y)$.*

Remark 3.5. In Theorem 3.4 one can also find (T_n) as in that theorem such that moreover for any $x \neq y, x E y$, there is a unique n such that $T_n(x) = y$. This is equivalent to saying that the Borel graph $E \setminus \Delta_X$ has countable Borel edge chromatic number and follows from the general result [KST, 4.10] (see also [Ke11, 3.7]).

Although Theorem 3.3 always guarantees the existence of a Borel action of a countable group that generates a given countable Borel equivalence relation, it is not always clear how to find a "natural" such action that generates a specific equivalence relation of interest. Considering the examples in the list of Examples 3.2, E_0 is generated by an action as follows. Identify $2^{\mathbb{N}}$ with the compact product group $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ and consider the translation action of its countable dense subgroup $(\mathbb{Z}/2\mathbb{Z})^{<\mathbb{N}}$. The induced equivalence relation is clearly E_0. It follows from a general result of [GPS, 3.9] that E_0 is generated by a continuous $\mathbb{Z}$-action, i.e., is generated by a single homeomorphism. A more direct construction is given in [Cl1]. (The action of $\mathbb{Z}$ on $2^{\mathbb{N}}$ induced by the **odometer** map, i.e., addition of 1 modulo 2 with right carry, generates an equivalence relation in which one class consists of the eventually constant sequences and the others coincide with the E_0-classes.) Joshua Frisch pointed out that if $T\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is the homeomorphism given by $01x \to 10x, 00x \to 0x, 1x \to 11x$, for $x \in 2^{\mathbb{N}}$, and U is the homeomorphism given by $0x \to 1x, 1x \to 0x$, for $x \in 2^{\mathbb{N}}$, then the group $\langle T, U \rangle$ generates E_t. On the other hand, as opposed to E_0, E_t cannot be generated by a single homeomorphism, as such would have an invariant probability Borel measure (by the amenability of $\mathbb{Z}$) but it can be shown that any Borel action of a countable group that generates E_t cannot have such an invariant measure (see the paragraph preceding Corollary 5.7). We will see in Chapter 8 that E_t can be generated by a single Borel automorphism.

The Vitali equivalence relation and the commensurability relation are clearly generated by actions of $(\mathbb{Q}, +)$ and $(\mathbb{Q}^+, \cdot)$, resp. On the other hand it is not clear how to explicitly find actions that generate $E_{0,a}, E_{t,a}, \equiv_T, \equiv_A$ and isomorphism of subshifts.

Definition 3.6. *A finite equivalence relation E is of* **type** *n if every E-class has cardinality $\leq n$.*

The following is an immediately corollary of Theorem 3.4:

Corollary 3.7. *If E is a countable Borel equivalence relation, then there is a sequence (E_n) of Borel equivalence relations of type 2 such that $E = \bigvee_n E_n = \bigcup_n E_n$.*

It is natural to ask whether in Corollary 3.7 one can find finitely many *finite* Borel equivalence relations $(E_n)_{n<N}$ with $E = \bigvee_{n<N} E_n$. This is how-

ever ruled out by the theory of cost, see [Ga1]. On the other hand the following is shown in [JKL]:

Theorem 3.8 ([JKL, 1.21]). *For every countable Borel equivalence relation E, there is a countable Borel equivalence relation F such that $E \sim_B F$ and F is of the form $F = G \vee H$, where G, H are Borel equivalence relations of type 2,3, resp. Moreover this fails if we require that such F, G, H can be found, where G, H are of type 2. However one can write such an F as $F = G \vee H \vee K$, with G, H, K of type 2.*

In fact it is shown in [JKL, 1.21] that the equivalence relations of the form $G \vee H$, with G, H of type 2, are exactly the hyperfinite ones, see Chapter 8. Here a countable Borel equivalence relation E is **hyperfinite** if it can be written as $E = \bigcup_n E_n$, where (E_n) is an increasing sequence ($E_n \subseteq E_{n+1}$, for each n) of finite Borel equivalence relations.

3.3 Induced actions

There is a very useful construction, called the **inducing construction**, due to Mackey, that allows for a pair of groups $G \le H$, to extend an action of a group G to an action of H.

Theorem 3.9 (Mackey; see, e.g., [BK, 2.3.5]). *Let H be a Polish group and $G \le H$ a closed subgroup. Let a be a Borel action of G on a standard Borel space X. Then there is a standard Borel space Y such that $X \subseteq Y$ and X is a Borel subset of Y, with the following properties:*
    (i) For $x \in X$ and $g \in G$, $a(g, x) = b(g, x)$;
    (ii) Every orbit of H on Y contains exactly one orbit of G on X;
    (iii) $E_a \sqsubseteq_B E_b$ and $E_b \le_B E_a$, therefore $E_a \sim_B E_b$;
    (iv) If a is a free action, so is b.

The action b is called the **induced action** of a and is denoted by $\mathrm{IND}_G^H(a)$.

3.4 Closure properties

We record below some simple closure properties of the class of countable Borel equivalence relations.

Proposition 3.10. *(i) If E is a countable Borel equivalence relation on X and $A \subseteq X$ is Borel, then $E \restriction A$ is also countable Borel.*

(ii) If F is a countable Borel equivalence relation and $E \sqsubseteq_B F$, then E is also countable Borel.

(iii) If $E \subseteq F$ are Borel equivalence relations and F is countable, so is E. If E is countable and $[F : E] \leq \aleph_0$, then F is countable.

(iv) If E, F are countable Borel equivalence relations, then so is $E \times F$.

(v) If each $E_n, n \in \mathbb{N}$, is a countable Borel equivalence relation, then so is $\bigoplus_n E_n$. More generally, If Y is a standard Borel space and $(E_y)_{y \in Y}$ is a family of countable Borel equivalence relations on a standard Borel space X such that $\{(x, u, y) \in X^2 \times Y : (x, u) \in E_y\}$ is Borel, then $\bigoplus_y E_y$ is countable Borel.

(vi) If each $E_n, n \in \mathbb{N}$, is a countable Borel equivalence relation, then so is $\bigvee_n E_n$.

For a standard Borel space and a Borel map $T \colon X \to X$, let E_T be the smallest equivalence relation containing the graph of T. If T is countable-to-1, then E_T is countable Borel (being equal to $E_{t,a}$, where a is the Borel action of $(\mathbb{N}, +, 0)$ generated by T, i.e., $1 \cdot x = T(x)$; see Examples 3.2, 1). More generally, let T_n be a sequence of countable-to-1 Borel maps from X to X and let $E_{(T_n)}$ be the smallest equivalence relation containing the graphs of all T_n. Then $E_{(T_n)} = \bigvee_n E_{T_n}$ is countable Borel.

3.5 Complete sections and vanishing sequences of markers

If E is a countable Borel equivalence relation on a standard Borel space X and A is a Borel complete section for A, then we view $A \cap [x]_E$ as putting a set of markers on the E-class of x in a uniform Borel way, so sometimes we call such an A a **marker set**. Finding appropriate marker sets plays an important role in the study of countable Borel equivalence relations.

The simplest situation is when a Borel transversal can be found.

Definition 3.11. *A Borel equivalence relation E on a standard Borel space X is called **smooth** if there is a Borel function $f \colon X \to Y$, Y a standard Borel space, such that $xEy \iff f(x) = f(y)$, i.e., $E \leq_B \Delta_Y$.*

For example any finite Borel equivalence relation is smooth. We now have the following basic fact:

Proposition 3.12. *The following are equivalent for a countable Borel equivalence relation E:*

(i) E is smooth;

(ii) E admits a Borel transversal;

(iii) The space X/E with the **quotient Borel structure** Σ_E *(i.e., $A \subseteq X/E \in \Sigma_E \iff \tilde{A} = \bigcup A \subseteq X$ is Borel) is standard.*

For example, the equivalence relations in Examples 3.2, 1)–5), except possibly for $E_{0,a}, E_{t,a}$, for some a, are not smooth.

We will next discuss a very different characterization of smoothness.

A **mean** on a countable set S is a positive linear functional $\varphi : \ell^\infty(S) \to \mathbb{C}$, which assigns the value 1 to the constant 1 function. Let E be a countable Borel equivalence relation on a standard Borel space X. An **assignment of means** is a map which associates to each equivalence class $[x]_E$ a mean $\varphi_{[x]_E}$ on $[x]_E$. A map $x \mapsto f_x \in \ell^\infty([x]_E)$ is Borel if the function $f : E \to \mathbb{C}$ defined by $f(x, y) = f_x(y)$ is Borel. Finally an assignment of means $[x]_E \mapsto \varphi_{[x]_E}$ is Borel (in the weak sense) if for each Borel map $x \mapsto f_x \in \ell^\infty([x]_E)$, the function $x \mapsto \varphi_{[x]_E}(f_x)$ is Borel.

Theorem 3.13 ([KM2])**.** *Let E be a countable Borel equivalence relation. Then the following are equivalent:*

(i) E is smooth;

(ii) E admits a Borel assignment of means.

Remark 3.14. An analog of Theorem 3.13 in the Baire category context is also proved in [KM2]. On the other hand, we will see in Section 9.4 that in the measure theoretic context the situation is quite different, since smoothness is replaced in this case by hyperfiniteness.

The next, very useful, result guarantees the existence of appropriate markers even in the non-smooth situation. An equivalence relation E is called **aperiodic** if every E-class is infinite.

Theorem 3.15 (The Marker Lemma, [SlSt])**.** *Let E be an aperiodic countable Borel equivalence relation on a standard Borel space X. Then E admits a vanishing sequence of Borel markers, i.e., there is a sequence of complete Borel sections (A_n), with $A_0 \supseteq A_1 \supseteq A_2 \ldots$ and $\bigcap_n A_n = \emptyset$.*

From this we also have the following:

Corollary 3.16. *Let E be an aperiodic countable Borel equivalence relation on a standard Borel space X. Then E admits a pairwise disjoint sequence of Borel markers, i.e., there is a sequence of complete Borel sections (B_n), with $B_n \cap B_m = \emptyset$, if $m \neq n$.*

For a proof, see, e.g., [CM2, 1.2.6]. A generalization of Theorem 3.15 to transitive Borel binary relations with countably infinite vertical sections can be found in [Mi3].

It is also clear that for any finite set of aperiodic countable Borel equivalence relations there is a common vanishing sequence of Borel markers. On the other hand, concerning common vanishing sequences of Borel markers for infinite sets of aperiodic countable Borel equivalence relations, we have the following results:

Proposition 3.17. *(i)* (Miller) *There is a sequence of aperiodic Borel equivalence relations (F_n) such that if $A \subseteq X$ is a Borel complete section for all F_n, then A is comeager. Thus for any sequence of Borel sets (A_m) such that each A_m is a complete section for all F_n, $\bigcap_m A_m$ is comeager. In particular the sequence (F_n) has no common vanishing sequence of Borel markers.*

(ii) (Marks) *Let (F_n) be a sequence of aperiodic countable Borel equivalence relations on a standard Borel space X and let μ be a probability Borel measure on X. Then there is a decreasing sequence of Borel sets (A_m) such that each A_m is a complete section for every F_n and $\mu(\bigcap_n A_n) = 0$.*

Proof. (i) Let F_n be the subequivalence relation of E_0 defined by $x F_n y \iff x E_0 y$ & $(x_i)_{i<n} = (y_i)_{i<n}$. Then if a Borel set $A \subseteq X$ is a complete section for F_n, A is nonmeager in every basic nbhd $N_s = \{x \in 2^{\mathbb{N}} : (x_i)_{i<n} = s\}$, where $s \in 2^n$. Thus if a Borel set A is a complete section for all F_n, A must be comeager.

(ii) Let $(A_{n,m})$ be a vanishing sequence of Borel markers for E_n such that $\mu(A_{n,m}) \leq \frac{1}{2^{n+m}}$. Put $A_m = \bigcup_n A_{n,m}$. $\qquad\qquad\square$

An important "dual" question (especially because of its connection to Borel combinatorics, see [M1]) is whether two countable Borel equivalence relations can have disjoint complete sections. Here we have the following results:

Theorem 3.18 ([M1, Section 4]). *(i) There are aperiodic countable Borel equivalence relations E, F on a standard Borel space X such that there is no Borel set $A \subseteq X$ with A a complete section for E and $X \setminus A$ a complete section for F.*

(ii) For any two countable Borel equivalence relations E, F on a standard Borel space X such that all E-classes have cardinality are least 3 and all F-classes have cardinality at least 2, and for every Borel probability measure μ on X, there is Borel $A \subseteq X$ such that A meets μ-almost every E class (i.e., $\mu([A]_E) = 1$) and $X \setminus A$ meets μ-almost every F class.

(iii) For any two countable Borel equivalence relations E, F on a Polish space X such that all E-classes have cardinality at least 3 and all F-classes have cardinality at least 2, there is Borel $A \subseteq X$ such that A meets comeager many E-classes (i.e., $[A]_E$ is comeager) and $X \setminus A$ meets comeager many F-classes.

Finally for certain countable Borel equivalence relations, and especially those generated by shift actions of groups (see Section 6.3), there are several interesting results concerning the topological structure of vanishing sequences of markers and the local structure of complete sections, see [GJS1], [GJKS], [M3] and [CMa].

3.6   Maximal finite partial subequivalence relations

Another useful tool in studying countable Borel equivalence relations is the existence of appropriate finite partial subequivalence relations. Here by a **partial subequivalence relation** of an equivalence relation E on a space X, we mean an equivalence relation F on a subset $A \subseteq X$ such that $F \subseteq E$. A *finite* partial subequivalence relation is abbreviated as **fsr**.

Let now X be a standard Borel space and denote by $[X]^{<\infty}$ the standard Borel space of finite subsets of X. If E is an equivalence relation on X, we denote by $[E]^{<\infty}$ the subset of $[X]^{<\infty}$ consisting of all finite sets that are contained in a single E-class. If E is Borel, so is $[E]^{<\infty}$. For each set $\Phi \subseteq [E]^{<\infty}$, an fsr F of E defined on the set $A \subseteq X$ is Φ-**maximal**, if every F-class is in Φ and every finite set S disjoint from A is not in Φ. We now have the following that is proved using a result from Borel combinatorics.

Theorem 3.19 ([KM1, 7.3]). *If E is a countable Borel equivalence relation and $\Phi \subseteq [E]^\infty$ is Borel, then there is a Borel Φ-maximal fsr of E.*

The following is a typical application of this result.

Corollary 3.20. *Let (M_n) be a sequence of positive integers ≥ 2. Then for each aperiodic countable Borel equivalence relation E, there is an increasing sequence of finite Borel subequivalence relations (E_n) of E, such that each E_n-class has exactly $M_0 M_1 \cdots M_n$ elements.*

Proof. It is shown in [KM1, 7.4] (using Theorem 3.19) that given a positive integer M, every aperiodic countable Borel equivalence contains a finite subequivalence relation all of whose classes have cardinality M. One can then define E_n inductively as follows: Given E_n, let X_n be a Borel transversal for E_n. Apply this fact to $E \restriction X_n$ to find a finite subequivalence relation $F_n \subseteq E \restriction X_n$, each of whose classes has cardinality M_{n+1} and then take $E_{n+1} = E_n \vee F_n$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

If in Corollary 3.20 we let $F = \bigcup_n E_n$, then F is an aperiodic hyperfinite Borel subequivalence relation of E.

3.7 Compressibility

Recall that a set C is called **Dedekind infinite** if there is an injection $f\colon C \to C$ such that $f(C) \subsetneqq C$, i.e., C can be compressed to a proper subset of itself. The following is an analog of this concept in the context of countable Borel equivalence relations.

Definition 3.21. *Let E be a countable Borel equivalence relation on a standard Borel space X. We say that E is **compressible** if there is an injective Borel map $f\colon X \to X$ such that for each E-class C, $f(C) \subsetneqq C$. A Borel set $A \subseteq X$ is compressible if $E \restriction A$ is compressible.*

For any countable Borel equivalence relation E on a standard Borel space X and Borel sets $A, B \subseteq X$, we let

$\quad A \sim_E B \iff \exists f\colon A \to B(f \text{ is a Borel bijection and } \forall x \in A(f(x)Ex)).$

In particular, $A \sim_E B \implies E \restriction A \cong_B E \restriction B$. We also put

$$A \preceq_E B \iff \exists \text{ Borel } C \subseteq B(A \sim_E C)$$

and

$A \prec_E B \iff \exists \text{ Borel } C \subseteq B(A \sim_E C, B \setminus C \text{ a complete section of } E|[B]_E)$

The standard Borel Schroeder-Bernstein argument shows that

$$A \sim_E B \iff A \preceq_E B \,\&\, B \preceq_E A$$

Note also that a Borel set A is compressible iff $A \prec_E A$.

We also have the following, which is part of the proof of Theorem 5.6; see also [BK, 4.5.1]:

Proposition 3.22. *Let E be a countable Borel equivalence relation on a standard Borel space X. Let A, B be two complete Borel sections for E.Then there is a partition $X = P \sqcup Q$ into E-invariant Borel sets such that $A \cap P \prec_E B \cap P$ and $B \cap Q \preceq_E A \cap Q$.*

A Borel set $A \subseteq X$ is called **E-paradoxical** if there are disjoint Borel subsets $B, C \subseteq A$ such that $A \sim_E B, A \sim_E C$.

The following result, for which we refer to [DJK, Section 2] and references therein to [CN1], [CN2], [N1], [N2], gives a number of equivalent formulations of compressibility.

Proposition 3.23. *Let E be a countable Borel equivalence relation on a standard Borel space X. Then the following are equivalent:*

(i) E is compressible;

(ii) There is a sequence of pairwise disjoint complete Borel sections (A_n) of E such that $A_i \sim_E A_j$ for each i, j;

(iii) There is an infinite partition $X = A_0 \sqcup A_1 \sqcup \cdots$ into complete Borel sections such that $A_i \sim_E A_j$ for each i, j;

(iv) The space X is E-paradoxical;

(v) $E \cong_B E \times I_{\mathbb{N}}$;

(vi) There is a smooth aperiodic Borel subequivalence relation $F \subseteq E$.

We call $E \times I_{\mathbb{N}}$ the **amplification** of E. Thus the compressible equivalence relations are those that are Borel isomorphic to their amplifications.

Remark 3.24. In contrast to Proposition 3.23 (iii), from [KM1, 7.4] (see also Corollary 3.20) it follows that for *any* aperiodic countable Borel equivalence relation E on a standard Borel space X and for any $n \geq 1$, there is a finite partition $X = A_0 \sqcup A_1 \sqcup \cdots \sqcup A_{n-1}$ into complete Borel sections such that $A_i \sim_E A_j$ for each $i, j < n$. See also [Sl5, 1.8.5].

Another characterization of compressibility is the following, where for a countable Borel equivalence relation E on a standard Borel space X, a Borel set $A \subseteq X$ is called **_E_-syndetic** if for some $n > 0$ there are Borel sets $A_i, i < n$, such that $A \sim_E A_i, \forall i < n$, and $X = \bigcup_{i<n} A_i$.

Proposition 3.25 ([Sl2, Proposition 10.2]). *Let E be an aperiodic countable Borel equivalence relation on a standard Borel space X. Then the following are equivalent:*
(i) E is compressible;
(ii) For any two Borel syndetic sets $A, B \subseteq X$, $A \sim_E B$.

In Chapter 5 we will also see Nadkarni's characterization of compressibility in terms of invariant measures.

It is easy to see that $E_t, E_v, E_c, \equiv_T, \equiv_A$ are compressible and so is the eventual equality relation $E_0(\mathbb{N})$ on $\mathbb{N}^{\mathbb{N}}$ (i.e., $x E_0(\mathbb{N}) y \iff \exists m \forall n \geq m(x_n = y_n)$. On the other hand, E_0 is not compressible (see Section 5.3).

The following is also a basic fact concerning compressible sets, see [N1, 5.7] or [DJK, 2.2]..

Proposition 3.26. *Let E be a countable Borel equivalence relation on a standard Borel space X. If a Borel set $A \subseteq X$ is compressible, then we have that $A \sim_E [A]_E$. Thus $[A]_E$ is also compressible.*

The next result deals with embeddability for compressible relations.

Proposition 3.27. *Let E, F be countable Borel equivalence relations.*
(i) If E is compressible, then $E \sqsubseteq_B F \iff E \sqsubseteq_B^i F$.
(ii) If both E, F are compressible, then $E \leq_B F \iff E \sqsubseteq_B^i F$. In particular, if both E, F are compressible, then $E \sim_B F \iff E \simeq_B F \iff E \cong_B F$.

Part (i) of Proposition 3.27 follows from Proposition 3.26. For part (ii), see [CK, 5.23].

The following gives another connection between reducibility and embeddability.

Proposition 3.28. *Let E, F be countable Borel equivalence relations on standard Borel spaces X, Y, resp. Then*

$$E \leq_B F \iff E \sqsubseteq_B F \times I_{\mathbb{N}}.$$

Proof. If $E \sqsubseteq_B F \times I_{\mathbb{N}}$, then clearly $E \leq_B F$, since $F \times I_{\mathbb{N}} \leq_B F$. If now $E \leq_B F$, let $f\colon X \to Y$ be a Borel reduction of E to F. Then $f(X) = A$ is a Borel subset of Y and there is sequence of Borel maps $g_n\colon A \to X$ such that $(g_n(y))$ enumerates $f^{-1}(y)$ for each $y \in A$. Let $g\colon X \to Y \times \mathbb{N}$ be defined by $g(x) = (f(x), i)$, where i is least such that $g_i(f(x)) = x$. This witnesses that $E \sqsubseteq_B F \times I_{\mathbb{N}}$. □

Finally it turns out that generically *every* aperiodic countable Borel equivalence relation is compressible.

Theorem 3.29 ([KM1, 13.3]). *Let E be an aperiodic countable Borel equivalence relation on a Polish space X. Then there is an invariant comeager Borel set $C \subseteq X$ such that $E \restriction C$ is compressible.*

For a stronger result involving graphings of equivalence relations, see [CKM, Section 4]. Also for related results about semigroup actions, see [Mi7].

Remark 3.30. We say that an infinite countable group G is **dynamically compressible** if every aperiodic E generated by a Borel action of G can be Borel reduced to a compressible aperiodic F induced by a Borel action of G. It is shown in [FKSV] that every infinite countable amenable group is dynamically compressible and the same is true for any countable group that contains a non-abelian free group. However there are infinite countable groups that fail to satisfy these two conditions but they are still dynamically compressible (see again [FKSV]). It is not known if *every* infinite countable group is dynamically compressible.

Remark 3.31. For a connection between compressibility and cardinal algebras (discussed in Chapter 13) see Remark 13.5.

3.8 Borel cardinalities and a Schroeder-Bernstein type theorem

If E, F are countable Borel equivalence relations on standard Borel spaces X, Y, resp., then $E \leq_B F$ means that there is an injective map

$$f\colon X/E \to Y/F$$

which has a Borel lifting. We can interpret this as meaning that the **Borel cardinality** of X/E is at most that of Y/F, in

$$|X/E|_B \leq |Y/F|_B.$$

We also let

$$|X/E|_B = |Y/F|_B \iff |X/E|_B \leq |Y/F|_B \ \& \ |F/Y|_B \leq |X/E|_B,$$

so that $|X/E|_B = |Y/F|_B \iff E \sim_B F$, and

$$|X/E|_B < |Y/F|_B \iff |X/E|_B \leq |Y/F|_B \ \& \ |F/Y|_B \nleq |X/E|_B,$$

so that $|X/E|_B < |Y/F|_B \iff E <_B F$.

The next result provides analogs of the classical Schroeder-Bernstein theorem that in particular show that $X/E, Y/F$ have the same Borel cardinality (i.e., $|X/E|_B = |Y/F|_B$) iff there is a bijection between X/E and Y/F with Borel lifting.

Theorem 3.32. *Let E, F be countable Borel equivalence relations on standard Borel spaces X, Y, resp. Then the following are equivalent:*

(i) $E \sim_B F$;

(ii) There are Borel sets $A \subseteq X, B \subseteq Y$ which are complete sections of E, F, resp., such that $E \restriction A \cong_B F \restriction B$;

(iii) $E \times I_{\mathbb{N}} \cong_B F \times I_{\mathbb{N}}$;

(iv) There is a bijection $f \colon X/E \to Y/F$ with Borel lifting (in which case f^{-1} has also a Borel lifting);

(v) There are decompositions $X = X_1 \sqcup X_2, Y = Y_1 \sqcup Y_2$ into invariant Borel sets and Borel complete sections $A_2 \subseteq X_2, B_1 \subseteq Y_1$ of $E \restriction X_2, F \restriction Y_1$, resp., such that $E \restriction X_1 \cong_B F \restriction B_1$ and $F \restriction Y_2 \cong_B E \restriction A_2$.

For the proof of the equivalence of parts (i)–(iv) see [DJK, 2.6] and for (v) see [Mi1]. Alternatively, as pointed out by Ronnie Chen, one can see that (iv) implies (v) as follows: Let Z be the disjoint union of the spaces X, Y and define the Borel equivalence relation R on Z by gluing together any E-class C with the F-class $f(C)$, where f is as in (iv). Then X, Y are complete Borel sections for R and an application of Proposition 3.22 gives (v).

It was asked in [DJK, page 201] whether Borel bireducibility, for aperiodic countable Borel equivalence relations, is also equivalent to Borel

biembeddability. This is equivalent to asking whether $E \times I_{\mathbb{N}} \sqsubseteq_B E$ holds for all aperiodic countable Borel equivalence relations E. This was disproved by Simon Thomas, using methods of ergodic theory.

Theorem 3.33 ([T2])**.** *There is an aperiodic countable Borel equivalence relation E such that it is not the case that $E \times I_2 \sqsubseteq_B E$.*

Other proofs of this theorem can be found in [HK4, 3.9] and [CM2, Theorem H].

For a countable Borel equivalence relation E on a standard Borel space X, recall that the quotient Borel structure on X/E, Σ_E, is the σ-algebra on X/E defined by: $A \in \Sigma_E \iff \tilde{A} = \bigcup A (\subseteq X)$ is Borel. We say that two countable Borel equivalence relations E, F on X, Y, resp., are **quotient Borel isomorphic** if there is a bijection of X/E to F/Y that takes Σ_E to Σ_F. Denote this by $E \cong_B^q F$. It is easy to see that $E \sim_B F \implies E \cong_B^q F$.

Problem 3.34. *Is it true that $E \sim_B F \iff E \cong_B^q F$?*

Finally we note the following result about liftings.

Proposition 3.35 ([Mi5, page 169])**.** *Let E, F be countable Borel equivalence relations on standard Borel spaces X, Y, resp., with F aperiodic. Let $f \colon X/E \to Y/F$ be a countable-to-1 function. Then if f has a Borel lifting, it has a finite-to-1 Borel lifting.*

3.9   Weak Borel reductions

We now consider a weaker notion of Borel reduction.

Definition 3.36. *Let E, F be countable Borel equivalence relations. A **weak Borel reduction** of E to F is a countable-to-1 Borel homomorphism f from E to F. We denote this by $f \colon E \leq_B^w F$. If such an f exists, we say that E is **weakly Borel reducible** to F, in symbols $E \leq_B^w F$.*

Clearly when $E \subseteq F$, the identity map is a weak Borel reduction of E to F. Of course a Borel reduction is also a weak Borel reduction. It turns out that weak reducibility is just the combination of inclusion and reduction. More precisely we have the following result, attributed to Kechris and Miller in [T12].

Theorem 3.37 (see [T12, Section 4]). *Let E, F be countable Borel equivalence relations on uncountable standard Borel spaces X, Y, resp. Then the following are equivalent:*

(i) $E \leq_B^w F$;

(ii) There is a countable Borel equivalence relation $E' \supseteq E$ with $E' \leq_B F$;

(iii) There is a Borel subequivalence relation $F' \subseteq F$ such that $E \sim_B F'$.

The notion of weak reduction is in general weaker than reduction. In fact we have the following stronger result, proved by using methods of ergodic theory.

Theorem 3.38 ([A6]). *There exist countable Borel equivalence relations E, F such that $E \subseteq F$ but $E \nleq_B F$ and $F \nleq_B E$.*

Other proofs of this result were given in [HK4, 3.8], [T12, Section 5], [H12] (see also [Mi11, 6.1]) and [CM2, Theorem G].

3.10   The full group and the automorphism group

To each countable Borel equivalence relation E one can assign a group of Borel automorphisms of the underlying space that actually determines it up to Borel isomorphism, at least in the aperiodic case.

Definition 3.39. *Let E be a countable Borel equivalence relation on a standard Borel space X. The **full group** of E, in symbols $[E]_B$, is the group of all Borel automorphisms T of X such that $T(x)Ex, \forall x$.*

Sometimes the full group is called the **inner automorphism group** of E and denoted by $\mathrm{Inn}_B(E)$.

We then have the following "Borel" analog of the classical theorem of Dye on (measure theoretic) full groups of ergodic, measure preserving countable Borel equivalence relations; see, e.g., [Ke10, 4.1].

Theorem 3.40 ([MR1]). *Let E, F be aperiodic countable Borel equivalence relations. Then the following are equivalent:*

(i) $E \cong_B F$;

(ii) $[E]_B, [F]_B$ are isomorphic (as abstract groups).

Moreover for any (algebraic) isomorphism $\varphi \colon [E]_B \to [F]_B$, there is a Borel isomorphism $f \colon E \cong_B F$ such that for $T \in [E]$, we have $\varphi(T) = f \circ T \circ f^{-1}$.

In fact in [MR1] it is shown that Theorem 3.40 holds for more general, than countable Borel, equivalence relations.

A further study of full groups can be found in [Me], [Mi5, Chapter 1], [Mi2] and [Mi16]. In particular one can characterize compressibility in terms of the algebraic properties of the full group.

Let G be a group. Then G has the **Bergman property** if for any increasing sequence $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq G$ with $\bigcup_n A_n = G$, there exist n, k such that $G = (A_n)^k$.

We now have:

Theorem 3.41 ([Mi5, page 91], [Mi16, Theorem 9]). *Let E be an aperiodic countable Borel equivalence relation, Then the following are equivalent:*

(i) E is compressible;

(ii) $[E]_B$ has the Bergman property.

Finally Rosendal has shown that the full group of any aperiodic countable Borel equivalence relation on an uncountable standard Borel space (in fact any countable Borel equivalence relation with uncountably many classes of cardinality at least 3) admits no second countable Hausdorff topology in which it becomes a topological group. The proof is similar to that of [Ro, Theorem 1].

The normalizer of $[E]_B$ in the group of all Borel automorphisms of X is denoted by $N_B[E]$. It consists exactly of all Borel automorphisms of E, i.e., all Borel automorphisms T of X such that $xEy \iff T(x)ET(y)$, for all $x, y \in X$. For that reason sometimes $N_B[E]$ is called the **automorphism group** of E and denoted by $\mathrm{Aut}_B(E)$. Moreover by Theorem 3.40, for aperiodic E, the group $N_B[E]$ can be identified with the group of automorphisms of the group $[E]_B$. The quotient group $N_B[E]/[E]_B$ is called the **outer automorphism group** of E, in symbols $\mathrm{Out}_B(E)$.

Consider now a Borel action a of a countable group G on X by automorphisms of E, i.e., for each $g \in G$, $x \mapsto g \cdot x$ is in $N[E]$. We abbreviate this by writing $a \colon G \curvearrowright_B (X, E)$. Then we can define the following equivalence relation on X, denoted by $E(a)$:

$$xE(a)y \iff \exists g \in G(g \cdot xEy).$$

Clearly $E(a) = E \vee E_a$. We call E_a the **expansion** of E by the action a. The following is due to J. Frisch and F. Shinko:

Proposition 3.42. *Every countable Borel equivalence relation is Borel bireducible to an expansion of a smooth countable Borel equivalence relation,*

Proof. Let E be a countable Borel equivalence relation on X and let a be a Borel action of a countable group G on X with $E = E_a$. Consider the action b of G on $X \times G$ given by: $g \cdot (x, h) = (g \cdot x, gh)$. Let also F on $X \times G$ be defined by : $(x, g)F(y, h) \iff x = y$, so that F is smooth and clearly G acts by automorphisms of F. Then note that $(x, g)F(b)(y, h) \iff xEy$, so that $E \sim_B F(b)$. □

If $E \subseteq F$ are countable Borel equivalence relation on X, we say that E is **normal** in F (or F is normal over E) if F is an expansion of E, i.e., there is $a \colon G \curvearrowright_B (X, E)$ with $F = E(a)$. Thus up to bireducibility every countable Borel equivalence relation is normal over a smooth one.

3.11   Actions on quotient spaces

Let E, F be countable Borel equivalence relations on X, Y, resp. We call a map $f \colon X/E \to Y/F$ **Borel** if it has a Borel lifting $\tilde{f} \colon X \to Y$. We denote by $\mathrm{Sym}_B(X/E)$ the **Borel symmetric group** of X/E, i.e., the group of Borel permutations of X/E. Let G be a countable group. A **Borel action** of G on X/E, in symbols $G \curvearrowright_B X/E$, is an action of G on X/E by Borel permutations. For such an action, we let $E^{\vee G}$ be the countable Borel equivalence relation given by

$$xE^{\vee G}y \iff \exists g \in G(g \cdot [x]_E = [y]_E).$$

Note that an action $a \colon G \curvearrowright_B (X, E)$ gives canonically a Borel action of G on X/E given by $g \cdot [x]_E = [g \cdot x]_E$. Then it is easy to check that $E^{\vee G} = E(a)$.

1) Given countable Borel equivalence relations $E \subseteq F$ on X, we say that F/E is **generated by a Borel action** if there is $G \curvearrowright_B X/E$ such that $F = E^{\vee G}$, i.e., there is an analog of the Feldman-Moore Theorem for X/F over X/E. In [dRM1] it is shown that there is a countable set of obstructions for being generated by a Borel action. Namely, there is a sequence of pairs $E_n \subseteq F_n$ of countable Borel equivalence relations on $2^{\mathbb{N}}$, where F_n/E_n is not generated by a Borel action, such that if $E \subseteq F$ are countable Borel equivalence relations on a Polish space X and F/E is not generated by a

Borel action, then there is some n and a continuous embedding of $2^{\mathbb{N}}$ into X, which simultaneously embeds E_n to E and F_n to F.

Remark 3.43. In [dRM2] obstructions are also obtained for analogs of the Glimm-Effros dichotomy theorem and the Lusin-Novikov theorem for quotient spaces.

2) We next consider ergodicity of actions $G \curvearrowright_B X/E$. Given a pair $E \subseteq F$ as in 1) above, F/E is **ergodic** if there is no Borel partition $X = A \sqcup B$ with A, B E-invariant, complete F-sections. The action $G \curvearrowright_B X/E$ is ergodic if $E^{\vee G}/E$ is ergodic. Ben Miller in [Mi5] showed the following results concerning ergodicity.

For a countable group G, let $F_0(G)$ be the countable Borel equivalence relation on $G^{\mathbb{N}}$ given by

$$(g_0, g_1, g_2, \ldots)\, F_0(G)\, (h_0, h_1, h_2, \ldots) \iff \exists m\, \forall k > m\, [g_0 \cdots g_k = h_0 \cdots h_k].$$

There is an action $G \curvearrowright_B (G^{\mathbb{N}}, F_0(G))$ defined by

$$g \cdot (g_0, g_1, g_2, \ldots) = (g \cdot g_0, g_1, g_2, \ldots),$$

inducing an action $G \curvearrowright_B G^{\mathbb{N}}/F_0(G)$. Then $G^{\mathbb{N}}/F_0(G)$ is ergodic. Let E be a countable Borel equivalence relation on a Polish space X, and let $G \curvearrowright_B X/E$ be a free action. Then $E^{\vee G}/E$ is ergodic iff there is a G-equivariant Borel injection of $G^{\mathbb{N}}/F_0(G)$ to X/E induced by a continuous embedding of $G^{\mathbb{N}}$ to X (see [Mi5, Theorem 7.2]). If $E^{\vee G}$ is hyperfinite, then there is a G-equivariant Borel injection of X/E to $G^{\mathbb{N}}/F_0(G)$ (see [Mi5, Theorem 8.1]).

3) In [FKS] the lifting problem for actions on quotients is studied. Let E be a countable Borel equivalence relation on X. Denote by p_E the canonical surjective homomorphism of $\mathrm{Aut}_B(E)$ onto $\mathrm{Out}_B(E)$. There is also a canonical homomorphism of $\mathrm{Aut}_B(E)$ into $\mathrm{Sym}_B(X/E)$ which sends $T \in \mathrm{Aut}_B(E)$ to the permutation $[x]_E \mapsto [T(x)]_E$. This has kernel $\mathrm{Out}_B(E)$ and therefore it gives an injective homomorphism i_E from $\mathrm{Out}_B(E)$ into $\mathrm{Sym}_B(X/E)$. Its range consists of the so-called **outer permutations**. An action $G \curvearrowright_B (X, E)$, which can be viewed as a homomorphism from G into $\mathrm{Aut}_B(E)$, gives via p_E a homomorphism from G into $\mathrm{Out}_B(E)$ and finally through i_E to a homomorphism from G to $\mathrm{Sym}_B(X/E)$, i.e., an action $G \curvearrowright_B X/E$. One considers here the question of when a homomorphism

G to $\mathrm{Sym}_B(X/E)$ lifts to a homomorphism of G to $\mathrm{Aut}_B(E)$, i.e., when an action $G \curvearrowright_B X/E$ lifts to an action $G \curvearrowright_B (X,E)$. The first result is the following:

Theorem 3.44 ([FKS, Theorem 1.1]). *Let E be a compressible countable Borel equivalence relation. Then every Borel action $G \curvearrowright_B X/E$ has a lift $G \curvearrowright_B (X,E)$.*

This has the following corollary:

Corollary 3.45 ([FKS, Corollary 1.2]). *Let E be an aperiodic countable Borel equivalence relation on a Polish space X. Then for any action $G \curvearrowright_B X/E$, there is a comeager $E^{\vee G}$-invariant Borel subset $Y \subseteq X$ such that $G \curvearrowright_B Y/E$ lifts.*

On the other hand there are examples of E for which not every element of $\mathrm{Sym}_B(X/E)$ is outer and therefore there are actions $G \curvearrowright_B X/E$, i.e., homomorphims of G into $\mathrm{Sym}_B(X/E)$ which do not even lift to homomorphisms of G into $\mathrm{Out}_B(E)$ and thus much less to actions $G \curvearrowright_B (X,E)$.

Call then an action $G \curvearrowright_B X/E$ **outer** if it acts by outer permutations. The next question is then to find out when an *outer* action $G \curvearrowright_B X/E$ lifts to an action $G \curvearrowright_B (X,E)$. Here we have the following result:

Theorem 3.46 ([FKS, Theorem 1.3]). *Outer actions of amenable groups and amalgamated free products of finite groups have a lift.*

However there is a limit to the groups G that can have the lifting property for outer actions. Below a countable group G is called **treeable** if it admits a free Borel action on a standard Borel space X that has an invariant probability measure (see Chapter 5) and for which the associated orbit equivalence relation E_G^X is treeable (see Section 10.1). For example, every amenable group and every free group is treeable but infinite property (T) groups and products of an infinite group and a non-amenable group are not treeable.

Proposition 3.11.1 ([FKS, Proposition 1.4]). *If every outer action of a countable group G lifts, then G is treeable.*

Thus if $\mathcal{G}$ is the class of countable groups that have the lifting property for outer actions, then $\mathcal{G}$ contains all the amenable groups and all amalgamated free products of finite groups and it is closed under subgroups

(see [FKS, Proposition 4.7]) and free products. Finally all groups in $\mathcal{G}$ are treeable.

It is not known whether one can characterize the class of groups that have this lifting property,

3.12 The Borel classes of countable Borel equivalence relations

In a recent preprint, Lecomte [Lec] characterizes when a countable Borel equivalence relation belongs to the Borel classes $\mathbf{\Sigma}^0_\xi$ or $\mathbf{\Pi}^0_\xi$ ($\xi \geq 1$) in terms of a dichotomy. (In fact he proves a more general result for Borel equivalence relations with F_σ classes.)

Below we state his result for $\xi \geq 3$, in which case the dichotomy is somewhat easier to state. For each Polish space X and $\mathbf{\Gamma}$ one of the classes $\mathbf{\Sigma}^0_\xi$ or $\mathbf{\Pi}^0_\xi$ ($\xi \geq 1$), let $\mathbf{\Gamma}(X)$ be the class of subsets of X in $\mathbf{\Gamma}$. We also let $\check{\mathbf{\Gamma}}$ be the dual class of $\mathbf{\Gamma}$, i.e., $\check{\mathbf{\Gamma}}$ is $\mathbf{\Pi}^0_\xi$ or $\mathbf{\Sigma}^0_\xi$, resp. Let $H = 2 \times 2^{\mathbb{N}}$ and for $\xi \geq 3$ and $\mathbf{\Gamma}$ as above, let $C \subseteq 2^{\mathbb{N}}$ be such that $C \cap N_s \in \check{\mathbf{\Gamma}}(N_s) \setminus \mathbf{\Gamma}$, for each $s \in \bigcup_n 2^n$. (Such C are shown to exist.) Define the equivalence relation $E_3^{\mathbf{\Gamma}}$ on H by:

$$(i, x) E_3^{\mathbf{\Gamma}} (j, y) \iff (i, x) = (j, y) \text{ or } (x = y \in C).$$

Clearly $E_3^{\mathbf{\Gamma}}$ is a countable Borel equivalence relation and $E_3^{\mathbf{\Gamma}} \in \check{\mathbf{\Gamma}}(H^2) \setminus \mathbf{\Gamma}$. Then we have:

Theorem 3.47 ([Lec, Theorem 1.4]). *Let $\mathbf{\Gamma}$ be one of the classes $\mathbf{\Sigma}^0_\xi$ or $\mathbf{\Pi}^0_\xi$, for $\xi \geq 3$, let X be a Polish space and E a countable Borel equivalence relation on X. Then exactly one of the following holds:*
  (i) $E \in \mathbf{\Gamma}(X^2)$;
  (ii) $E_3^{\mathbf{\Gamma}} \sqsubseteq_c E$.

4. Essentially countable relations

4.1 Essentially countable and reducible to countable equivalence relations

It turns out that the scope of the theory of countable Borel equivalence relations is much wider as it encompasses many other classes of equivalence relations up to Borel bireducibility.

Definition 4.1. *A Borel equivalence relation E is* **essentially countable** *if it is Borel bireducible to a countable Borel equivalence relation. It is called* **reducible to countable** *if it is Borel reducible to a countable Borel equivalence relation.*

Remark 4.2. In the literature the term "essentially countable" is often used for what we called here "reducible to countable".

These two notions are distinct.

Theorem 4.3 ([H4])**.** *There is a Borel equivalence relation which is reducible to countable but not essentially countable.*

On the other hand, for many naturally occurring Borel equivalence relations the notions coincide. To explain this we need a definition first.

Definition 4.4. *Let E be a Borel equivalence relation on a standard Borel space X. Then E is* **idealistic** *if there is a map $C \in X/E \mapsto \mathcal{I}_C$, assigning to each E-class C a σ-ideal $\mathcal{I}_C$ of subsets of C, with $C \notin \mathcal{I}_C$, such that $C \mapsto \mathcal{I}_C$ is Borel in the following (weak) sense: For each Borel set $A \subseteq Y \times X$, Y a standard Borel space, the set $A_{\mathcal{I}} \subseteq Y \times X$ defined by $(y,x) \in A_{\mathcal{I}} \iff \{x' \in [x]_E : (y,x') \in A\} \in \mathcal{I}_{[x]_E}$ is Borel.*

A typical example of an idealistic E is a Borel equivalence relation induced by a Borel action of a Polish group (see [Ke2, page 285]). In the next result we will also need the following definition:

Definition 4.5. *Let E be an equivalence relation on a space X. A **complete countable section** for E is a complete section S such that for each $x \in X$, $S \cap [x]_E$ is countable.*

If a Borel equivalence relation E admits a complete countable Borel section, it is clearly essentially countable. Now we have:

Theorem 4.6. *Let E be an idealistic Borel equivalence relation. Then the following are equivalent:*
    (i) E is reducible to countable;
    (ii) E is essentially countable;
    (iii) E admits a complete countable Borel section.
    In fact if $E \leq_B F$, where F is a countable Borel equivalence relation on a standard Borel space Y, then there is an F-invariant Borel set $B \subseteq Y$, such that $E \sim_B F \upharpoonright B$.

For a proof, see the more general result in [KMa, 3.7, 3.8] (and the corrections posted in http://www.math.caltech.edu/~kechris/).

The following characterization of reducibility to countable is often useful in establishing this property.

Proposition 4.7 (Kechris). *Let E be a Borel equivalence relation on a standard Borel space X. Then the following are equivalent:*
    (i) E is reducible to countable;
    (ii) There is a standard Borel space Y and a Borel function $f \colon X \to Y$ such that (a) $f([x]_E)$ is countable, $\forall x \in X$, and (b) $\neg(xEy) \implies f([x]_E) \cap f([y]_E) = \emptyset$.

A proof can be found in [H2, 5.2] and also in [Ka, 7.6.1] and [CLM, 7.1].

Hjorth has found a dichotomy for the property of essential countability of Borel equivalence relations induced by Borel actions of Polish groups. Let G be a Polish group and $a \colon G \times X \to X$ a continuous action of G on a Polish space X. This action is **stormy** if for every non-empty open set $U \subseteq G$ and any $x \in X$, the map $g \mapsto g \cdot x$ from U to $U \cdot x$ in not an open map.

We now have:

Theorem 4.8 ([H5, Theorem 1.3]). *Let G be a Polish group and $a \colon G \times X \to X$ a Borel action of G on a standard Borel space X. Then if the associated equivalence relation E_a is Borel, exactly one of the following holds:*

(i) E_a is reducible to countable;

(ii) There is a stormy action b of G on a Polish space Y and a Borel embedding of the action b to the action a, i.e., a Borel injection $F \colon Y \to X$ such that $F(g \cdot y) = g \cdot F(y)$.

Moreover if the action a on a Polish space X is continuous, F can be taken to be continuous too.

We next proceed to discuss various classes of essentially countable Borel equivalence relations.

4.2 Actions of locally compact groups and lacunary sections

A rich source of essentially countable Borel equivalence relations comes from actions of locally compact groups. Before we state the main result here, we introduce some additional concepts.

Definition 4.9. *Let G be a topological group and let $a \colon G \times X \to X$ be an action of G on a space X. A complete section S of E_a is **lacunary** if there is a neighborhood U of the identity of G such that for all $s \in S$, $U \cdot s \cap S = \{s\}$. In this case, we also say that S is **U-lacunary**. A complete section S is called **cocompact** if there is a compact neighborhood U of the identity of G such that $U \cdot S = X$. Again in this case we say that S is **U-cocompact**.*

Note that if G is second countable, then any lacunary complete section is countable.

If a Polish locally compact group G acts in a Borel way on a standard Borel space X, the induced equivalence relation is Borel (note, for example, that by [BK, 5.2.1] we can assume that X is a Polish space and the action is continuous, in which case the induced equivalence relation is K_σ). We now have:

Theorem 4.10 ([Ke2]). *Let G be a Polish locally compact group and let $a \colon G \times X \to X$ be a Borel action of G on a standard Borel space X. For any compact neighborhood U of the identity of G, E_a has a complete U-lacunary Borel section S. In particular E_a is essentially countable.*

A measure theoretic version of this result (where null sets are neglected) was proved in [FHM] (and for the free action case in [Fo]) and the case $G = \mathbb{R}$ of Theorem 4.10 was proved in [Wa] (while the measure theoretic version in the case of $\mathbb{R}$ was proved in [Am]).

Other proofs of Theorem 4.10 can be found in [Ke8, pages 244–245] and [H5, 2.2].

One can also make the lacunary sections to be cocompact. This was shown independently by Conley and Dufloux (unpublished).

Theorem 4.11. *(i)* (Conley, Dufloux) *Let G be a Polish locally compact group and let $a\colon G \times X \to X$ be a Borel action of G on a standard Borel space X. Then E_a has a complete lacunary cocompact Borel section.*

(ii)[Sl1, 2.4] *In fact, let U be a symmetric compact neighborhood of the identity of G and put $V = U^2$. Then for any complete V-lacunary Borel section S of E_a, there is a maximal (under inclusion) complete V-lacunary Borel section $T \supseteq S$ of E_a and thus T is V-cocompact.*

A measure theoretic version of such a result for free actions can be also found in [KPV, 4.2].

One can formulate these results in the language of descriptive combinatorics. Let G be a Polish locally compact group and let $a\colon G \times X \to X$ be a Borel action of G on a standard Borel space X. Let U be a symmetric compact neighborhood of the identity of G and put $V = U^2$. Define the Borel graph whose set of vertices is X and distinct $x, y \in X$ are connected by an edge iff $y \in V \cdot x$. Then Theorem 4.10 and Theorem 4.11 imply that in this graph there exists a maximal independent set which is Borel.

The following gives a more detailed analysis of the equivalence relation induced by a locally compact group action.

Theorem 4.12 ([Ke4]). *Let G be a Polish locally compact group and let $a\colon G \times X \to X$ be a Borel action of G on a standard Borel space X. Then there is a (unique) decomposition $X = A \sqcup B$ of X into invariant Borel sets such that $E_a \upharpoonright A$ is countable and $E_a \upharpoonright B \cong_B F \times I_\mathbb{R}$, where $F = E_a \upharpoonright S$, with S a countable complete Borel section of $E_a \upharpoonright B$.*

Corollary 4.13 ([Ke4]). *The map $E \mapsto E \times I_\mathbb{R}$ induces a bijection between countable Borel equivalence relations, up to Borel bireducibility, and equivalence relations induced by Borel actions of Polish locally compact groups with uncountable orbits, up to Borel isomorphism.*

One interesting application of Theorem 4.10 is in the proof of the result in [HK2] that isomorphism (conformal equivalence) of Riemann surfaces, and in particular complex domains (open connected subsets of $\mathbb{C}$), is an essentially countable Borel equivalence relation. We will discuss this in more detail in Section 12.2, 5).

Concerning lacunary sections as in Theorem 4.10, it is of interest to obtain in certain situations additional information about their structure. For the case of free Borel actions of $\mathbb{R}$ on standard Borel spaces, each orbit is an (affine) copy of $\mathbb{R}$, so if S is a complete lacunary section, it makes sense to talk about the distance between consecutive members of S in the same orbit. We now have the following result that provides a purely Borel strengthening of a classical result of Rudolph [Ru] in the measure theoretic context and again neglecting null sets.

Theorem 4.14 ([Sl2]). *Let α, β be two rationally independent positive reals. Then any free Borel action of $\mathbb{R}$ admits a complete lacunary Borel section such that the distance between any two consecutive points in the same orbit belongs to $\{\alpha, \beta\}$.*

The paper [Sl3] further characterizes the sets of positive reals A with the property that in any free Borel action of $\mathbb{R}$ there is a complete lacunary Borel section such that the distance between any two consecutive points in the same orbit belongs to A. The papers [Sl1] and [Sl4] use complete lacunary and cocompact sections to prove classification results for $\mathbb{R}^n$-actions.

In [Mi13] the following generalization of lacunarity was introduced. A Borel action a of a Polish group on a standard Borel space X is called **σ-lacunary** if X can be decomposed into countably many invariant Borel sets on each of which the induced action admits a complete lacunary Borel section. Clearly for any such action, E_a is essentially countable. In fact the converse holds as well.

Theorem 4.15 ([Gr1]). *Let G be a Polish group and let a be a Borel action of G on a standard Borel space. Then the following are equivalent:*
  (i) The action is σ-lacunary.
  (ii) E_a is essentially countable.

Finally we consider the question of whether Theorem 4.10 actually characterizes Polish locally compact groups.

Problem 4.16. *Let G be a Polish group with the property that all the equivalence relations induced by Borel actions of G on standard Borel spaces are Borel and essentially countable. Is the group locally compact?*

In [Tho] it is shown that every such group must be CLI, i.e., admit a complete left-invariant metric. However there are many CLI groups that are not locally compact.

An affirmative answer to Problem 4.16 has been obtained for certain classes of Polish groups:

(i) [So] All separable Banach spaces, viewed as groups under addition;

(ii) [Ma] All *abelian* isometry groups of separable locally compact metric spaces;

(iii) [KMPZ] All isometry groups of separable locally compact metric spaces. This class of groups includes those in (ii) and all non-archimedean Polish groups,

A related result characterizing Polish compact groups was also proved in [So]: A Polish group is compact iff all the equivalence relations induced by Borel actions of G on standard Borel spaces are Borel and smooth.

4.3 On the existence of complete countable Borel sections

We note here that the existence of a complete countable Borel section for a Borel equivalence relation is in general stronger than being essentially countable. The standard example is as follows: Let $X \subseteq \mathbb{R} \times \mathbb{R}$ be a Borel set which projects onto $\mathbb{R}$ but admits no Borel uniformization (see, e.g., [Ke6, 18.17]). Let E on X be defined by $(x, y)E(x', y') \iff x = x'$. Then $E \sim_B \Delta_{\mathbb{R}}$ but E admits no complete countable Borel section. The following result gives a characterization of the existence of complete countable Borel sections.

We call a Borel equivalence relation E **ccc idealistic** if it satisfies Definition 4.4 with the σ-ideals $\mathcal{I}_C$ being in addition ccc (i.e., any pairwise disjoint collection of subsets of C not in $\mathcal{I}_C$ is countable). For example, a Borel equivalence relation induced by a Borel action of a Polish group is ccc idealistic.

Below for measures μ, ν, on the same space, $\mu \sim \nu$ denotes **measure equivalence**, i.e., having the same null sets. Finally we call a Borel equiv-

alence relation E **σ-smooth** if it can be written as $E = \bigcup_n E_n$, where each E_n is a smooth Borel equivalence relation.

Theorem 4.17 ([Ke2, 1.5])**.** *Let E be a Borel equivalence relation on a standard Borel space X. Then the following are equivalent:*

(i) E admits a complete countable Borel section;

(ii) (a) E is σ-smooth and (b) E is ccc idealistic.

(iii) As in (ii) but with (b) replaced by (b)*: There is a Borel assignment $x \mapsto \mu_x$ of probability Borel measures to points $x \in X$ such that $\mu_x([x]_E) = 1$ and $xEy \implies \mu_x \sim \mu_y$.

A generalization of the measure theoretic result in [FHM] is proved in [R]. It states that if E is a Borel equivalence relation on a standard Borel space X, μ a probability Borel measure on X, and there is a Borel assignment $x \mapsto \mu_x$ of probability Borel measures to points $x \in X$ such that $\mu_x([x]_E) = 1$ and $xEy \implies \mu_x \sim \mu_y$, then E admits a complete countable Borel section, μ-a.e.. It is unknown if there is a purely Borel version of this result;, i.e., whether condition (a) is necessary in Theorem 4.17, (iii). It is known that condition (a) is necessary in Theorem 4.17, (ii); see the discussion in [Ke2, Section 1, (III), (IV)].

Note that if E is a Borel equivalence relation which is reducible to countable, then in Theorem 4.17, (a) is automatically satisfied by Corollary 3.7, so for such equivalence relations the existence of a countable complete Borel section is equivalent to condition (b) and also to condition (b)* and by Theorem 4.6 also to the condition that E is idealistic.

Another characterization has been found in [H10]. Below a Borel equivalence relation E on a standard Borel space X is called **treeable** if there is a Borel acyclic graph on X such that the E-classes are exactly its connected components. It is called **σ-treeable** if it can be written as $E = \bigcup_n E_n$, where each E_n is Borel and treeable. We now have:

Theorem 4.18 ([H10])**.** *Let E be a Borel equivalence relation which is reducible to countable. Then the following are equivalent:*

(i) E admits a complete countable Borel section;

(ii) E is σ-treeable.

Moreover the following holds, which shows that in Theorem 4.17, (ii), condition (a) can be replaced by σ-treeability.

Theorem 4.19 ([H10])**.** *Let E be a Borel equivalence relation on a standard Borel space X. Then the following are equivalent:*

    (i) E admits a complete countable Borel section;

    (ii) (a) E is σ-treeable and (b) E is ccc idealistic.

It is also shown in [H10] that every σ-treeable smooth Borel equivalence relation admits a Borel transversal.

Finally in [CLM], dichotomy theorems are proved characterizing when a treeable Borel equivalence relation admits a complete countable Borel section and also when the equivalence relation E_T, for a Borel function T, admits a complete countable Borel section.

4.4   Actions of non-archimedean Polish groups

Recall that a Polish group is **non-archimedean** if it has a neighborhood basis at the identity consisting of open subgroups. Equivalently these are (up to topological group isomorphism) the closed subgroups of the infinite symmetric group S_∞ of all permutations of $\mathbb{N}$ with the pointwise convergence topology and also the automorphism groups of countable structures (in the sense of model theory); see [BK]. It turns out that one can characterize exactly which Borel actions of such groups induce Borel equivalence relations that are essentially countable. To formulate this we need the following definition:

Definition 4.20. *Let E be a Borel equivalence relation on a standard Borel space and Λ a class of sets in Polish spaces, closed under continuous preimages. Then E is **potentially** Λ if there is an equivalence relation F, in some Polish space, which is in the class Λ, such that $E \leq_B F$. This is equivalent to saying that there is a Polish topology τ on X inducing its Borel structure such that E is in the class Λ (in the product space $(X^2, \tau \times \tau)$)*

For example, it turns out that a Borel equivalence relation is smooth iff it potentially $\mathbf{\Pi}^0_1$ iff it is potentially $\mathbf{\Pi}^0_2$ (see [HKL]). We now have:

Theorem 4.21 ([HK1, 3.8]), [HKLo, 4.1])**.** *Let G be a Polish non-archimedean group and let $a\colon G \times X \to X$ be a Borel action of G on a standard Borel space X. Then the following are equivalent:*

    (i) E_a is essentially countable;

(ii) E_a is potentially $\mathbf{\Sigma}_2^0$;
(iii) E_a is potentially $\mathbf{\Sigma}_3^0$.

This fails for arbitrary Polish groups (see, e.g., [HK1, Remark in page 236]) and also for Polish non-archimedean groups if $\mathbf{\Sigma}_3^0$ is replaced by $\mathbf{\Pi}_3^0$ (see [HKLo]; a specific example is the equivalence relation E_{ctble}, on the G_δ subspace of injective sequences in $\mathbb{R}^{\mathbb{N}}$, given by $(x_n) E_{ctble} (y_n) \iff \{x_n \colon n \in \mathbb{N}\} = \{y_n \colon n \in \mathbb{N}\}$, which is $\mathbf{\Pi}_3^0$ but not essentially countable).

4.5 Logic actions and the isomorphism relation on the countable models of a theory

Fix a countable relational language $L = \{R_i\}_{i \in I}$, where R_i has arity n_i. We denote by $X_L = \mathrm{Mod}_{\mathbb{N}}(L)$ the space of L-structures with universe $\mathbb{N}$. Thus X_L can be identified with the compact metrizable space $\prod_{i \in I} 2^{\mathbb{N}^{n_i}}$. We let $\mathbb{A} \cong \mathbb{B}$ be the isomorphism relation between structures in X_L. This is the equivalence relation induced by the so-called **logic action** of the infinite symmetric group S_∞ of all permutations of $\mathbb{N}$ on X_L, given by $g \cdot \mathbb{A} = \mathbb{B}$ iff g is an isomorphism of $\mathbb{A}$ with $\mathbb{B}$. For each sentence $\sigma \in L_{\omega_1 \omega}$, let

$$\mathrm{Mod}(\sigma) = \{\mathbb{A} \in X_L \colon \mathbb{A} \models \sigma\}$$

be the set of models of σ. This is a Borel invariant under isomorphism subset of X_L and, by the classical theorem of Lopez-Escobar, every such Borel subset of X_L is of the form $\mathrm{Mod}(\sigma)$, see [Ke6, 16.8]. Denote by $\cong_\sigma$ the isomorphism relation restricted to $\mathrm{Mod}(\sigma)$. It is shown in [BK, 2.7.3] that if $a \colon S_\infty \times X \to X$ is a Borel action of S_∞ on a standard Borel space X, then a is Borel isomorphic to the logic action on some $\mathrm{Mod}(\sigma)$.

Remark 4.22. In cases where we want to consider languages with function symbols, we will replace them by their graphs.

We next state model theoretic criteria for essential countability (and smoothness) of $\cong_\sigma$. Let F be a countable fragment of $L_{\omega_1 \omega}$, see [B]. For any L-structure $\mathbb{A}$, we denote by $\mathrm{Th}_F(\mathbb{A})$ the set of sentences in F that hold in $\mathbb{A}$. We say that a countable L-structure $\mathbb{A}$ is $\aleph_0$-**categorical for** F if every countable L-structure $\mathbb{B}$ for which $\mathrm{Th}_F(\mathbb{A}) = \mathrm{Th}_F(\mathbb{B})$, $\mathbb{B}$ is isomorphic to $\mathbb{A}$. We now have:

Theorem 4.23 ([HK1, 4.2]). *Let $\sigma \in L_{\omega_1\omega}$. Then the following are equivalent:*

(i) $\cong_\sigma$ is Borel and smooth;

(ii) There is a countable fragment F of $L_{\omega_1\omega}$ containing σ, such that every countable model $\mathbb{A}$ of σ is $\aleph_0$-categorical for F.

Theorem 4.24 ([HK1, 4.3]). *Let $\sigma \in L_{\omega_1\omega}$. Then the following are equivalent:*

(i) $\cong_\sigma$ is Borel and essentially countable;

(ii) There is a countable fragment F of $L_{\omega_1\omega}$ containing σ, such that for every countable model $\mathbb{A} = \langle A, \ldots \rangle$ of σ, there is $n \geq 1$ and a finite sequence $\bar{a} \in A^n$ such that $\langle \mathbb{A}, \bar{a} \rangle$ is $\aleph_0$-categorical for F.

Using this last result, one can easily prove the essential countability of the isomorphism relation on the following structures: finitely generated groups (or more generally finitely generated structures in some countable language), connected locally finite graphs, locally finite trees, finite transcendence degree over $\mathbb{Q}$ fields, torsion-free abelian groups of finite rank, etc.

Remark 4.25. (i) For the case of torsion-free abelian groups of finite rank $\leq n$, one can also directly see that the isomorphism relation is essentially countable, since it is Borel bireducible to the equivalence relation on the space of subgroups of $\langle \mathbb{Q}^n, + \rangle$ induced by the action of $\mathrm{GL}_n(\mathbb{Q})$ on this space.

(ii) Also in the case of finitely generated groups, one can also directly see that the isomorphism relation is essentially countable by using the space of finitely generated groups, see, e.g., [T8, Section 2].

We conclude with the following open problem of Hjorth and Kechris:

Problem 4.26. *Let σ be a first-order theory, i.e., the conjunction of countably many first-order sentences. Is it possible for $\cong_\sigma$ to be Borel, non-smooth and essentially countable?*

A negative answer has been obtained in [Mar] for first-order theories with uncountably many types.

4.6   Another example

Let $\mathbb{U}$ be the **Urysohn metric space** and $F(\mathbb{U})$ the standard Borel space of closed subsets of $\mathbb{U}$ with the Effros Borel structure. We view $F(\mathbb{U})$ as the

standard Borel space of Polish metric spaces. Let $\mathcal{M}$ be a class of Polish metric spaces closed under isometries. We call $\mathcal{M}$ a **Borel** class if $\mathcal{M} \cap F(\mathbb{U})$ is Borel in $F(\mathbb{U})$. Denote by $\cong^{iso}$ the equivalence relation of isometry on $F(\mathbb{U})$ and we let $\cong^{iso}_{\mathcal{M}}$ be its restriction to $\mathcal{M} \cap F(\mathbb{U})$, i.e, the equivalence relation of isometry for spaces in $\mathcal{M}$. See [GK] for the study of this equivalence relation on various classes of Polish metric spaces. The following is a special case of a more general result of Hjorth; see [GK, 7.1]. Recall that a metric space is **proper** (or **Heine-Borel**) if every closed bounded set is compact. It is easy to see that the class of proper Polish spaces is Borel.

Theorem 4.27 (Hjorth; see [GK, 7.1]). *Let $\mathcal{M}$ be the class of proper Polish spaces. Then the relation $\cong^{iso}_{\mathcal{M}}$ is an essentially countable Borel equivalence relation. The same holds for any Borel class $\mathcal{M}$ of connected, locally compact Polish metric spaces.*

4.7 Dichotomies involving reducibility to countable

For the theorems below we use the following terminology and notation. The equivalence relation E_2 on $2^{\mathbb{N}}$ is defined by

$$x E_2 y \iff \sum_{\{n \in \mathbb{N}:\, x_n \neq y_n\}} \frac{1}{n+1} < \infty$$

It can be shown that E_2 is Borel bireducible to the equivalence relation induced by the translation action of $\ell^1 = \{(x_n) \in \mathbb{R}^{\mathbb{N}}:\, \sum_n |x_n| < \infty\}$ on $\mathbb{R}^{\mathbb{N}}$, see [Ka, 6.2.4]. We also let

$$E_3 = E_0^{\mathbb{N}}.$$

A Polish group is **tsi** if it admits a compatible two-sided invariant metric. For non-archimedean groups this is equivalent to admitting a nbhd basis at the identity consisting of normal open subgroups.

We now have the following dichotomy theorems:

Theorem 4.28 ([H2, 0.4]). *Let E be a Borel equivalence relation on a standard Borel space. If $E \leq_B E_2$, then exactly one of the following holds:*
    (i) E is reducible to countable;
    (ii) $E_2 \leq_B E$.

Theorem 4.29 ([HK3, 8.1]). *Let E be a Borel equivalence relation on a standard Borel space. If $E \leq_B E_a$, where a is a Borel action of a non-archimedean tsi Polish group G with E_a Borel, then exactly one of the following holds:*
   (i) E is reducible to countable;
   (ii) $E_3 \leq_B E$.

In [Gr1, 1.2] it is shown that for $E = E_a$ alternative (i) in Theorem 4.29 is equivalent to the following statement:
   (i)′ There is a sequence (N_n) of open normal subgroups of G and for each n a Borel action a_n of G/N_n on a standard Borel space such that $E_a \leq_B \bigoplus_n E_{a_n}$.
   Also for generalizations of Theorem 4.29, again in the case $E = E_a$, for *arbitrary* tsi Polish groups, see [Mi13, 4.1] and [Gr2].
   Finally the following dichotomy is established in [dRM2]:

Theorem 4.30 ([dRM2]). *Let E be a Borel equivalence relation on a Polish space which is a countable union of smooth Borel subequivalence relations. Then exactly one of the following holds:*
   (i) E is reducible to countable,
   (ii) $E_1 \sqsubseteq_c E$.

4.8   Canonization

In the book [KSZ] the authors study canonization theorems, which analyze the behavior of equivalence relations on "large sets". For example, Silver's Theorem 6.1 implies that if E is a Borel equivalence relation on an uncountable Polish space, E will be trivial on a perfect nonempty set P, i.e., $E \restriction P = \Delta_P$ or $E \restriction P = P^2$. Chapters 4, 7 and 8 in [KSZ] deal with problems of canonization related to countable and reducible to countable Borel equivalence relations. For example, a result of Mathias, see [KSZ, 8.17], asserts that for every countable Borel equivalence relation E on the space of all infinite subsets of $\mathbb{N}$, there is an infinite subset $A \subseteq \mathbb{N}$ such that E restricted to the set of all infinite subsets of A is hyperfinite, in fact is contained in E_0 (where we view here E_0 as the equivalence relation of finite symmetric difference on the space of subsets of $\mathbb{N}$).

5. Invariant and quasi-invariant measures

5.1 Terminology and notation

For the rest of this paper we adopt the following terminology and notation: A **measure** on a standard Borel space X is a σ-finite Borel measure μ. If $\mu(X) < \infty$, μ is called **finite** and if $\mu(X) = 1$ it is called a **probability** measure. If μ is a measure on X and $Y \subseteq X$ is a Borel set, then $\mu \upharpoonright Y$ is the measure on Y which is the restriction of μ to the Borel subsets of Y. A measure μ is **absolutely continuous** to a measure ν, in symbols $\mu \ll \nu$, if for every Borel set $A \subseteq X$, $\nu(A) = 0 \implies \mu(A) = 0$, and μ, ν are **equivalent**, in symbols $\mu \sim \nu$, if $\mu \ll \nu$ & $\nu \ll \mu$. The equivalence class of a measure under this equivalence relation is called its **measure class**. Note that every measure is equivalent to a probability measure.

5.2 Invariant measures

If G is a countable group which acts in a Borel way on a standard Borel space X, then G acts on the set of measures on X by $g \cdot \mu(A) = \mu(g^{-1} \cdot A)$. We say that μ is **invariant** under this action if for all $g \in G$, $g \cdot \mu = \mu$.

Assume now that E is a countable Borel equivalence relation on a standard Borel space X. Denote below by $[[E]]_B$ the **full pseudogroup** of E, which is the set of all Borel bijections $f \colon A \to B$ between Borel subsets A, B of X such that $\forall x \in A(f(x)Ex)$. Thus $[E]_B \subseteq [[E]]_B$. Note that if E is induced by a Borel action of a countable group G on X, then $f \colon A \to B$ as above is in $[[E]]_B$ iff there is a countable decomposition $A = \bigsqcup_n A_n$, and group elements (g_n) such that for each n and $x \in A_n$, $f(x) = g_n \cdot x$.

Note now the following simple fact, where for a Borel function $T\colon X \to Y$ on standard Borel spaces X, Y and measure μ on X, $T_*\mu$ is the **push-forward measure** on Y, defined by $T_*\mu(B) = \mu(T^{-1}(B))$, for every Borel set $B \subseteq Y$.

Proposition 5.1. *Let E be a countable Borel equivalence relation on a standard Borel space X. Then the following are equivalent for each measure μ on X:*

(i) For some countable group G and Borel action a of G on X such that $E_a = E$, μ is invariant under this action;

(ii) For every countable group G and every Borel action a of G on X such that $E_a = E$, μ is invariant under this action;

(iii) For every $f\colon A \to B$ in $[[E]]_B$, $\mu(A) = \mu(B)$;

(iv) For every $T \in [E]_B$, $T_*\mu = \mu$.

Definition 5.2. *Let E be a countable Borel equivalence relation on a standard Borel space X and μ a measure on X. Then μ is called **E-invariant** if it satisfies the equivalent conditions of Proposition 5.1. Also μ is **E-ergodic** if for every E-invariant Borel set A, we have $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.*

Below a measure μ is called **nonatomic** if every singleton has measure 0.

Proposition 5.3 (see [DJK, 3.2]). *Let E be a countable Borel equivalence relation on a standard Borel space X and A a complete Borel section for E. Let ν be a measure on A such that ν is $E \upharpoonright A$-invariant. Then there is a unique E-invariant measure μ on X such that for all Borel sets $B \subseteq A$, $\mu(B) = \nu(B)$. If ν is nonatomic or ergodic, so is μ.*

In particular if E has an invariant (nonatomic, ergodic measure) and $E \sqsubseteq_B F$, then the same holds for F.

Note that E_0 admits an invariant, nonatomic, ergodic probability measure, namely the usual product measure on $2^{\mathbb{N}}$. Now the General Glimm-Effros Dichotomy, proved in [HKL] asserts that a Borel equivalence relation E is not smooth iff $E_0 \sqsubseteq_B E$. The special case of this for countable Borel equivalence relations was already proved in [E1], [E2] and [We1]. So we have the following characterization:

Theorem 5.4 ([E1], [E2], [We1]). *Let E be a countable Borel equivalence relation on a standard Borel space X. Then the following are equivalent:*

(i) E is not smooth;

(ii) There is a nonatomic, E-ergodic, E-invariant measure.

Corollary 5.5. *Every countable Borel equivalence relation on an uncountable standard Borel space admits a nonatomic invariant measure.*

5.3   Invariant probability measures and the theorem of Nadkarni

The following result characterizes the existence of E-invariant probability measures.

Theorem 5.6 ([N2]; see also [BK, 4.5])**.** *Let E be a countable Borel equivalence relation on a standard Borel space X. Then the following are equivalent:*
    (i) E is not compressible;
    (ii) There is an E-invariant probability measure;
    (iii) There is an E-ergodic, E-invariant probability measure.

A proof of this result can be also found in [Ke5, 4.G] and [Sl5, 2.3-2.8].

For example E_t, and eventual equality $E_0(\mathbb{N})$ on $\mathbb{N}^{\mathbb{N}}$, being compressible, do not admit an invariant probability measure and E_0, having an invariant probability measure, is not compressible.

Corollary 5.7. *Let E be a countable Borel equivalence relation on a standard Borel space X. Then for any Borel set $A \subseteq X$ the following are equivalent:*
    (i) $[A]_E$ is compressible;
    (ii) For any E-invariant probability measure μ, $\mu(A) = 0$.

It follows from Theorem 3.29 that for every aperiodic countable Borel equivalence relation E on a Polish space X, there is a comeager invariant Borel set C such that $E \upharpoonright C$ admits no invariant probability measure. For a related result involving stationary measures, see [CKM, Corollary 18].

Remark 5.8. When G is a unimodular Polish locally compact group, a is a free Borel action on a standard Borel space X and $Y \subseteq X$ is a complete lacunary cocompact Borel section, then there is a canonical correspondence between finite invariant measures for $E_a \upharpoonright Y$ and finite invariant measures for the action a, see [Sl1, Section 4] and [KPV, Section 4].

5.4   Ergodic invariant measures and the Ergodic Decomposition Theorem

For each standard Borel space X, denote by $P(X)$ the standard Borel space of probability measures on X, which is generated by the maps $\mu \mapsto \mu(A)$, for Borel sets A; see [Ke6, 17.23, 17.24]. For a countable Borel equivalence relation E, denote by INV_E the set of E-invariant probability measures. Also let EINV_E be the set of E-ergodic, E-invariant probability measures. The following is an important property of such measures:

Proposition 5.9. *Let E be a countable Borel equivalence relation on a standard Borel space X and let $A, B \subseteq X$ be Borel sets. Then there is $f \in [[E]]_B, f \colon C \to D$, such that $C \subseteq A, D \subseteq B$ and $\mu(A \setminus C) = 0$, $\forall \mu \in \mathrm{EINV}_E$ with $\mu(A) \leq \mu(B)$.*

A proof can be found, for example, in [KM1, 7.10].

The next results apply as well to Borel actions of Polish locally compact groups but we will restrict here attention to actions of countable groups or equivalently to countable Borel equivalence relations.

Theorem 5.10 ([Fa], [Va])**.** *Let E be a countable Borel equivalence relation on a standard Borel space X. Then*

(i) INV_E, EINV_E are Borel sets in $P(X)$ and EINV_E is the set of the extreme points of the convex set INV_E (under the usual operation of convex combination of probability measures);

(ii) $\mathrm{INV}_E \neq \emptyset \iff \mathrm{EINV}_E \neq \emptyset$.

The following result is known as the **Ergodic Decomposition Theorem** for invariant measures.

Theorem 5.11 ([Fa], [Va])**.** *Let E be a countable Borel equivalence relation on a standard Borel space X and assume that $\mathrm{INV}_E \neq \emptyset$. Then there is a Borel surjection $\pi \colon X \to \mathrm{EINV}_E$ such that*

(i) π is E-invariant;

(ii) If $X_e = \pi^{-1}(\{e\})$, for $e \in \mathrm{EINV}_E$, then $e(X_e) = 1$ and e is the unique E-invariant probability measure concentrating on X_e;

(iii) If $\mu \in \mathrm{INV}_E$, then $\mu = \int \pi(x) \, \mathrm{d}\mu(x) = \int e \, \mathrm{d}\pi_*\mu(e)$.

Moreover this map is unique in the following sense: If π, π' satisfy (i)-(iii), then for any $\mu \in \mathrm{INV}_E$, $\pi(x) = \pi'(x), \mu$-a.e. (x)(equivalently by Corollary 5.7, the set $\{x \colon \pi(x) \neq \pi'(x)\}$ is compressible).

Corollary 5.12. *Let E be a countable Borel equivalence relation on a standard Borel space X and let $\mu, \nu \in \mathrm{INV}_E$. Then $\mu = \nu$ iff for every E-invariant Borel set $A \subseteq X$, $\mu(A) = \nu(A)$.*

Proofs of Theorem 5.10 and Theorem 5.11 can be also found in [Ke5, 3.K], [Sl5, 2.9] and [Ch3].

In [Ch2] a connection is found between the ergodic decomposition of a countable Borel equivalence relation E on a standard Borel space X and the topological ergodic decomposition of continuous (in Polish topologies on X that generate its Borel structure) actions of countable groups G that generate E. Here the **topological ergodic decomposition** of an action of a group G on a topological space X is the equivalence relation on X, where two points of X are equivalent if the closures of their orbits coincide.

5.5 Quasi-invariant measures

If G is a countable group which acts in a Borel way on a standard Borel space X and μ is a measure on X, then μ is **quasi-invariant** under this action if for all $g \in G$, $g \cdot \mu \sim \mu$. Note that if μ is quasi-invariant under the action and $\nu \sim \mu$, then ν is also quasi-invariant under this action. Thus for every quasi-invariant measure there is an equivalent quasi-invariant probability measure.

We now have the following analog of Proposition 5.1.

Proposition 5.13. *Let E be a countable Borel equivalence relation on a standard Borel space X. Then the following are equivalent for each measure μ on X:*

(i) For some countable group G and Borel action a of G on X such that $E_a = E$, μ is quasi-invariant under this action;

(ii) For every countable group G and every Borel action a of G on X such that $E_a = E$, μ is quasi-invariant under this action;

(iii) For every $f : A \to B$ in $[[E]]_B$, $\mu(A) = 0 \iff \mu(B) = 0$;

(iv) For every $T \in [E]_B$, $T_*\mu \sim \mu$;

(v) For every Borel $A \subseteq X$, $\mu(A) = 0 \iff \mu([A]_E) = 0$.

Definition 5.14. *Let E be a countable Borel equivalence relation on a standard Borel space X and μ a measure on X. Then μ is called **E-quasi-invariant** if it satisfies the equivalent conditions of Proposition 5.13.*

The following two results show that studying the structure of countable Borel equivalence relations with respect to arbitrary measures can often be reduced to that of quasi-invariant measures.

Proposition 5.15. *Let E be a countable Borel equivalence relation on a standard Borel space X and μ a measure on X. Then there is an E-quasi-invariant measure $\bar{\mu}$ such that:*
   (i) $\mu \ll \bar{\mu}$ and if ν is E-quasi-invariant with $\mu \ll \nu$, then $\bar{\mu} \ll \nu$;
   (ii) For any Borel E-invariant set $A \subseteq X$, $\mu(A) = \bar{\mu}(A)$;
   (iii) If μ is nonatomic or ergodic, so is $\bar{\mu}$.

Proof. Let E be generated by a Borel action of a countable group G, let $G = \{g_n\}$ and put $\bar{\mu} = \sum_n 2^{-n-1}(g_n \cdot \mu)$. $\qquad\qquad\square$

Proposition 5.16 (Woodin, see [Mi5, page 191], [Mi14, 1.3]). *Let E be a countable Borel equivalence relation on a standard Borel space X and let μ be a probability measure on X. Then there is complete Borel section $Y \subseteq X$ with $\mu(Y) = 1$ such that if $A \subseteq Y$ is a Borel set with $\mu(A) = 0$, then $\mu([A]_E) = 0$. Therefore $\mu \upharpoonright Y$ is $E \upharpoonright Y$-quasi-invariant.*

The following is an analog of Proposition 5.3:

Proposition 5.17 (see [DJK, 3.3]). *Let E be a countable Borel equivalence relation on a standard Borel space X and A a complete Borel section for E. Let ν be a probability measure on A such that ν is $E \upharpoonright A$-quasi-invariant. Then there is an E-quasi-invariant probability measure μ on X such that for all Borel sets $B \subseteq A$, $\mu(B) = \mu(A)\nu(B)$. If ν is nonatomic or ergodic, so is μ.*

And the following is an analog of Theorem 5.4:

Theorem 5.18 ([E1], [E2], [We1]). *Let E be a countable Borel equivalence relation on a standard Borel space X. Then the following are equivalent:*
   (i) E is not smooth;
   (ii) There is a nonatomic, E-ergodic, E-quasi-invariant measure.

The following classical results of Hopf and Hajian-Kakutani characterize the existence of an invariant probability measure equivalent to a given quasi-invariant measure (compare with Theorem 5.6). Below if a countable group G acts in a Borel way on a standard Borel space X, a **weakly wandering** Borel set for this action is a Borel set $A \subseteq X$ such that for some sequence (g_n) of elements of G, we have $g_n \cdot A \cap g_m \cdot A = \emptyset, \forall m \neq n$.

Theorem 5.19 (Hopf, see [N3, Section 10]; Hajian-Kakutani, see [HaK]).
*Let E be a countable Borel equivalence relation on a standard Borel space X and
μ an E-quasi-invariant measure. Then the following are equivalent:*

(i) There is an E-invariant probability measure ν such that $\mu \sim \nu$;

(ii) (Hopf) *There is no Borel set A with $\mu(A) > 0$ such that $E \restriction A$ is compressible;*

(iii) (Hajian-Kakutani) *Let G be a countable group and let a be a Borel action
of G on X such that $E_a = E$. Then there is no weakly wandering set of μ-positive
measure for this action.*

Other related characterizations can be found in [Ke5, 2.83-2.85].

5.6   The space of quasi-invariant measures

Let E be a countable Borel equivalence relation on a standard Borel space
X. We denote by QINV_E, EQINV_E, ERG_E, the spaces of E-quasi-invariant,
E-ergodic and E-quasi-invariant, E-ergodic probability measures on X,
resp. The following is a special case of a more general result concerning
Borel actions of Polish locally compact groups proved in [Di].

Theorem 5.20 ([Di]). *Let E be a countable Borel equivalence relation on a standard Borel space X. Then QINV_E, EQINV_E, ERG_E are Borel sets in $P(X)$.*

The set EQINV_E is invariant under measure equivalence $\sim$, which is a
Borel equivalence relation, and [DJK, 4.1] shows that if E is not smooth,
measure equivalence on EQINV_E (even restricted to nonatomic measures)
is not smooth.

The structure of QINV_E and EQINV_E under absolute continuity depends only on the bireducibility type of E.

Proposition 5.21 ([DJK, 4.2]). *Let E, F be countable Borel equivalence relations
on standard Borel spaces. If $E \sim_B F$, then there are Borel maps $\varphi \colon \mathrm{QINV}_E \to
\mathrm{QINV}_F$ and $\psi \colon \mathrm{QINV}_F \to \mathrm{QINV}_E$ such that*

(i) $\psi(\varphi(\mu)) \sim \mu$ and $\varphi(\psi(\nu)) \sim \nu$;

(ii) $\mu \ll \nu \iff \varphi(\mu) \ll \varphi(\nu)$;

(iii) φ, ψ map ergodic measures to ergodic measures.

We also have the following related fact:

Proposition 5.22 ([Ke5, 4.34])**.** *Let E, F be countable Borel equivalence relations on standard Borel spaces such that $E \sqsubseteq_B F$. Then there is a Borel injection $\varphi \colon \mathrm{QINV}_E \to \mathrm{QINV}_F$ such that $\mu \ll \nu \iff \varphi(\mu) \ll \varphi(\nu)$ and φ maps ergodic measures to ergodic measures.*

The paper [Ke12] studies the descriptive complexity of the Borel sets $\mathrm{QINV}_\Gamma = \mathrm{QINV}_E$, $\mathrm{EQINV}_\Gamma = \mathrm{EQINV}_E$, where E is the equivalence relation induced by the shift action of a countable group Γ on $X = (2^{\mathbb{N}})^\Gamma$. The following is shown, where the sets below belong to the space $P(X)$, which is given the usual compact metrizable topology (see, e.g., [Ke6, Section 17.E]) that generates its standard Borel structure. Also $\mathbb{F}_\infty$ is the free group with a countably infinite set of generators.

Theorem 5.23 ([Ke12])**.** *(i) For each infinite countable group Γ, the set QINV_Γ is $\mathbf{\Pi}_3^0$-complete and EQINV_Γ is $\mathbf{\Pi}_3^0$-hard.*
   (ii) The set $\mathrm{EQINV}_\mathbb{Z}$ is $\mathbf{\Pi}_3^0$-complete.
   (iii) There is a countable ordinal $3 \le \alpha_\infty \le \omega + 2$ such that $\mathrm{EQINV}_{\mathbb{F}_\infty}$ is $\mathbf{\Pi}_{\alpha_\infty}^0$-complete.

The exact value of α_∞ is unknown.

5.7   The cocycle of a quasi-invariant measure

Let E be a countable Borel equivalence relation on a standard Borel space X and let μ be a probability measure. We define two measures M_l, M_r on the space E (viewed as a Borel subset of X^2) as follows:

$$M_l(A) = \int |A_x| \, \mathrm{d}\mu(x),$$

where $A \subseteq E$ is Borel, $A_x = \{y \in X : (x, y) \in A\}$, and $|B|$ is the cardinality of B, which is equal to ∞ if B is infinite. For any nonnegative real-valued Borel φ,

$$\int \varphi(x, y) \, \mathrm{d}M_l(x, y) = \int \sum_{y \in [x]_E} \varphi(x, y) \mathrm{d}\mu(x).$$

Let also

$$M_r(A) = \int |A^y| \, \mathrm{d}\mu(y),$$

where $A^y = \{x \in X : (x, y) \in A\}$. Note that for $f \colon A \to B$ in $[[E]]_B$,

$$M_l(\mathrm{graph}(f)) = \mu(A), \, M_r(\mathrm{graph}(f)) = \mu(B),$$

thus clearly M_l, M_r are σ-finite. Moreover, μ is E-quasi-invariant iff $M_l \sim M_r$ and μ is E-invariant iff $M_r = M_l$.

Assume now that μ is E-quasi-invariant. Consider then the Radon-Nikodym derivative,

$$\rho_\mu(x, y) = (dM_l/dM_r)(x, y),$$

for $(x, y) \in E$. Therefore ρ_μ is a Borel map from E to $\mathbb{R}^+$ such that for any nonnegative real-valued $\varphi \colon E \to \mathbb{R}$, we have

$$\int \varphi(x, y) \, \mathrm{d}M_l(x, y) = \int \varphi(x, y)\rho_\mu(x, y) \, \mathrm{d}M_r(x, y),$$

thus for every Borel set $A \subseteq E$:

$$M_l(A) = \int_A \rho_\mu(x, y) \, \mathrm{d}M_r(x, y),$$

and ρ_μ is uniquely determined M_r-a.e. by this property. Also $\rho_\mu^{-1} = \mathrm{d}M_r/\mathrm{d}M_l$, M_l-a.e.

For any $f \colon A \to B$ in $[[E]]_B$ and Borel $C \subseteq B$, we have

$$\mu(f^{-1}(C)) = \int_C \rho_\mu(f^{-1}(y), y) \, \mathrm{d}\mu(y),$$

thus if $T \in [E]_B$, then $(\mathrm{d}T_*\mu/\mathrm{d}\mu)(x) = \rho_\mu(T^{-1}(x), x)$, μ-a.e. (x).

A map $\rho : E \to G$, where G is a group, is called a **cocycle** if it satisfies the **cocycle identity**

$$\rho(x, z) = \rho(y, z)\rho(x, y),$$

for $xEyEz$. If this cocycle identity holds only on an E-invariant Borel set A with $\mu(A) = 1$, then ρ is a **cocycle a.e.** We now have that $\rho_\mu : E \to \mathbb{R}^+$ (the multiplicative group of positive reals) is a cocycle a.e. We thus call ρ_μ the **cocycle associated** with the E-quasi-invariant measure μ.

Proofs of the facts mentioned here can be found in [KM1, Section 8].

5.8   Existence of quasi-invariant probability measures with a given cocycle

Again generically there are no quasi-invariant probability measures with a given cocycle (compare with the paragraph following Corollary 5.7). Below for a countable Borel equivalence relation E on a standard Borel space X and $\rho\colon E \to \mathbb{R}^+$ a Borel cocycle, we say that E is $\boldsymbol{\rho}$**-aperiodic** if for every $x \in X$, $\sum_{y \in [x]_E} \rho(y, x) = \infty$. Also we say that a probability measure μ on X is $\boldsymbol{\rho}$**-invariant** if it is E-quasi-invariant and $\rho_\mu(x, y) = \rho(x, y)$, for all xEy in an E-invariant Borel set A with $\mu(A) = 1$. For more information about such measures, see [Mi2, Section 18] and [Mi14].

Theorem 5.24 ([KM1, 13.1]). *Let E be a countable Borel equivalence relation on a Polish space X and let $\rho\colon E \to \mathbb{R}^+$ be a Borel cocycle such that E is ρ-aperiodic. Then there is an E-invariant comeager Borel set $C \subseteq X$ such that $\mu(C) = 0$, for any ρ-invariant probability measure μ.*

In the papers [Mi2, Section 20], [Mi9], [Mi12], [Mi14] Miller obtains analogs of Nadkarni's Theorem 5.6 by characterizing when, for a given countable Borel equivalence relation E on a standard Borel space X and Borel cocycle $\rho\colon E \to \mathbb{R}^+$, there exits a ρ-invariant probability measure. Analogous results for measures (as opposed to probability measures) were obtained in [Mi2, Section 18], [Mi8], [Mi14].

5.9   An ergodic decomposition theorem for quasi-invariant measures with a given cocycle

The following is a generalization of the Ergodic Decomposition Theorem Theorem 5.11 to measures having a given cocycle (Theorem 5.11 is the special case of the constant value 1 cocycle).

Let E be a countable Borel equivalence relation on a standard Borel space X. For each Borel cocycle $\rho\colon E \to \mathbb{R}^+$, let INV_ρ, resp., EINV_ρ, be the spaces of ρ-invariant, resp. E-ergodic, ρ-invariant, probability measures on X.

Theorem 5.25 ([Di]). *Let E be a countable Borel equivalence relation on a standard Borel space X and $\rho\colon E \to \mathbb{R}^+$ a Borel cocycle. Then*

(a) INV_ρ, EINV_ρ *are Borel sets in* $P(X)$ *and* EINV_ρ *is the set of the extreme points of the convex set* INV_ρ *(under the usual operation of convex combination of probability measures);*

(b) $\text{INV}_\rho \neq \emptyset \iff \text{EINV}_\rho \neq \emptyset$.
Moreover there is a Borel surjection $\pi\colon X \to \text{EINV}_\rho$ *such that*

(i) π *is E-invariant;*

(ii) If $X_e = \pi^{-1}(\{e\})$, *for* $e \in \text{EINV}_\rho$, *then* $e(X_e) = 1$ *and* e *is the unique* ρ*-invariant measure concentrating on* X_e;

(iii) If $\mu \in \text{INV}_\rho$, *then* $\mu = \int \pi(x) \, \mathrm{d}\mu(x) = \int e \, \mathrm{d}\pi_*\mu(e)$.

Moreover this map is unique in the following sense: If π, π' *satisfy (i)-(iii), then for any* $\mu \in \text{INV}_\rho$, $\pi(x) = \pi'(x)$, μ*-a.e.* (x).

Another proof of this result is given in [Mi9, 5.2].

5.10   An ergodic decomposition theorem with respect to an arbitrary probability measure

The following result is a special case of a more general theorem concerning analytic equivalence relations, see [Ke5, 3.J] and [LM, 3.2].

Proposition 5.26 (Kechris). *Let* E *be a countable Borel equivalence relation on a standard Borel space* X *and let* $\mu \in P(X)$. *Then there is a Borel E-invariant map* $\pi\colon X \to P(X)$ *such that letting* $\pi(x) = \mu_x$, *we have:*

(i) μ_x *is E-ergodic;*

(ii) $\{y \in X \colon \mu_y = \mu_x\}$ *has* μ_x*-measure 1;*

(iii) $\mu = \int \mu_x \, \mathrm{d}\mu(x)$.

5.11   Measures agreeing on invariant sets

Recall form Corollary 5.12 that if E is a countable Borel equivalence relation and $\mu, \nu \in \text{INV}_E$, then μ, ν agree on the E-invariant Borel sets iff $\mu = \nu$. In fact it turns out that one can characterize exactly when two *arbitrary* probability measures agree on the E-invariant Borel sets. The following result is due to [Th, Theorem 1], where it is proved more generally for equivalence relations induced by Borel actions of Polish locally compact groups. See also [Kh, Section 3.4].

Theorem 5.27 ([Th, Theorem 1]). *Let a be a Borel action of a countable group G on a standard Borel space X and put $a(g, x) = g \cdot x$ and $E = E_a$. Let μ, ν be two probability measures on X. Then the following are equivalent:*

(i) For each E-invariant Borel set $A \subseteq X$, $\mu(A) = \nu(A)$;

(ii) There is a probability measure ρ on E such that $s_*\rho = \mu, t_*\rho = \nu$, where $s(x, y) = x, t(x, y) = y$.

(iii) There is a probability measure σ on $G \times X$ such that $u_*\sigma = \mu, v_*\sigma = \nu$, where $u(g, x) = x, v(g, x) = g \cdot x$;

(iv) There is a Borel map $x \mapsto \mu_x$ from X to $P(G)$ such that for every Borel set $A \subseteq X$, $\nu(A) = \int \mu_x(\{g \in G \colon g \cdot x \in A\}) \, \mathrm{d}\mu(x)$.

(v) There is a map $g \mapsto \mu_g$ from G to the set of measures on X such that $\mu = \sum_g \mu_g$ and $\nu = \sum_g g \cdot \mu_g$.

In [Sh], statement (v) of Theorem 5.27 is generalized to the context of cardinal algebras, leading to an algebraic proof of this statement.

6. Smoothness, E_0 and E_∞

We will now start studying the hierarchical order $\leq_B$ of Borel reducibility on countable Borel equivalence relations.

6.1 Smoothness

The simplest Borel equivalence relations are the smooth ones and they are easy to classify up to Borel bireducibility. Below for $n \geq 1$, Δ_n is the equality relation on a set of cardinality n.

First we recall the following dichotomy result of Silver:

Theorem 6.1 ([Si]). *Let E be a Borel (or even Π^1_1) equivalence relation on a standard Borel space X. Then exactly one of the following holds:*
(i) There are only countably many E-classes;
(ii) $\Delta_\mathbb{R} \sqsubseteq_B E$.

We now have as an immediate consequence:

Corollary 6.2. *If E is a smooth Borel equivalence relation, then $E \sim_B \Delta_n$, for some $n \geq 1$, $E \sim_B \Delta_\mathbb{N}$ or $E \sim_B \Delta_\mathbb{R}$.*

Moreover the smooth countable Borel equivalence relations form an initial segment in $\leq_B$.

Corollary 6.3. *We have that*

$$\Delta_1 <_B \Delta_2 <_B \cdots <_B \Delta_n <_B \cdots <_B \Delta_\mathbb{N} <_B \Delta_\mathbb{R}$$

and every Borel equivalence E is either Borel bireducible to an equivalence relation in that list or else $\Delta_\mathbb{R} <_B E$.

Remark 6.4. Given a pair $E_1 \subseteq E_2$ of countable Borel equivalence relations on a standard Borel space X and a pair $F_1 \subseteq F_2$ of countable Borel equivalence relations on a standard Borel space Y, we say that (E_1, E_2) is **simultaneously Borel reducible** to (F_1, F_2) if there is a Borel function $f \colon X \to Y$ such that $f \colon E_1 \leq_B F_1, f \colon E_2 \leq_B F_2$. The paper [Sc1] contains a classification of pairs $E_1 \subseteq E_2$ of smooth countable Borel equivalence relations up to simultaneous Borel bireducibility.

6.2   The simplest non-smooth relation

There is a least, in the sense of Borel reducibility, Borel equivalence relation $>_B \Delta_\mathbb{R}$.

Theorem 6.5 (The General Glimm-Effros Dichotomy, [HKL])**.** *Let E be a Borel equivalence relation on a standard Borel space. Then exactly one of the following holds:*

(i) E is smooth;

(ii) $E_0 \sqsubseteq_B E$.

We note that for the case of a countable Borel equivalence relation E this result is already included in [E1], [E2] and [We1].

We can thus extend the initial segment given in Corollary 6.3.

Theorem 6.6. *We have that*

$$\Delta_1 <_B \Delta_2 <_B \cdots <_B \Delta_n <_B \cdots <_B \Delta_\mathbb{N} <_B \Delta_\mathbb{R} <_B E_0$$

and every Borel equivalence E is either Borel bireducible to an equivalence relation in that list or else $E_0 <_B E$.

The countable Borel equivalence relations which are $\leq_B E_0$ are exactly the hyperfinite ones and will be studied in detail in Chapter 8.

6.3   The most complicated relation

At the other end of the spectrum there is a most complicated, in terms of Borel reducibility, countable Borel equivalence relation.

For each countable group G and standard Borel space X, denote by $s_{G,X}$ the **shift action** of G on the space X^G:

$$(g \cdot p)_h = p_{g^{-1}h},$$

for $p \in X^G$, $g, h \in G$. Let $E(G, X) = E_{s_{G,X}}$ be the associated equivalence relation. Below let $\mathbb{F}_n$, $n = 1, 2, \ldots$, be the **free group** with n generators. We also let $\mathbb{F}_\infty$ be the free group with a countably infinite set of generators.

If a is a Borel action of a countable group G on a standard Borel space X, which we can assume is a Borel subset of $\mathbb{R}$, the map

$$x \in X \mapsto (g^{-1} \cdot x)_{g \in G} \in \mathbb{R}^G$$

is a Borel embedding of the action a to the shift action $s_{G,\mathbb{R}}$, so in particular $E_a \sqsubseteq_B^i E(G, \mathbb{R})$.

Definition 6.7. *A countable Borel equivalence relation E is **invariantly universal** if for every countable Borel equivalence relation F, $F \sqsubseteq_B^i E$.*

Since by Theorem 3.3 every countable Borel equivalence relation is induced by a Borel action of $\mathbb{F}_\infty$, it follows that the equivalence relation $E(\mathbb{F}_\infty, \mathbb{R})$ is invariantly universal. Clearly there is a unique, up to Borel isomorphism, invariantly universal countable Borel equivalence relation and it will be denoted by E_∞.

Definition 6.8. *We say that a countable Borel equivalence relation E is **universal** if for any countable Borel equivalence relation F, we have $F \leq_B E$, i.e., $E \sim_B E_\infty$.*

Another example of a universal countable Borel equivalence relation is the following:

Proposition 6.9 ([DJK, 1.8]). $E(\mathbb{F}_2, 2) \sim_B E_\infty$

In [T20] S. Thomas studies what he calls E_0-extensions, i.e., countable Borel equivalence relations of the form $E(a)$, where E, on some space X, is Borel isomorphic to E_0 and $a \colon G \curvearrowright_B (X, E)$. Up to Borel isomorphism these are exactly the countable Borel equivalence relations that are normal over E_0. For consistency with our terminology, we will call these $\boldsymbol{E_0}$-**expansions**. He shows in [T15, Theorem 1.2] that for $\mathbb{F}_2$, and in fact any

countable group G containing $\mathbb{F}_2$, the following E_0-expansion is universal: Let E be the analog of E_0 on the space 2^G (i.e., $pEq \iff \{g\colon p(g) \neq q(g)\}$ is finite), so that $E \cong_B E_0$. Let $s = s_{G,2}$ be the shift action of G. Clearly this action is by automorphisms of E. Then the E_0-expansion $E(s)$ is universal.

We will study universal countable Borel equivalence relations in Chapter 12.

6.4   Intermediate relations

The interval $[E_0, E_\infty]$ in the Borel reducibility order $\leq_B$ is not trivial, as one can prove, for example, using the results in [A1] and [SlSt]. In fact we have the following, where by $F(G, X)$ we denote the restriction of $E(G, X)$ to the (invariant Borel) **free part** $F(X^G) = \{p \in X^G\colon \forall g \neq 1_G(g \cdot p \neq p)\}$ of the shift action $s_{G,X}$.

Theorem 6.10 ([JKL, Section 3]). $E_0 <_B F(\mathbb{F}_2, 2) <_B E_\infty$

The relation $F(\mathbb{F}_2, 2)$ is an example of a treeable countable Borel equivalence relation, a concept that we will study in detail in Chapter 10.

We call countable Borel equivalence relations E such that $E_0 <_B E <_B E_\infty$ **intermediate**. In the next Chapter 7 we will see that they contain many interesting examples and have a very rich structure.

It should be pointed out that all known proofs of existence of intermediate countable Borel equivalence relations use measure theoretic methods of ergodic theory. We will see in Chapter 8 that generically, in the sense of Baire category, all countable Borel equivalence relations are $\leq_B E_0$.

7. Rigidity and incomparability

7.1 The complex structure of Borel reducibility

By the early 1990's a small finite number of intermediate countable Borel equivalence relations were known and they were linearly ordered under $\leq_B$. This lead to the following basic problems: Are there infinitely many, up to Borel bireducibility? Does non-linearity occur here?

These problems were resolved in [AK], where it was shown that the structure of countable Borel equivalence relations under Borel reducibility is quite rich.

Theorem 7.1 ([AK]). *The partial order of Borel sets under inclusion can be embedded in the quasi-order of Borel reducibility of countable Borel equivalence relations, i.e., there is a map $A \mapsto E_A$ from the Borel subsets of $\mathbb{R}$ to countable Borel equivalence relations such that $A \subseteq B \iff E_A \leq_B E_B$.*

In particular it follows that any Borel partial order can be embedded in the quasi-order of Borel reducibility of countable Borel equivalence relations, Under the **Continuum Hypothesis (CH)**, any partial order of the size of the continuum can be embedded in the partial order of inclusion of subsets of $\mathbb{N}$ modulo finite sets. It follows that for every quasi-order $\leq$ on a set X of the size of the continuum, there is a map $x \mapsto E_x$ from X to countable Borel equivalence relations such that $x \leq y \iff E_x \leq_B E_y$, i.e., $\leq_B$ on countable Borel equivalence relations is a universal quasi-order of the size of the continuum.

Other proofs of Theorem 7.1 can be found in [HK4], [CM2] and [T12].

Another indication of the complexity of $\leq_B$ on countable Borel equivalence relations is contained in the next result. To formulate it, fix a coding (parametrization) of countable Borel equivalence relation by reals. This consists of a $\mathbf{\Pi}_1^1$ subset C of $\mathbb{R}$ and a surjective map $c \mapsto E_c$ from C to the

set of countable Borel equivalence relations on $\mathbb{R}$, satisfying some natural definability conditions; see [AK, Section 5]. Let

$$C_{\leq} = \{(c,d) \in C^2 : E_c \leq_B E_d\}, C_{\sim} = \{(c,d) \in C^2 : E_c \sim_B E_d\}.$$

Theorem 7.2 ([AK]). *The sets $C_{\leq}, C_{\sim}$ are $\mathbf{\Sigma_2^1}$-complete.*

Concerning the equivalence relation $C_{\sim}$, it follows from Theorem 7.1 that every Borel equivalence relation can be Borel reduced to $C_{\sim}$ and in [G3] this was extended to Σ_1^1 equivalence relations. It appears to be unknown if it also holds for Π_1^1 equivalence relations.

The proof of Theorem 7.1 used methods of ergodic theory, more precisely Zimmer's cocycle superrigidity theory for ergodic actions of linear algebraic groups and their lattices.

The key point is that there is a phenomenon of set theoretic rigidity analogous to the measure theoretic rigidity phenomena discovered by Zimmer; see [Z2]. Informally this can be described as follows:

- **(Measure theoretic rigidity)** Under certain circumstances, when a countable group acts preserving a probability measure, the equivalence relation associated with the action together with the measure "encode" or "remember" significant information about the group (and the action).

- **(Borel theoretic rigidity)** Such information is simply encoded in the Borel cardinality of the (quotient) orbit space.

7.2   Cocycle reduction

A basic idea in establishing rigidity results in the measure theoretic, as well as in the descriptive context, is cocycle reduction, which will discuss next in the framework of Borel reducibility.

Let E be a countable Borel equivalence relation on a standard Borel space X and Γ a countable group. Two Borel cocycles $\alpha\colon E \to \Gamma, \beta\colon E \to \Gamma$ are **cohomologous** or **equivalent**, in symbols $\alpha \sim \beta$, if there is a Borel map $f\colon X \to \Gamma$ such that $xEy \implies \beta(x,y) = f(y)\alpha(x,y)f(x)^{-1}$. If G is a countable group acting in a Borel way on a standard Borel space X, a Borel cocycle of this action into Γ is a Borel map $\alpha\colon G \times X \to \Gamma$ such that

$\alpha(g_1 g_2, x) = \alpha(g_1, g_2 \cdot x)\alpha(g_2, x)$, for $g_1, g_2 \in G$. Two such cocycles α, β are equivalent, in symbols $\alpha \sim \beta$, if there is a Borel map $f \colon X \to \Gamma$ such that $\beta(g, x) = f(g \cdot x)\alpha(g, x)f(x)^{-1}$. Note that if α is a cocycle of the associated to this action equivalence relation E, then any cocycle α of E into Γ gives a cocycle β of the action, namely $\beta(g, x) = \alpha(x, g \cdot x)$ Conversely if the action is free, any cocycle of the action gives rise as above to a cocycle of the equivalence relation.

A simple example of a cocycle of an action of G on X is a homomorphism $h \colon G \to \Gamma$, which can be identified with the cocycle $\alpha(g, x) = h(g)$.

Now let F be a countable Borel equivalence on a standard Borel space Y, which is induced by a *free* Borel action b of a countable group Γ on Y, i.e., $F = E_b$. Let $\varphi \colon E \to_B F$ be a Borel homomorphism. Then there is a canonical Borel cocycle $\alpha \colon E \to \Gamma$ associated to φ, namely $\alpha(x, y) = \gamma$, where γ is the unique element of Γ such that $\gamma \cdot \varphi(x) = \varphi(y)$, i.e., $xEy \implies \varphi(y) = \alpha(x, y) \cdot \varphi(x)$. Note now that if $a \sim \beta$, via the Borel function $f \colon X \to \Gamma$, then β is also associated to a Borel homomorphism $\psi \colon E \to_B F$, such that $\varphi(x)F\psi(x)$ for every $x \in X$, namely $\psi(x) = f(x) \cdot \varphi(x)$. Similarly, if $E = E_a$ for a Borel action a of a countable group G on X, we have an associated to φ cocycle of the action, given by $\varphi(g \cdot x) = \alpha(g, x) \cdot \varphi(x)$.

A cocycle reduction result, for some given action or equivalence relation, shows that certain cocycles of the action or equivalence relation into certain groups are equivalent to ones that are much simpler in some sense, e.g., are group homomorphisms or have a "small range". When such a cocycle result is applied to the cocycle coming from a homomorphism of equivalence relations as above, in particular to a reduction, it can be used to replace the given homomorphism with another one that has additional structure.

For example, let a be a Borel action of a countable group G on a standard Borel space X and b a free Borel action of a countable group Γ on a standard Borel space Y. Put $E = E_a, F = E_b$. Let $\varphi \colon E \to_B F$ be a Borel homomorphism and let $\alpha \colon G \times X \to \Gamma$ be the associated to φ cocycle of the action a as above. If α is equivalent to a homomorphism $h \colon G \to \Gamma$, let ψ be the associated to the cocycle h as above homomorphism of E to F. Then we have that $\psi(g \cdot x) = h(g) \cdot \psi(x)$, which is a very strong property, that can be ruled out in a given situation, thereby ruling out the existence of the original homomorphism φ. This therefore gives a basic technique for showing that an equivalence relation cannot be reduced to another one.

In practice such cocycle reduction results are actually established in

a measure theoretic context, i.e., in ergodic theory. Suppose we have a countable Borel equivalence relation E on a standard Borel space X with an invariant probability measure μ. Then we can define the above notions of cocycle for E, μ by requiring that the cocycle identity holds only on an E-invariant Borel set of μ-measure 1, i.e., we consider cocycles a.e.. Moreover we identify two such cocycles if they agree μ-a.e. Analogously we define equivalence of cocycles, neglecting again μ-null sets. We can also similarly define cocycles of group actions with an invariant measure.

To illustrate these ideas let us mention a cocycle reduction result due to Popa [Po], which is usually referred to as **cocycle superrigidity**.

Theorem 7.3 ([Po]). *Let G be an infinite countable group with property* (T) *and consider the shift action $s = s_{G,[0,1]}$ of G on $[0,1]^G$ with the invariant product measure λ^G, where λ is Lebesgue measure on $[0,1]$. Then for any countable group Γ, any Borel cocycle of this action into Γ is equivalent, a.e., to a homomorphism of G into Γ.*

Such cocycle reduction results are used to prove measure theoretic rigidity results in the sense of Section 7.1. For example, here is an application of Theorem 7.3. Below if a is a Borel action of a countable group G on a standard Borel space X with invariant probability measure μ and if b is a Borel action of a countable group Γ on a standard Borel space Y with invariant probability measure ν, we say that a, b are **orbit equivalent** if there are invariant Borel sets $X_0 \subseteq X, Y_0 \subseteq Y$ with $\mu(X_0) = 1, \nu(Y_0) = 1$ and a measure preserving Borel bijection $T \colon X_0 \to Y_0$, such that $T \colon E_a | X_0 \cong_B E_b | Y_0$. We now have:

Theorem 7.4 ([Po]). *Let G be a simple infinite countable group with property* (T) *and let $s = s_{G,[0,1]}$ be its shift action on $X = [0,1]^G$ with the product measure $\mu = \lambda^G$. Let Γ be a countable group and let a be a free Borel action of Γ on a standard Borel space Y with invariant probability measure ν. If s, a are orbit equivalent, then there is an isomorphism $h \colon G \to \Gamma$ and a Borel isomorphism $T \colon (X, \mu) \to (Y, \nu)$ such that $T(g \cdot x) = h(g) \cdot T(x)$, μ-a.e. (x).*

Thus in Theorem 7.4, the equivalence relation E_a and the measure μ determine completely the group and the action.

In the Borel context, a method that has been frequently employed to solve a problem of showing that one countable Borel equivalence relation cannot be Borel reducible to another, ultimately comes down to an application of a cocycle reduction theorem in a measure theoretic context, usually

after considerable technical work that often employs sophisticated methods of ergodic theory or other subjects depending on the context. Such a technique has been used in the proof of Theorem 7.1 and also in all the results that will be mentioned below in this section.

Other methods, concerning the problem of showing failure of Borel reducibility, were used, e.g., in [HK4, Chapters 6,7 and Appendices B3, B4] (based on earlier work of Furstenberg, Zimmer and Adams), Hjorth [H3], [H12], [Ke9], [ET], [Mi11], [CM2] and [CM3].

In the rest of this section, we will discuss concrete natural instances of Borel theoretic rigidity phenomena that allow us to distinguish up to Borel bireducibility countable Borel equivalence relations or show that one cannot Borel reducible to another.

7.3 Actions of linear groups

The proof of Theorem 7.1 was based on the following result, which uses cocycle reduction results of Zimmer; see [Z2] and references contained therein.

Definition 7.5. *If E is a Borel equivalence relation on a standard Borel space X, μ is a probability measure on X and F is a Borel equivalence relation on a standard Borel space Y, we say that E is $\boldsymbol{\mu}$, $\boldsymbol{F}$-ergodic if for any $f\colon E \to_B F$, there is a Borel E-invariant set $A \subseteq X$, with $\mu(A) = 1$, such that f maps A into a single F-class.*

Theorem 7.6 ([AK, 4.5]). *Let for each nonempty set of primes S, let $G_S = \mathrm{SO}_7(\mathbb{Z}[S^{-1}])$ be the group of 7×7 orthogonal matrices with determinant 1 with coefficients in the ring $\mathbb{Z}[S^{-1}]$ of rationals whose denominators, in reduced form, have prime factors in S. Then*

$$S \not\subseteq T \implies F(G_S, 2) \text{ is } \mu_S, F(G_T, 2)-\text{ergodic},$$

where μ_S is the usual product measure on 2^{G_S}.
In particular,

$$S \subseteq T \iff F(G_S, 2) \leq_B F(G_T, 2).$$

Another set theoretic rigidity result proved in [AK] is the following:

Theorem 7.7 ([AK, Section 7, (i)]). *Consider the canonical action of* $\mathrm{GL}_n(\mathbb{Z})$ *on* $\mathbb{R}^n/\mathbb{Z}^n$ *and let* G_n *be the associated countable Borel equivalence relation. Then*

$$m < n \iff G_m <_B G_n.$$

In particular, the Borel cardinality of the orbit space of this action "encodes" the dimension n.

Fix an integer $n > 1$. For each nonempty set of primes S consider the compact group $H_S^n = \prod_{p \in S} \mathrm{SL}_n(\mathbb{Z}_p)$. The group $\mathrm{SL}_n(\mathbb{Z})$ can be viewed as a (dense) subgroup of H_S via the diagonal embedding. Denote by E_S^n the equivalence relation on H_S^n induced by the translation action of $\mathrm{SL}_n(\mathbb{Z})$ on H_S^n. Then we have:

Theorem 7.8 ([T4], for $n \geq 3$; [I2] for $n = 2$).

$$S = T \iff E_S^n \leq_B E_T^n.$$

Next for each prime p and $n \geq 2$, consider the projective space $\mathrm{PG}(n - 1, \mathbb{Q}_p)$ over the field $\mathbb{Q}_p$ of p-adic numbers, i.e., the space of 1-dimensional vector subspaces of the n-dimensional vector space $\mathbb{Q}_p^n$. Then the group $\mathrm{GL}_n(\mathbb{Z})$ acts in the usual way on $\mathrm{PG}(n-1, \mathbb{Q}_p)$. Denote by F_p^n the associated countable Borel equivalence relation. Then we have:

Theorem 7.9 ([T4], for $n \geq 3$; [I2] for $n = 2$).

$$p = q \iff F_p^n \leq_B F_q^n.$$

In [I2] further such set theoretic rigidity results are proved for similar actions of non-amenable subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

For a finite set S of primes, a prime number p and $n \geq 2$, denote by $F_{p,S}^n$ the countable Borel equivalence relation induced by the action of $\mathrm{SL}_n(\mathbb{Z}[S^{-1}])$ on $\mathrm{PG}(n-1, \mathbb{Q}_p)$. Also for a set of primes J such that $S \cap J = \emptyset$, view $\mathrm{SL}_n(\mathbb{Z}[S^{-1}])$ as a subgroup of $\prod_{p \in J} \mathrm{SL}_n(\mathbb{Z}_p)$ and let $F_{S,J}^n$ be the associated equivalence relation induced by the translation action. Then we have:

Theorem 7.10 ([T6],[T4]). *Let* $n \geq 2$.

(i) If p, q *are primes and* S, T *are nonempty finite sets of primes with* $p \notin S, q \notin T$*, then*
$$(p, S) = (q, T) \iff F_{p,S}^n \leq_B F_{q,T}^n;$$

(ii) If S, T are finite nonempty sets of primes and J, K are nonempty sets of primes with $S \cap J = \emptyset, T \cap K = \emptyset$, then

$$(S, J) = (T, K) \iff F_{S,J}^n \leq_B F_{T,K}^n.$$

For $n = 2$ this result is contained in [T6], while for $n \geq 3$ it is implicit in [T4].

For other set theoretic rigidity results for actions of linear groups as above, see also [T1], [Co2], [Co4], [I2], [C].

7.4   Actions of product groups

There are also set theoretic rigidity results concerning actions of product groups.

Theorem 7.11 ([HK4, Theorem 1]). *For each nonempty set S of odd primes, let $H_S = (\star_{p \in S}(\mathbb{Z}/p\mathbb{Z} \star \mathbb{Z}/p\mathbb{Z})) \times \mathbb{Z}$. Then*

$$S \nsubseteq T \implies F(H_S, 2) \text{ is } \mu_S, F(H_T, 2)-\text{ergodic},$$

where μ_S is the usual product measure on 2^{H_S}.
  In particular,

$$S \subseteq T \iff F(H_S, 2) \leq_B F(H_T, 2).$$

Among several other set theoretic (and measure theoretic) rigidity results proved in [HK4], we state the following.

Theorem 7.12 ([HK4, Theorem 7]). *Let for $n \geq 1$, $S_n = F(\mathbb{F}_2^n, 2)$ and $R_n = F(\mathbb{F}_2, 2)^n$ (thus $R_1 = S_1$). Then*

$$R_1 <_B R_2 <_B \cdots <_B R_n <_B \cdots , S_1 <_B S_2 <_B \cdots <_B S_n <_B \cdots ,$$

and for each n,
$$R_n \leq S_n$$

but for $2 \leq n < m$,
$$S_n \nleq_B R_m, R_m \nleq_B S_n.$$

7.5   Torsion-free abelian groups of finite rank

We next discuss rigidity results in algebra. The classification problem for torsion-free abelian groups of finite rank is a classical problem in group theory. For rank 1 the problem was solved by Baer in 1937 but no reasonable classification has been found for rank at least 2; see [T3], [T5] for more on the history of this problem. Denote below by $\cong_n$ the isomorphism relation for torsion-free abelian groups of rank $n \geq 1$. As explained in Remark 4.25, we can view this (up to Borel bireducibility) as a countable Borel equivalence relation. Baer's result now implies the following:

Theorem 7.13 (Baer). $\cong_1 \sim_B E_0$

Using methods from ergodic theory, Hjorth in [H1] showed that $\cong_1 <_B \cong_2$ and finally Thomas proved by such methods the following:

Theorem 7.14 ([T3]). *For each $n \geq 1$, $\cong_n <_B \cong_{n+1}$.*

In particular, the Borel cardinality of the set of isomorphism classes of torsion-free abelian groups of rank n encodes the dimension n. Also one can interpret this result as a strong indication of the non-existence of a reasonable classification for the rank at least 2 case.

For other discussions of these results, see [T5], [Co4], [Q] and [TS].

Denote by $\cong_n^*$ the isomorphism relation on torsion-free abelian groups of rank n that are rigid (i.e., the only automorphisms are the identity and the map $a \mapsto -a$). Then we have:

Theorem 7.15 ([AK]). *For each $n \geq 1$, $\cong_n^* <_B \cong_{n+1}^*$.*

Theorem 7.16 ([T3]). *For each $n \geq 1$, $\cong_{n+1}^* \not\leq_B \cong_n$.*

For each set of primes S, an abelian group is **S-local** if it divisible by any prime $p \notin S$. Denote by $\cong_n^S$ the isomorphism relation on the S-local torsion-free abelian groups of rank n. Also let $\cong_n^p = \cong_n^S$, where $S = \{p\}$. Then we have:

Theorem 7.17 ([T3]). *For each prime p and $n \geq 1$, $\cong_n^p <_B \cong_{n+1}^p$.*

Theorem 7.18 ([T13]). *For sets of primes S, T, and $n \geq 2$,*

$$S \subseteq T \iff \cong_n^S \leq_B \cong_n^T .$$

In [T3, 5.7] it is shown that if E_p is the equivalence relation induced by the action of $GL_2(\mathbb{Q})$ on $\mathrm{PG}(1, \mathbb{Q}_p)$, then $E_p \sim_B \,\cong^p_2$, so if $p \neq q$, then E_p, E_q are incomparable in $\leq_B$.

We also have:

Theorem 7.19 ([Co2]). *Let $n, m \geq 3$ and $p \neq q$ be primes. Then*

$$\cong^p_m \,\not\leq_B\, \cong^q_n, \ \cong^q_n \,\not\leq_B\, \cong^p_m \,.$$

For other related results, see [Co1] and [Co3].

7.6 Multiples of an equivalence relation

For any countable Borel equivalence relation E and $n \geq 1$, recall that nE is the direct sum of n copies of E. Note that $nE_0 \sim_B E_0$ and $nE_\infty \sim_B E_\infty$. On the other hand we have:

Theorem 7.20 ([T4, 4.9]). *There is a countable Borel equivalence relation E such that*

$$E_0 <_B E <_B 2E <_B 3E \cdots$$

See also [HK4, 3.9] and [Q, 3.4.10].

8.   Hyperfiniteness

8.1   Characterizations and classification

Recall the following definition:

Definition 8.1. *A countable Borel equivalence relation E is* **hyperfinite** *if $E = \bigcup_n E_n$, where $E_n \subseteq E_{n+1}$ and each E_n is a finite Borel equivalence relation.*

We next state a number of equivalent characterizations of hyperfiniteness.

Theorem 8.2. *Let E be a countable Borel equivalence relation on a standard Borel space X. Then the following are equivalent:*

(i) E is hyperfinite;

(ii) $E = \bigcup_{n \geq 1} E_n$, with (E_n) an increasing sequence of finite Borel equivalence relations, such that each E_n-class has cardinality at most n;

(iii) $E = \bigcup_{n \geq 1} E_n$, with (E_n) an increasing sequence of smooth Borel equivalence relations;

(iv) There is a Borel action a of $\mathbb{Z}$ on X such that $E = E_a$;

(v) There is a Borel binary relation $R \subseteq X^2$ such that $xRy \implies xEy$ and for each E-class C, $R \restriction C = R \cap C^2$ is a linear ordering on C of order type $\mathbb{Z}$ or finite;

(vi) $E \sqsubseteq_B E_0$;

(vii) $E \leq_B E_0$.

In Theorem 8.2, the equivalence of (i) and (ii) is due to Weiss [We1], (iv) $\implies$ (i) is due to Weiss [We1] and Slaman-Steel [SlSt], and (i) $\implies$ (iv) is due to Slaman-Steel [SlSt]. The equivalence of (i), (iii), (vi) and (vii) is due to Dougherty-Jackson-Kechris [DJK]. Another proof of the equivalence of (i) and (vi), due to Hjorth, can be found in [Ts1].

We thus have the following complete classification of hyperfinite Borel equivalence relations up to Borel bireducibility.

Corollary 8.3 ([DJK]). *Every hyperfinite Borel equivalence relation is Borel bireducible to exactly one in the list:*

$$\Delta_1 <_B \Delta_2 <_B \cdots <_B \Delta_n <_B \cdots <_B \Delta_{\mathbb{N}} <_B \Delta_{\mathbb{R}} <_B E_0$$

Moreover if E is a non-smooth hyperfinite Borel equivalence relations, then $E \simeq_B E_0$.

Hyperfinite Borel equivalence relations have been also completely classified up to Borel isomorphism. The main result is the following, where for any set S, $\mathrm{card}(S)$, is the cardinality of S.

Theorem 8.4 ([DJK]). *Let E, F be aperiodic, non-smooth hyperfinite Borel equivalence relations. Then*

$$E \cong_B F \iff \mathrm{card}(\mathrm{EINV}_E) = \mathrm{card}(\mathrm{EINV}_F).$$

Another version of the proof can be also found in [Sl5, 3.7].

Note that for any countable Borel equivalence relation E,

$$\mathrm{card}(\mathrm{EINV}_E) \in \{0, 1, 2, \ldots, n, \ldots, \aleph_0, 2^{\aleph_0}\}.$$

The tail equivalence relation E_t is hyperfinite, see [DJK, Section 8], and, being compressible, it has no invariant probability measure. Also E_0 has a unique invariant, and thus ergodic, probability measure, nE_0 has exactly n ergodic invariant probability measures, $\mathbb{N}E_0$ has $\aleph_0$ such measures and $F(\mathbb{Z}, 2)$ has $2^{\aleph_0}$ such measures (consider product measures corresponding to the $(p, 1-p)$ measure on $\{0, 1\}$ for $0 < p < 1$). Another example of a hyperfinite Borel equivalence relation with $2^{\aleph_0}$ ergodic invariant probability measures is $\mathbb{R}E_0$, where for each equivalence relation E, $\mathbb{R}E = E \times \Delta_{\mathbb{R}}$. We thus have:

Corollary 8.5 ([DJK]). *Every aperiodic, non-smooth hyperfinite Borel equivalence relation is Borel isomorphic to exactly one in the list:*

$$E_t \sqsubset_B^i E_0 \sqsubset_B^i 2E_0 \sqsubset_B^i \cdots \sqsubset_B^i nE_0 \sqsubset_B^i \cdots \sqsubset_B^i \mathbb{N}E_0 \sqsubset_B^i \mathbb{R}E_0 \cong_B F(\mathbb{Z}, 2).$$

Below let for each countable Borel equivalence relation E:

(i) $c_n(E) = \mathrm{card}(\{C \in X/E \colon \mathrm{card}(C) = n\})$, for $1 \leq n \leq \aleph_0$;
(ii) $s(E) = 0$, if E is smooth; $= 1$, otherwise;
(iii) $t(E) = \mathrm{card}(\mathrm{EINV}_E)$.

Corollary 8.6 ([DJK])**.** *The list* $(c_n)_{1 \leq n \leq \aleph_0}, s, t$ *is a complete list of invariants for Borel isomorphism of hyperfinite Borel equivalence relations.*

A hyperfinite Borel equivalence E is **invariantly universal hyperfinite** if for every hyperfinite Borel equivalence relation F, $F \sqsubseteq^i_B E$. It is easy to see that $E(\mathbb{Z}, \mathbb{R})$ has this property. Clearly an invariantly universal hyperfinite equivalence relation is unique up to Borel isomorphism and it will be denoted by $E_{\infty h}$. The next corollary gives another manifestation of $E_{\infty h}$.

Corollary 8.7. $E_{\infty h} \cong_B \bigoplus_{n \geq 1} \mathbb{R}\Delta_n \oplus \mathbb{R}E_0$

The proof of Theorem 8.4 uses, among other tools, the following classical result of Dye in ergodic theory.

Theorem 8.8 ([Dy])**.** *Let* E, F *be hyperfinite Borel equivalence relations on standard Borel spaces* X, Y*, resp., and let* $\mu \in \mathrm{EINV}_E$*,* $\nu \in \mathrm{EINV}_F$ *be nonatomic. Then there is an* E*-invariant Borel set* $X_0 \subseteq X$*, an* F*-invariant Borel set* $Y_0 \subseteq Y$ *and a measure preserving Borel isomorphism* $T \colon E|X_0 \cong_B F|Y_0$*.*

Equivalently this theorem says that any two Borel $\mathbb{Z}$-actions which have nonatomic, ergodic invariant probability measures are orbit equivalent. For a proof, see, e.g., [KM1, Section 7].

An important ingredient in the proof of Theorem 8.8 is the classical Rokhlin's Lemma. We state it below in a strong uniform version proved in [GW]. Other references for this are [Sl5, 3.4.3] and the Corrections and Updates of [KM1, 7.5] that are posted in:

http://www.math.caltech.edu/∼kechris/

Below a Borel automorphism T of a standard Borel space is called **aperiodic** if all its orbits are infinite.

Theorem 8.9 ([GW, 7.9])**.** *Suppose* T *is an aperiodic Borel automorphism of a standard Borel space* X*,* $n \geq 1$*, and* $\epsilon > 0$*. Then there is a Borel complete section* $A \subseteq X$ *of the equivalence relation induced by* T *such that*

(I) $T^i(A) \cap T^j(A) = \emptyset$, *if* $0 \leq i < j < n$,
and for any T*-invariant probability measure* μ*,*

(ii) $\mu(X \setminus \bigcup_{i<n} T^i(A)) < \epsilon$.

Suppose E, F are countable Borel equivalence relations on standard Borel spaces X, Y, resp. Let $\mu \in \mathrm{EQINV}_E$, $\nu \in \mathrm{EQINV}_F$. We say that (E, μ) is **isomorphic** to (F, ν) if there is an E-invariant Borel set $X_0 \subseteq X$ and an F-invariant Borel set $Y_0 \subseteq Y$ with $\mu(X_0) = \nu(Y_0) = 1$ and $T \colon E|X_0 \cong_B F|Y_0$ such that $T_*\mu \sim \nu$. The **Dye-Krieger Classification** provides a classification up to isomorphism of such (E, μ) for hyperfinite E. One says that (E, μ) is of type $\mathbf{I_n}$, for $1 \leq n \leq \aleph_0$, if μ is atomic concentrating on an E-class of cardinality n. Up to isomorphism there is exactly one (E, μ) of type I_n.

From now on assume that μ is nonatomic. Then (E, μ) is of type $\mathbf{II_1}$ if there is $\nu \sim \mu$ with $\nu \in \mathrm{EINV}_E$ and it is of type $\mathbf{II_\infty}$ if there is an infinite E-invariant measure $\nu \sim \mu$ (i.e., $\nu(X) = \infty$). Otherwise (E, μ) is of type $\mathbf{III}$. Dye's Theorem implies that, up to isomorphism, there is exactly one (E, μ) of type II_1 and exactly one (E, μ) of type II_∞. The type III equivalence relations are further subdivided into classes $\mathbf{III_\lambda}$, for $\lambda \in [0, 1]$. Krieger showed that for $\lambda > 0$, there is a unique, up to isomorphism, (E, μ) of type III_λ, while there is a bijection between isomorphism classes of III_0 equivalence relations and free Borel actions of $\mathbb{R}$ with nonatomic ergodic quasi-invariant probability measure up to isomorphism (of the actions). For a proof of Krieger's results, see [Kr] and [KW].

Another corollary of Theorem 8.4 is the following:

Corollary 8.10 ([DJK, 9.7]). *Let E be an aperiodic hyperfinite Borel equivalence relation, Then for any sequence (M_n) of positive integers ≥ 2, there is an increasing sequence (E_n) of finite Borel subequivalence relations of E such that $E = \bigcup_n E_n$ and every E_n-class has cardinality $M_0 M_1 \cdots M_n$.*

Compare this with Corollary 3.20 and Theorem 8.2, (ii).

A further application of Theorem 8.4 is a classification of non-smooth Borel equivalence relations induced by Borel actions of $\mathbb{R}$.

Theorem 8.11 ([Ke4, Theorem 3]). *Let a, b be two Borel actions of $\mathbb{R}$ on standard Borel spaces. Let c_a be the cardinality of the singleton orbits of the action a and similarly for b. Then if E_a, E_b are not smooth, $E_a \cong_B E_b \iff c_a = c_b$.*

In particular, for any two actions of $\mathbb{R}$ as above with uncountable orbits and E_a, E_b not smooth, $E_a \cong_B E_b$. Using Theorem 4.12 and [JKL, 1.15, 1.16] this holds as well for any such actions of a Polish locally compact group, which is compactly generated of polynomial growth

An analog of Theorem 8.4 for classification of non-smooth E_a, where a is a free Borel action of $\mathbb{R}^n$, up to Lebesgue Orbit Equivalence, i.e., Borel isomorphism that preserves Lebesgue measure on each orbit, is given in [Sl1] and [Sl2]. For results concerning time-change equivalence of free Borel actions of $\mathbb{R}^n$, see [MR2] and [Sl4].

An interesting illustration of Theorem 8.4 is found in [DJK, Section 10], which provides a classification (up to Borel isomorphism) of Lipschitz homeomorphisms of $2^{\mathbb{N}}$.

For countable Borel equivalence relations E, F we put $E \subseteq_B F$ if there is $E' \subseteq F$ with $E \cong_B E'$. Then $\subseteq_B$ is a quasi-order called the **Borel inclusion order**. Denote by $\subset_B$ its strict part. Using Theorem 8.4 the following was shown in [FKSV].

Below for a quasi-order $\preceq$ with strict part $\prec$ on a set Q and $q, r \in Q$, we say that r is a **successor** to q if $q \prec r$ and $(s \prec r \implies s \preceq q)$.

Theorem 8.12 ([FKSV, 2.10]). *(i)* $\mathbb{R}E_0 \subset_B \mathbb{N}E_0 \subset_B \cdots \subset_B 3E_0 \subset_B 2E_0 \subset_B E_0 \subset_B E_t$, *each equivalence relation in this list is a successor in $\subseteq_B$ of the one preceding it and $\mathbb{N}E_0$ is the infimum in $\subseteq_B$ of the $nE_0, n \in \mathbb{N} \setminus \{0\}$.*
(ii) $\mathbb{R}I_{\mathbb{N}} \subset_B E_t$ *and E_t is a successor of $\mathbb{R}I_{\mathbb{N}}$ in $\subseteq_B$.*

The following is an immediate corollary of Theorem 8.12:

Corollary 8.13 ([FKSV, 2.11]). *Let E, F be aperiodic hyperfinite Borel equivalence relations on uncountable standard Borel spaces. Then*

$$E \subseteq_B F \iff \operatorname{card}(\operatorname{EINV}_E) \geq \operatorname{card}(\operatorname{EINV}_F).$$

The next result is a version of the Glimm-Effros Dichotomy (see Theorem 6.5) for the inclusion order $\subseteq_B$ instead of $\sqsubseteq_B$. It is an immediate corollary of Theorem 8.12 and Theorem 8.16 below.

Corollary 8.14 ([FKSV, 2.12]). *Let E be an aperiodic countable Borel equivalence relation. Then exactly one of the following holds:*
(i) E is smooth,
(ii) $\mathbb{R}E_0 \subseteq_B E$.

Finally we mention the following characterization of smoothness for aperiodic hyperfinite Borel equivalence relations.

Theorem 8.15 ([KST, 1.1]). *Let E be an aperiodic hyperfinite Borel equivalence relation on a standard Borel space X. Then the following are equivalent:*

(i) E is smooth;

(ii) For every partition of X into Borel sets $X = A \sqcup B$ such that both A, B have infinite intersection with every E-class, we have that $A \sim_E B$.

8.2   Hyperfinite subequivalence relations

Recall from Corollary 3.20 that every aperiodic countable Borel equivalence relation contains an aperiodic hyperfinite Borel subequivalence relation. The following is a strengthening of this result:

Theorem 8.16. *Let E be an aperiodic countable Borel equivalence relation. Then there is an aperiodic hyperfinite Borel equivalence relation $F \subseteq E$ such that $\mathrm{INV}_E = \mathrm{INV}_F$, and so $\mathrm{EINV}_E = \mathrm{EINV}_F$, and E, F have the same ergodic decomposition (as in Theorem 5.11) modulo compressible sets (for E or equivalently F).*

For a proof, see [Ke5, 5.66 and paragraph following it]. The proof is based on the following result related to Theorem 8.8.

Theorem 8.17 (see [Ke5, 5.26]). *Let E be an aperiodic countable Borel equivalence relation on a standard Borel space X and let $\nu \in \mathrm{EINV}_E$. Then there is an E-invariant Borel set $X_0 \subseteq X$ and an E_0-invariant Borel set $Y_0 \subseteq 2^{\mathbb{N}}$ with $\nu(X_0) = \mu(Y_0) = 1$, where μ is the usual product measure on $2^{\mathbb{N}}$, and a measure preserving Borel isomorphism $T \colon Y_0 \to X_0$ such that $x, y \in Y_0, x E_0 y \implies T(x) E T(y)$.*

The fact that every aperiodic countable Borel equivalence relation contains an aperiodic hyperfinite Borel equivalence relation admits the following generalization.

Theorem 8.18 ([CM2, 2.5.1]). *Let E be a countable Borel equivalence relation on a standard Borel space and $\rho \colon E \to \mathbb{R}^+$ a Borel cocycle for which E is ρ-aperiodic. Then there is a hyperfinite Borel subequivalence relation $F \subseteq E$ for which F is $\rho \upharpoonright F$-aperiodic.*

From Theorem 8.18 the following is also derived:

Theorem 8.19 ([CM2, 2.5.3]). *Let E be a countable Borel equivalence relation on a standard Borel space X and let $\mu \in \mathrm{QINV}_E$ be such that there is no Borel set of μ-positive measure A for which $E \upharpoonright A$ is smooth. Then there is a hyperfinite Borel subequivalence relation $F \subseteq E$ with the same property.*

Actually in [CM2, 2.5.1, 2.5.3] stronger statements are proved concerning Borel graphs.

8.3   Generic hyperfiniteness

A Borel equivalence relation is **essentially hyperfinite** if it is Borel bireducible with a hyperfinite Borel equivalence relation and it is **reducible to hyperfinite** if it is Borel reducible to a hyperfinite Borel equivalence relation. In view of Theorem 8.2 and Theorem 6.5 a Borel equivalence relation E is essentially hyperfinite iff it is reducible to hyperfinite.

The following result shows that essential hyperfiniteness always holds generically.

Theorem 8.20 ([HK1, 6.2]). *Let E be a Borel equivalence relation on a Polish space X such that E is reducible to countable. Then there is a comeager E-invariant Borel set $C \subseteq X$ such that $E \upharpoonright C$ is essentially hyperfinite. In particular, for any countable Borel equivalence relation E on a Polish space X, there is an E-invariant Borel set $C \subseteq X$ such that $E \upharpoonright C$ is hyperfinite.*

This extends an earlier result of [SWW] and its proof also uses the following result, which is an analog of Proposition 5.16.

Proposition 8.21 (Woodin, see [HK1, 6.5]). *Let E be a countable Borel equivalence relation on a Polish space X. Then there is a comeager complete Borel section $C \subseteq X$ such that for any meager Borel set $A \subseteq C$, $[A]_E$ is meager.*

Another proof of Theorem 8.20 for countable E can be found in [KM1, 12.1].

The second part of the following result was originally proved in [SWW] and the first part by Woodin. A countable Borel equivalence relation E on a Polish space X is called **generically ergodic** if every E-invariant Borel set is meager or comeager. It is called **generic** if the E-saturation of a meager set is meager,

Theorem 8.22 ([SWW]; Woodin, see [Ke5, 5.46]). *Let E be a a generically ergodic countable Borel equivalence relation on a perfect Polish space X. Then there is a dense G_δ set $X_0 \subseteq X$, an E_0-invariant dense G_δ set $Y_0 \subseteq 2^\mathbb{N}$ and a homeomorphism $f \colon X_0 \to Y_0$ which takes $E \upharpoonright X_0$ to $E_0 \upharpoonright Y_0$. If moreover E is generic, the set X_0 can be also E-invariant.*

We note that Theorem 8.20 fails for measure instead for category. For example, consider the equivalence relation $E(\mathbb{F}_2, 2)$ on $X = 2^{\mathbb{F}_2}$ with the usual product measure. Then it is a standard result that for any Borel set $A \subseteq X$ of positive measure, $E \upharpoonright A$ is not hyperfinite (see, e.g., Proposition 8.29).

In fact Theorem 8.20 fails for measure, even for *compressible* equivalence relations. There is indeed a compressible countable Borel equivalence relation E on a Polish space X and a probability measure μ on X for which there is no invariant Borel set $A \subseteq X$ with $\mu(A) = 1$ and $E|A$ hyperfinite; see [CG, Théoréme 2] and [Mo, Section 4].

8.4   Closure properties

Below we state the basic closure properties of hyperfiniteness:

Theorem 8.23 ([DJK, 5.2], [JKL, 1.3], Theorem 3.37). *(i) If E, F are countable Borel equivalence relations, F is hyperfinite and $E \leq^w_B F$, then E is hyperfinite.*

(ii) If $E \subseteq F$ are countable Borel equivalence relations, E is hyperfinite and every F-class contains only finitely many E-classes, then F is hyperfinite.

(iii) If each $E_n, n \in \mathbb{N}$, is a hyperfinite Borel equivalence relation, then $\bigoplus_n E_n$ is hyperfinite.

(iv) If E, F are hyperfinite Borel equivalence relations, then $E \times F$ is hyperfinite.

The fundamental open problem concerning closure properties of hyperfiniteness is the following:

Problem 8.24. *Let $E_n, n \in \mathbb{N}$, be hyperfinite Borel equivalence relations such that $E_n \subseteq E_{n+1}$ for every n. Is $\bigcup_n E_n$ hyperfinite?*

A very interesting example of an equivalence relation which is the union of an increasing sequence of hyperfinite Borel equivalence relations was discovered in [Sm]. Fix a countable standard model M of set theory

and let X be the space of Cohen generic reals over M, a G_δ subset of $2^\mathbb{N}$. On X define the following equivalence relation:

$$xEy \iff M[x] = M[y].$$

Then it is shown in [Sm] that E is the union of an increasing sequence of hyperfinite Borel equivalence relations. It is not known if E is hyperfinite.

The following notion was introduce in [BJ2]. Below for $x, y \in \mathbb{N}^\mathbb{N}$, we let

$$x \leq_* y \iff \exists m \forall n \geq m(x_n \leq y_n)$$

and recall that $E_0(\mathbb{N})$ is the eventual equality relation on $\mathbb{N}^\mathbb{N}$.

Let E be a countable Borel equivalence relation on a standard Borel space X. Then E is **Borel-bounded** if for every Borel function $f\colon X \to \mathbb{N}^\mathbb{N}$, there is a Borel function $g\colon X \to \mathbb{N}^\mathbb{N}$ such that $\forall x(f(x) \leq_* g(x))$ and $g\colon E \to_B E_0(\mathbb{N})$. Boykin-Jackson show that hyperfinite Borel equivalence relations are Borel bounded and that the closure properties (i), (ii), (iii) of Theorem 8.23 hold if hyperfiniteness is replaced by Borel-boundedness. However the analog of part (iv) remains open. They also show that if a Borel-bounded countable Borel equivalence relation is the union of an increasing sequence of hyperfinite Borel equivalence relations, then it is hyperfinite. It is unknown if *every* countable Borel equivalence relation is Borel-bounded. Thomas in [T11, 5.2] has shown that Martin's Conjecture on functions on Turing degrees implies that $\equiv_T$ is not Borel-bounded.

We recall here the statement of **Martin's Conjecture**. For $x, y \in 2^\mathbb{N}$, let $x \leq_T y \iff x$ is recursive in y. Then Martin's Conjecture states that every for Borel homomorphism $f\colon \equiv_T \to_B \equiv_T$, either there is $x \in 2^\mathbb{N}$ and $z_0 \in 2^\mathbb{N}$ such that for all $y \geq_T x$, $f(y) \equiv_T z_0$ or else there is $x \in 2^\mathbb{N}$ such that for all $y \geq_T x$, $y \leq_T f(y)$. Equivalently this is stated as follows. Let $\mathcal{D} = 2^\mathbb{N}/\equiv_T$ be the set of **Turing degrees** equipped with the partial order $\leq$ induced by $\leq_T$. A **cone** is a subset of the form $\{d \in \mathcal{D}\colon c \leq d\}$, for some $c \in \mathcal{D}$. Then Martin's Conjecture says that for every function $f\colon \mathcal{D} \to \mathcal{D}$ that has a Borel lifting, there is a cone C of Turing degrees such that either $f \upharpoonright C$ is constant or else $d \leq f(d)$, for all $d \in C$.

It is also unknown if every Borel-bounded countable Borel equivalence relation is hyperfinite.

For further results on a weakening on the notion of Borel boundedness that holds for all countable Borel equivalence relations and other related notions see [BJ2].

In [CS] the authors introduce and study certain properties of countable Borel equivalence relations that relate to cardinal characteristics of the continuum.

8.5 μ-**hyperfiniteness**

Let E be a countable Borel equivalence relation on a standard Borel space X and μ a probability measure on X. We say that E is $\boldsymbol{\mu}$-**hyperfinite** if there is an E-invariant Borel set A with $\mu(A) = 1$ such that $E \restriction A$ is hyperfinite. Also E is **measure-hyperfinite** if it is μ-hyperfinite for every probability measure μ.

We state now some equivalent conditions for μ-hypefiniteness.

Proposition 8.25. *Let E be a countable Borel equivalence relation on a standard Borel space X and let μ be a probability measure on X. Then the following are equivalent:*

(i) E is μ-hyperfinite;

(ii) For any $f_1, f_2, \ldots, f_n \in [[E]]_B$, $f_i \colon A_i \to B_i$, and any $\epsilon > 0$, there are $T_1, T_2, \ldots, T_n \in [E]_B$ such that the group generated by $T_1, T_2, \ldots, T_n$ is finite and for each $1 \le i \le n$, $\mu(\{x \in A_i \colon f_i(x) \ne T_i(x)\}) < \epsilon$;

(iii) Same as (ii) but with $f_1, f_2, \ldots, f_n \in [E]_B$;

(iv) Let $T_n \in [E]$ be such that $xEy \iff \exists n(T_n(x) = y)$. Define for any Borel subequivalence relation F of E and $n \ge 1$, $d_n(F, E) = \mu(\{x \in X \colon \exists i < n(\neg x F T_i(x))\})$. Then for each $\epsilon > 0, n \ge 1$, there is a finite Borel subequivalence relation F of E such that $d_n(F, E) < \epsilon$.

For the proof of Proposition 8.25, see [FM, Section 4], [Ke5, 5.K, A)], [Ts2] and [M4].

Using Proposition 8.25, one can see easily that the answer to Problem 8.24 is positive in the measure theoretic category.

Theorem 8.26 (Dye, Krieger)**.** *Let X be a standard Borel space and μ a probability measure on X. If $E_n, n \in \mathbb{N}$, is an increasing sequence of μ-hyperfinite Borel equivalence relations, then $\bigcup_n E_n$ is μ-hyperfinite.*

Another proof of this result can be given using the concept of Borel-boundedness. In fact one has the following more general result. Below if $\mathcal{C}$ is a class of countable Borel equivalence relations, denote by $\mathrm{HYP}(\mathcal{C})$ the

class of all countable Borel equivalence relations that can be written as the union of an increasing sequence of equivalence relations in $\mathcal{C}$. Then we have:

Theorem 8.27 ([BJ2, page 116]). *Let $\mathcal{C}$ be a class of countable Borel equivalence relations closed under subrelations. Then for any countable Borel equivalence relation E on a standard Borel space X and every probability measure μ on X, if $E = \bigcup_n E_n$, with (E_n) increasing and $E_n \in \mathrm{HYP}(\mathcal{C})$, then there is a Borel set $A \subseteq X$ such that $\mu(A) = 1$ and a countable Borel equivalence relation $F \in \mathrm{HYP}(\mathcal{C})$ such that $E \upharpoonright A = F \upharpoonright A$.*

We conclude with the following open problem.

Problem 8.28. *Does measure hyperfiniteness imply hyperfiniteness?*

8.6 Groups generating hyperfinite relations

By Theorem 8.2, (iv) every Borel action of the group $\mathbb{Z}$ generates a hyperfinite Borel equivalence relation. Which countable groups G have the property that all their Borel actions generate hyperfinite Borel equivalence relations? The following is a well-known fact, see, e.g., [JKL, 2.5 (ii)].

Proposition 8.29. *Let G be a countable group and let a be a free Borel action of G on a standard Borel space that admits an invariant probability measure. Then if E_a is hyperfinite, G is amenable.*

Since any countable group G admits a free Borel action with invariant probability measure, e.g., its shift action on 2^G (restricted to its free part) with the usual product measure, it follows that every non-amenable group has a Borel action that generates a non-hyperfinite equivalence relation. Moreover Ornstein and Weiss proved the following:

Theorem 8.30 ([OW]). *Let G be an amenable group and consider a Borel action a of G on a standard Borel space X. Then E_a is measure hyperfinite.*

This motivates the following problem of Weiss.

Problem 8.31 ([We1]). *Let G be a countable amenable group. Is it true that every Borel action of G generates a hyperfinite equivalence relation?*

Weiss proved a positive answer for the finitely generated abelian groups G. This was extended in [JKL] to all finitely generated nilpotent-by-finite groups, which by the result of Gromov are exactly the finitely generated groups of polynomial growth. In fact we have the following more general result concerning locally compact groups.

Let G be a Polish locally compact group and let μ_G be a right Haar measure. Let d be a positive integer. We say that G is **compactly generated of polynomial growth d** if there is a symmetric compact neighborhood K of the identity of G such that $G = \bigcup_n K^n$ and $\mu_G(K^n) \in O(n^d)$. For $c > 0$, G has the **mild growth property of order c** if there is an increasing sequence (K_n) of compact symmetric neighborhoods of the identity, such that: (a) $K_n^2 \subseteq K_{n+1}$; (b) for infinitely many n, $\mu_G(K_{n+4}) \leq c\mu_G(K_n)$; (c) $\bigcup_n K_n = G$.

It can be shown that if G is compactly generated of polynomial growth, then it has the mild growth property and if a Polish locally compact group G can be written as a union of an increasing sequence (G_n) of Polish locally compact groups that have the mild growth property of the same order c, then so does G, see [JKL, 1.15]. We now have:

Theorem 8.32 ([JKL, 1.16])**.** *Let G be a locally compact group with the mild growth property. Then any equivalence relation generated by a Borel action of G on a standard Borel space is essentially hyperfinite.*

In particular any equivalence relation generated by a Borel action of a finitely generated nilpotent-by-finite countable group is hyperfinite and the same holds for the group $\mathbb{Q}^n$.

The next step towards a positive answer to Weiss' problem was taken in [GJ].

Theorem 8.33 ([GJ])**.** *Let G be any countable abelian group. Then the equivalence relation generated by a Borel action of G on a standard Borel space is hyperfinite.*

Hjorth has raised the following problem:

Problem 8.34 (Hjorth)**.** *Suppose G is an abelian Polish group and let a be a Borel action of G on a standard Borel space. Is it true that if E is a reducible to countable Borel equivalence relation with $E \leq_B E_a$, then E is essentially hyperfinite?*

A positive answer has been obtained for non-archimedean groups.

Theorem 8.35 ([DG, 1.4]). *Let G be an abelian non-archimedean Polish group and let a be a Borel action of G on a standard Borel space. If E is a reducible to countable Borel equivalence relation with $E \leq_B E_a$, then E is essentially hyperfinite.*

Another proof of Theorem 8.35 for $E = E_a$ is given in [Gr1, 1.4].

Corollary 8.36 ([DG, 1.3]). *Let G be an abelian non-archimedean locally compact Polish group and let a be a Borel action of G on a standard Borel space. Then E_a is essentially hyperfinite.*

Moreover a positive answer also holds for any abelian locally compact Polish group.

Theorem 8.37 ([Cot]). *Let G be an abelian locally compact Polish group and let a be a Borel action of G on a standard Borel space. Then E_a is essentially hyperfinite.*

However Allison in [Al] has recently shown that the answer to Problem 8.34 is in general negative, see Section 10.4.

Further extending the methods of [GJ] the following was proved. Below a countable group G is **locally nilpotent** if all its finitely generated subgroups are nilpotent. This class of groups properly contains the class of countable nilpotent groups.

Theorem 8.38 ([ScSe]). *Let G be a locally nilpotent countable group. Then the equivalence relation generated by a Borel action of G on a standard Borel space is hyperfinite.*

It is also shown in [ScSe] that the equivalence relation induced by a free and continuous action of a countable locally nilpotent group on a zero-dimensional Polish space continuously embeds into E_0. For other such results concerning continuous embeddings and reductions, see [BJ1], [GJ] and [T9].

It is unknown if Weiss' Problem has a positive answer for all solvable groups. However in an important recent development, Conley, Jackson, Marks, Seward, and Tucker-Drob, in [CJMST2], introduce the novel concept of Borel asymptotic dimension for a Borel action of a countable group and show that when it is finite, then the induced equivalence relation is hyperfinite. They also show that every free Borel action of a polycyclic group

has finite Borel asymptotic dimension and thus the induced equivalence relation is hyperfinite. Combining this with work in [ScSe], the freeness assumption can be dropped and one has the following:

Theorem 8.39 ([CJMST2]). *Let G be a polycyclic group. Then the equivalence relation generated by a Borel action of G on a standard Borel space is hyperfinite.*

In particular the work in [CJMST2] provides the first examples of finitely generated groups of exponential growth that give a positive answer to Weiss' Problem. In fact several other classes of countable groups are shown to give a positive answer to Weiss' Problem, at least for free actions. Finally the methods introduced in [CJMST2] are used in this work to substantially simplify the proofs of earlier results, including Theorem 8.38.

Changing the point of view, a countable group G is called **hyperfinite generating** if for every aperiodic hyperfinite E there is a Borel action of G that generates E. In [FKSV] equivalent formulations of this property are provided and it is shown that all countable groups with an infinite amenable factor are hyperfinite generating, while no infinite countable group with property (T) has this property.

8.7   Examples

We will discuss here hyperfinite Borel equivalence relations that appear in various contexts.

1) Let $T\colon X \to X$ be a Borel function on a standard Borel space. Consider the Borel action a of the monoid $S = (\mathbb{N}, +, 0)$ on X given by $1 \cdot x = T(x)$. Then $E_{t,a} = E_T$ and $E_{0,a} = E_{0,T} \subseteq E_T$ is the equivalence relation $xE_{0,T}y \iff \exists n(T^n(x) = T^n(y))$. If T is countable-to-1, E_T and thus $E_{0,T}$ are hyperfinite, see [DJK, Section 8]. In particular, E_t is hyperfinite. In fact it turns out that for an arbitrary Borel function T the equivalence relations E_T and $E_{0,T}$ are **hypersmooth**, i.e, unions of an increasing sequence of Borel smooth relations, see [DJK, Section 8].

Let E be a Borel equivalence relation on a standard Borel space X and let $T\colon E \to_B E$. Put

$$xE_t(E, T)y \iff \exists m \exists n(T^n(x)\ E\ T^m(y)).$$

and

$$xE_0(E, T)y \iff \exists n(T^n(x)\ E\ T^n(y)).$$

Clearly $E_0(E,T) \subseteq E_t(E,T)$. The following is an open problem:

Problem 8.40. *If E is Borel hypersmooth (resp., Borel hyperfinite) and T is a Borel function (resp., countable-to-1 Borel function) are $E_0(E,T), E_t(E,T)$ hypersmooth (resp.hyperfinite)?*

As pointed out in [DJK, Section 8], a positive answer for $E_0(E,T)$ implies a positive answer to Problem 8.24 and gives another proof of Theorem 8.33. In the paper [CFW, Corollary 12] a positive answer is given to Problem 8.40, for the hyperfinite case, in the measure theoretic context, i.e., $E_t(E,T)$ is measure hyperfinite.
It is shown in [JKL, 1.21] that every equivalence relation of the form $E = F \vee G$, where F, G are finite Borel equivalence relations of type 2, is hyperfinite (see also the paragraph following Theorem 3.8)

2) The Vitali equivalence relation E_v is hyperfinite, see [My]. (This also follows from Theorem 8.2, (iii) and the fact that $\mathbb{Q} = \bigcup_{n \geq 1}(\mathbb{Z}/n!)$.) The commensurability relation E_c is hyperfinite. This follows from Theorem 8.33.

3) Consider the action of $\mathrm{GL}_2(\mathbb{Z})$ on $\mathbb{R} \cup \{\infty\}$ by fractional linear transformations (or equivalently the natural action of $\mathrm{GL}_2(\mathbb{Z})$ on the projective space $\mathrm{PG}(1,\mathbb{R})$). Then the associated equivalence relation is hyperfinite. Similarly consider the action of $\mathrm{GL}_2(\mathbb{Z})$ on the unit circle, where, identifying it with the set of rays $t\vec{x}$ $(t > 0)$, for $\vec{x} \in \mathbb{R}^2 \setminus \{0\}$, $A \in \mathrm{GL}_2(\mathbb{Z})$ acts on this ray to give the ray $tA(\vec{x})$ $(t > 0)$. This generates again a hyperfinite Borel equivalence relation. For a proof, see [JKL, 1.4, (C) and page 43].
The action of $\mathrm{GL}_2(\mathbb{Z})$ on the unit circle is also **productively hyperfinite** in the terminology of [CM2, 2.1], i.e., its product with *any* Borel action of $\mathrm{GL}_2(\mathbb{Z})$ on a standard Borel space also induces a hyperfinite Borel equivalence relation, see [CM2, 2.1.4]. Using this it is shown in [CM2, 2.2.3] that the action of $\mathrm{GL}_2(\mathbb{Z})$ on $\mathbb{R}^2$ also generates a hyperfinite Borel equivalence relation.

4) Consider any countable free group $\mathbb{F}_n$ with a fixed set of free generators. Let $\partial \mathbb{F}_n$ be the **boundary** of $\mathbb{F}_n$, i.e., the set of infinite reduced words (x_i), where each x_i is one of the generators or its inverse and $x_i x_{i+1} \neq 1$. Then $\mathbb{F}_n$ (viewed as the set of finite reduced words) acts on $\partial \mathbb{F}_n$ by left-concatenation and cancellation. The associated equivalence relation is Borel hyperfinite, see [JKL, 1.4, (E)].

It has been an interesting problem to extend this to the action of any hyperbolic group on its boundary. In [A4] a positive answer was obtained for any hyperbolic group but in the measure theoretic category, i.e., the associated equivalence relation is measure-hyperfinite. A positive answer in the Borel category has been obtained in [HSS] for any cubulated hyperbolic group. Finally in [MS] the problem was solved in full generality by showing that the answer is positive in the Borel category for any hyperbolic group. For other related results, see also [Marq] and [PS].

5) Following up on Remark 4.25 and Theorem 7.13, consider the multiplication action of the multiplicative group of non-zero rationals on the space of subgroups of $(\mathbb{Q}, +)$. The corresponding equivalence relation E is Borel bireducible to $\cong_1$ and thus to E_0. In fact it turns out that E restricted to the subgroups different from $\{0\}, \mathbb{Q}$ is Borel isomorphic to E_t.

Consider now the class of **Butler groups**, which are finite rank torsion-free abelian groups that can be expressed as finite (not necessarily direct) sums of rank 1 subgroups. In [T7] it is shown that the isomorphism equivalence relation on the class of Butler groups is essentially hyperfinite.

6) Recall here Examples 3.2, 5). The isomorphism relation on the space of all subshifts of $n^{\mathbb{Z}}$ is Borel bireducible to E_∞, see [Cl2], However there are various Borel classes of subshifts of $2^{\mathbb{Z}}$ for which the isomorphism relation turns out to be hyperfinite (and not smooth). These include:

(i) The class of non-degenerate rank-1 subshifts, see [GH];

(ii) The class of Toeplitz subshifts with separated holes; see [Kay]. See also [ST] for this result in the measurable context and for raising the question of whether isomorphism in the class of *all* Toeplitz subshifts is hyperfinite.

For each infinite countable group G, one can also consider the isomorphism equivalence relation for subshifts of k^G, $k \geq 2$. This is again a countable Borel equivalence relation and [GJS2, 9.4.3] shows that for all locally finite G, the isomorphism relation on the space of all subshifts is non-smooth and hyperfinite. On the other hand if G is not locally finite, isomorphism on the space of free subshifts is Borel bireducible to E_∞, see [GJS2, 9.4.9]. Finally in [ST, 1.2] it is shown that for residually finite, non-amenable G, the isomorphism relation on free, Toeplitz subshifts is not hyperfinite.

7) In the context of Section 4.6 it is shown in [GK, 8.2] that the isometry relation on proper ultrametric Polish spaces is hyperfinite and not smooth.

8) For essentially hyperfinite Borel equivalence relations that occur in the context of type spaces in model theory, see [KPS, 3.4, 3.6, 3.7, 3.9, 4.5]

9) Let $\mathbb{G} = (X, R)$ be a **Borel graph** on a standard Borel space X, i.e., X is the set of vertices and the set of edges $R \subseteq X^2$ is Borel. We let $E_{\mathbb{G}}$ be the equivalence relation whose equivalence classes are the connected components of $\mathbb{G}$. If $\mathbb{G}$ is **locally countable**, i.e., every vertex has countably many neighbors, then $E_{\mathbb{G}}$ is a countable Borel equivalence relation. In [Mi10] it is shown that if there is a Borel way to choose two ends of the graph in each connected component, then $E_{\mathbb{G}}$ is hyperfinite. Moreover it is shown in [Mi10] that if either there are no ends in each connected component or else there is a Borel way to choose at least three but finitely many ends in each connected component, then $E_{\mathbb{G}}$ is smooth. Finally, in the same paper it is shown that the class of all $E_{\mathbb{G}}$, where $\mathbb{G}$ is **locally finite**, i.e., every vertex has finitely many neighbors, and has one end in each connected component, coincides with the class of all aperiodic countable Borel equivalence relations.

10) Let H be a Polish group and $G \trianglelefteq H$ a countable normal subgroup. We will say that the quotient group H/G is hyperfinite if the coset equivalence relation of G in H is hyperfinite. It is shown in [FS] that H/G is always hyperfinite. In particular the outer automorphism group of any countable group is hyperfinite.

11) The three **Thompson groups** $F \leq T \leq V$ can be defined as follows. By a **T-tree** we mean a finite nonempty subtree S of $2^{<\mathbb{N}}$ such that if $s_1, \ldots, s_n$ are the terminal nodes of S, then $\forall x \in 2^{\mathbb{N}} \exists i \leq n(s_i \subseteq x)$. For such tree S, we put $|S| = n$ and order its leaves in lexicographical order $s_1 < \cdots < s_n$. For two such trees S, T with $|S| = |T| = n$, let $f_{S,T}$ be the homeomorphism of $2^{\mathbb{N}}$ defined by $f_{S,T}(s_i x) = t_i x$, for $x \in 2^{\mathbb{N}}, 1 \leq i \leq n$. The collection of all $f_{S,T}$ is a subgroup of the homeomorphism group of $2^{\mathbb{N}}$, the Thompson group F.

Now for each S, T as above and $1 \leq m \leq n$, let $c_m \colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ be the "cyclic" permutation $c_m(i) = (i + m - 1)$, if $(i + m - 1) \leq n$; $= (i + m - 1) - n$, otherwise. Then define the homeomorphism $f_{S,T,m}$ of $2^{\mathbb{N}}$ by $f_{S,T,m}(s_i x) = t_{c_m(i)} x$, for $1 \leq i \leq n$. The collection of all $f_{S,T,m}$ is a subgroup of the homeomorphism group of $2^{\mathbb{N}}$, the Thompson group T.

Finally for each $S. T$ as above and π any permutation of $\{1, \ldots, n\}$, define the homeomorphism $f_{S,T,\pi}$ of $2^{\mathbb{N}}$ by $f_{S,T,\pi}(s_i x) = t_{\pi(i)} x$, for $1 \leq i \leq n$.

The collection of all $f_{S,T,\pi}$ is a subgroup of the homeomorphism group of $2^{\mathbb{N}}$, the Thompson group V.

Each of these groups acts in the obvious way on $2^{\mathbb{N}}$ and we let E_F, E_T, E_V be the corresponding equivalence relations. Then it turns out that $E_F|(2^{\mathbb{N}} \setminus \{\bar{0}, \bar{1}\}) = E_t|(2^{\mathbb{N}} \setminus \{\bar{0}, \bar{1}\}), E_T = E_V = E_t$, where $\bar{i}$ is the constant sequence with value i.

8.8 Dichotomies for essential hyperfiniteness

Recall here the discussion and notation in Section 4.7. We also define the equivalence relation E_1 on $\mathbb{R}^{\mathbb{N}}$ by

$$(x_n)E_1(y_n) \iff \exists m \forall n \geq m(x_m = y_m).$$

Recall that a Borel equivalence relation E is hypersmooth if it can be written as the union of an increasing sequence of Borel smooth equivalence relations, Then E_1 is hypersmooth and for any hypersmooth Borel E, we have that $E \leq_B E_1$, see [KL]. Recall also from Theorem 8.2, (iii) that the hyperfinite Borel equivalence relations are exactly the hypersmooth countable Borel equivalence relations. We now have the following dichotomy:

Theorem 8.41 ([KL]). *Let E be a Borel equivalence relation such that $E \leq_B E_1$. Then exactly one of the following holds:*
   (i) E is essentially hyperfinite;
   (ii) $E \sim_B E_1$.

We also have an analogous dichotomy theorem for E_3.

Theorem 8.42 ([HK3]). *Let E be a Borel equivalence relation such that $E \leq_B E_3$. Then exactly one of the following holds:*
   (i) E is essentially hyperfinite;
   (ii) $E \sim_B E_3$.

Concerning E_2 the following is an open problem:

Problem 8.43. *Let E be a Borel equivalence relation such that $E \leq_B E_2$. Is it true that exactly one of the following holds?*
   (i) E is essentially hyperfinite;
   (ii) $E \sim_B E_2$.

By Theorem 4.28 this holds if "hyperfinite" is replaced by "reducible to countable". Also note that a positive answer to Problem 8.40 gives a positive answer to Problem 8.43.

8.9   Properties of the hyperfinite quotient space

Let E, F be countable Borel equivalence relations on standard Borel spaces X, Y, resp. A **Borel isomorphism** of X/E with Y/F is a bijection between X/E and Y/F that has a Borel lifting. It follows from Theorem 3.32 and Corollary 8.3 that if E, F are non-smooth and hyperfinite, then X/E and Y/F are Borel isomorphic. We can thus refer to $2^{\mathbb{N}}/E_0$ as *the* **hyperfinite quotient space**. We will discuss here some properties of this space.

1) An equivalence relation R on X/E is called Borel if its lifting $\tilde{R}$ on X, given by $x\tilde{R}y \iff [x]_E R[y]_E$, is Borel. Thus Borel equivalence relations on the hyperfinite space correspond to Borel equivalence relations containing E_0.

The papers [Mi4] and [Mi17] contain a classification of Borel equivalence relations on the hyperfinite space, all of whose classes have fixed cardinality $n \geq 1$, up to Borel isomorphism. The following is then a corollary of this classification:

Theorem 8.44 ([Mi4], [Mi17, Theorem 5]). *For each $n \geq 1$, there are only finitely many, up to Borel isomorphism, Borel equivalence relations on the hyperfinite quotient space all of whose classes have cardinality n.*

2) For each set X and $n \geq 1$, let $[X]^n = \{(x_i)_{i<n} \colon \forall i \neq j (x_i \neq x_j)\}$. Let now E be a Borel equivalence relation on a standard Borel space X. Then for $n \geq 1$, we say that E has the **Borel n-Jónsson property** if for all Borel (i.e., having a Borel lifting) functions $f \colon [X/E]^n \to X/E$ there is a Borel (i.e., having Borel lifting) set $A \subseteq X/E$ such that there is a Borel (i.e., having Borel lifting) bijection between X/E and A with $f([A]^n) \neq X/E$. Also E has the **Borel Jónsson property** if the above holds when $[X/E]^n$ is replaced by $\bigcup_{n \geq 1} [X/E]^n$.

As a special case of more general results proved in [HJ] and [CM], we have the following:

Theorem 8.45. *(i) ([HJ]) $\Delta_{\mathbb{R}}$ has the Borel Jónsson property and E_0 has the Borel 2-Jónsson property.*

(ii) ([CM]) E_0 *does not have the Borel 3-Jónsson property.*

The n-Jónsson property is related to another property called the Mycielski property. Let again $n \geq 1$. We say that E has the **n-Mycielski property** if for every comeager Borel set $C \subseteq X^n$, there is a Borel set $A \subseteq X$ such that $E \sim_B E \upharpoonright A$ and $[A]_E^n = \{(x_i)_{i<n} \in X^n \colon \forall i \neq j (\neg x_i E x_j)\} \subseteq C$. A classical result of Kuratowski and Mycielski (see, e.g., [Ke6, 19.1]) asserts that $\Delta_{\mathbb{R}}$ has the n-Mycielski property for every n. We now have (again as special cases of more general results):

Theorem 8.46. *(i)* ([HJ]) E_0 *has the Borel 2-Mycielski property.*
   (ii) ([CM]) E_0 *does not have the Borel 3-Mycielski property.*

3) Finally in [CJMST1] the authors study, under the **Axiom of Determinacy (AD)**, ultrafilters on the hyperfinite space and show that there is such an ultrafilter lying above, in the Rudin-Keisler order, the Martin ultrafilter on $\mathcal{D} = 2^{\mathbb{N}}/ \equiv_T$ (i.e., the ultrafilter generated by the cones), see [CJMST1, 1.8].

8.10   Effectivity of hyperfiniteness

The following problem, raised in [DJK, Section 5], is still open:

Problem 8.47. *Let E be a hyperfinite equivalence relation on $\mathbb{N}^{\mathbb{N}}$ which is Δ_1^1 (effectively Borel). Is there a Δ_1^1 automorphism of $\mathbb{N}^{\mathbb{N}}$ such that $E = E_T$? Equivalently is it true that $E = \bigcup_n E_n$, where (E_n) is a Δ_1^1 (uniformly in n) increasing sequence of finite equivalence relations?*

Miri Segal in her Ph.D.Thesis [Se] showed that the answer is positive in the measure theoretic context:

Theorem 8.48 ([Se]). *Let μ be a probability measure on $\mathbb{N}^{\mathbb{N}}$ and let E be a μ-hyperfinite equivalence relation on $\mathbb{N}^{\mathbb{N}}$ which is Δ_1^1. Then there is a $\Delta_1^1(\mu)$ E-invariant set A, with $\mu(A) = 1$, and a $\Delta_1^1(\mu)$ increasing sequence (E_n) of finite equivalence relations such that $E \upharpoonright A = \bigcup_n E_n \upharpoonright A$.*

A proof can be found in [Ts2], [CM2, 1.7.8] and [M4].

8.11 Bases for non-hyperfiniteness

Given a quasi-order $\leq$ on a set A, a **basis** for $\leq$ is a subset $B \subseteq A$ such that $\forall a \in A \exists b \in B (b \leq a)$. In this terminology, Theorem 6.5 implies that $\{E_0\}$ is the unique basis for the quasi-order $\leq_B$ on the non-smooth countable Borel equivalence relations. One can now ask whether there is a "reasonable basis" for $\leq_B$ on the non-hyperfinite countable Borel equivalence relations. This is a rather vague question but one can still formulate some precise problems.

Problem 8.49. *(i) Is there a countable basis for the quasi-order of Borel reducibility $\leq_B$ on the non-hyperfinite Borel equivalence relations?*

(ii) Consider the class $\mathcal{B}$ of all equivalence relations of the form E_a, where a is a free Borel action of $\mathbb{F}_2$ admitting an invariant probability measure. Is $\mathcal{B}$ a basis for the quasi-order of Borel reducibility $\leq_B$ on the non-hypefinite Borel equivalence relations?

Although these questions are still open, recent work in [CM2] shows that there are some severe obstacles towards a positive answer (at least for part (i)). Below we say that a countable Borel equivalence relation E on a standard Borel space X is **measure reducible** to a countable Borel equivalence relation F, in symbols $E \leq_M F$, if for every probability measure μ on X, there is a Borel set $A \subseteq X$ with $\mu(A) = 1$ such that $E \upharpoonright A \leq_B F$. Analogously we define the concept of **measure embeddability**, $E \sqsubseteq_M F$, and their strict counterparts $E <_M F$ and $E \sqsubset_M F$

Consider now the class of all countable Borel equivalence relations which are not measure hyperfinite. This is clearly closed upwards under $\leq_M$. We now have:

Theorem 8.50 ([CM2, 3.4.2]). *Any basis for the quasi-order of measure reducibility on the countable Borel equivalence relations which are not measure hyperfinite has the cardinality of the continuum.*

In particular this shows that every basis of cardinality less than the continuum for the quasi-order of Borel reducibility $\leq_B$ on the non-hyperfinite countable Borel equivalence relations must contain equivalence relations which are measure hyperfinite. It is an open problem whether such relations exist, see Problem 8.28. Also it follows from Theorem 8.50 that in Problem 8.49, (ii) one cannot replace $\mathcal{B}$ by a subset that has cardinality less than that of the continuum.

One can also ask the analog of Problem 8.49 for the quasi-order of weak Borel reducibility $\leq_B^w$.

Finally there are basis questions concerning inclusion of equivalence relations instead of reducibility. A version of Problem 8.49, ii) was considered in [KM1, 28.7]. Consider the class $\mathcal{C}$ of countable Borel equivalence relations E which are not μ-hyperfinite for every E-*invariant* probability measure μ and let $\mathcal{C}'$ be the subclass consisting of all equivalence relations of the form E_a, where a is a free Borel action of $\mathbb{F}_2$. Is $\mathcal{C}'$ a basis for $\mathcal{C}$ for the partial order of inclusion $\subseteq$? For a partial answer, see [KM1, 28.8].

The following is also a related well known problem in the measure theoretic context (see, e.g., [KM1, 28.14]). It is a dynamic version of the von Neumann-Day Problem that asked whether every non-amenable countable group contains a copy of $\mathbb{F}_2$ (the answer is negative as proved by Ol'shanskii).

Problem 8.51. *Let E be a countable Borel equivalence relation on a standard Borel space X and let μ be an E-invariant, E-ergodic probability measure on X. Then exactly one of the following holds:*

(i) E is μ-hyperfinite;

(ii) There is an E-invariant Borel set $A \subseteq X$ with $\mu(A) = 1$ and a free Borel action a of $\mathbb{F}_2$ on A such that $E_a \subseteq E \upharpoonright A$.

It was shown in [GL] that for any countable non-amenable group G, there is *some* standard Borel space X and probability measure ν on X, such that if E is induced by the shift action of G on X^G and $\mu = \nu^G$ is the product measure, then (ii) holds, with μ being also E_a-ergodic. It was shown in [Bo] that this holds for *every* (X, ν), when ν does not concentrate on one point.

Moreover it is shown in [BHI] that if E is a countable Borel equivalence relation on a standard Borel space X and μ is an E-invariant, E-ergodic probability measure on X such that E is not μ-hyperfinite, then there is a countable Borel equivalence relation F on a standard Borel space Y, an F-invariant, F-ergodic measure ν on Y, a Borel map $f \colon X \to Y$ such that $f_*\nu = \mu$ and a Borel F-invariant set $A \subseteq Y$ with $\nu(A) = 1$ such that for every $y \in A$, $f \upharpoonright [y]_F$ is a bijection with $[f(y)]_F$ and there is a free Borel action a of $\mathbb{F}_2$ on A such that $E_a \subseteq F \upharpoonright A$. More succinctly, this states that E has a class bijective extension in which (ii) of Problem 8.51 holds.

8.12 Hyper-Borel-finiteness

The following notion is introduced in [DaMa], which traces its origins in the paper [SlSt]. Let X be a standard Borel space and (f_n) a sequence of Borel functions $f_n \colon X \to X^{\mathbb{N}}$. A countable Borel equivalence relation E on X is called (f_n)**-finite** if for every $x \in X$ and every n, $f_n(x)(\mathbb{N})$ is not an infinite subset of $[x]_E$. Also E is called **hyper-(f_n)-finite** if E is the union of an increasing sequence of (f_n)-finite subequivalence relations. Finally E is called **hyper-Borel-finite** it is hyper-(f_n)-finite, for every sequence of Borel functions (f_n). Clearly every hyperfinite equivalence relation is hyper-Borel-finite but it is unknown whether the converse holds or in fact whether *every* countable Borel equivalence relation is hyper-Borel-finite. It is also unknown how this notion is related to the concept of Borel boundedness discussed in Section 8.4.

In [SlSt, Question 6] the question was raised of whether Turing equivalence $\equiv_T$ on $2^{\mathbb{N}}$ is hyper-(f_n)-finite, where (f_n) is the sequence enumerating the Turing reductions. It is shown in [DaMa, 3.4] that a positive answer is equivalent to the statement that every countable Borel equivalence relation is hyper-Borel-finite.

9. Amenability

9.1   Amenable relations

Let G be a Polish locally compact group and let λ be a left Haar measure. Recall that G is **amenable** if there is a finitely additive probability measure defined on all measurable subsets of G, vanishing on λ-null sets, that is invariant under left-translation. We say that a Polish locally compact group G satisfies the **Reiter condition** if there is a sequence (F_n) of Borel functions $F_n\colon G \to \mathbb{R}$ such that $F_n \geq 0$, $\|F_n\|_1 = 1$ and $\forall g \in G(\|F_n - g{\cdot}F_n\|_1 \to 0$, where for a function $F\colon G \to X$, X any set, $g \cdot F\colon G \to X$ and $g \cdot F(h) = F(g^{-1}h)$.

One of the many equivalent characterization of amenability is the following (see [Pa, 0.8, Problem 4.1]):

Theorem 9.1. *Let G be a Polish locally compact group. Then the following are equivalent:*
(i) G is amenable;
(ii) G satisfies the Reiter condition.

We can use an analog of the Reiter condition to define a notion of amenability for countable Borel equivalence relations (see [JKL], and [Kai] for such a definition in the measure theoretic context):

Definition 9.2. *Let E be a countable Borel equivalence relation. Then E is* **amenable** *if there is a sequence of Borel functions (f_n) with $f_n\colon E \to \mathbb{R}$, $f_n \geq 0$ such that letting $f_x^n(y) = f_n(x, y)$, we have $\forall x(f_x^n \in \ell^1([x]_E)$, $\|f_n^x\|_1 = 1)$ and $xEy \implies \|f_x^n - f_y^n\|_1 \to 0$.*

The following is now immediate, see, e.g., [JKL, 2.13]:

Proposition 9.3. *Let G be a countable amenable group and let a be a Borel action of G on a standard Borel space. Then E_a is amenable. In particular any hyperfinite Borel equivalence relation is amenable.*

Conversely we have the following, extending Proposition 8.29 (see, e.g., [JKL, 2.14]):

Proposition 9.4. *Let G be a countable group and let a be a free Borel action of G on a standard Borel space which admits an invariant probability measure. Then if E_a is amenable, G is amenable.*

The following strengthening of Problem 8.31 is also open:

Problem 9.5. *Let E be an amenable countable Borel equivalence relation. Is it true that E is hyperfinite?*

Proposition 9.3 can be generalized as follows for actions of Polish locally compact groups. Below a Borel equivalence relation is called **essentially amenable** (resp., **reducible to amenable**) if it is Borel bireducible (resp., reducible) to an amenable countable Borel equivalence relation.

Proposition 9.6. *Let G be an amenable Polish locally compact group and let a be a Borel action of G on a standard Borel space X. If $S \subseteq X$ is a countable complete Borel section of E_a, then $E_a|S$ is amenable and thus E_a is essentially amenable.*

Proof. Fix a Borel surjective function $\pi : X \to S$ such that $\pi(x) E_a x, \forall x \in X$. Let (F_n) be a sequence as in the definition of the Reiter condition for G. Let $E = E_a|S$ and for xEy, put $A_{xy} = \{g \in G \colon g^{-1} \cdot x \in \pi^{-1}(\{y\})\}$. Then define $f_n \colon E \to \mathbb{R}$ by

$$f_n(x, y) = \int_{A_{xy}} F_n(g) \, \mathrm{d}\lambda(g).$$

Since for each $x \in S$, $\{A_{xy} \colon xEy\}$ is a Borel partition of G, it is clear that $f_n \geq 0$ and $f_x^n \in \ell^1([x]_E), \|f_x^n\|_1 = 1$. Also given xEy, let $g \in G$ be such that $g \cdot x = y$. Then $\|f_x^n - f_y^n\|_1 \leq \|F_n - g \cdot F_n\|_1 \to 0$. $\qquad\square$

Again we have the following generalization of Problem 8.31:

Problem 9.7. *Let G be an amenable Polish locally compact group. Is it true that any Borel action of G generates an essentially hyperfinite equivalence relation?*

As an application of Proposition 9.6 one can give a stronger version of a result proved in [HK2, 5.C]. Let E be the Borel equivalence relation of isomorphism (conformal equivalence) of domains of the form $\mathbb{C}\setminus D$, for D a discrete subset of $\mathbb{C}$. As explained in [HK2, 5.C] this is Borel isomorphic to the equivalence relation induced by the action of the "$az+b$" group (where $a \in \mathbb{C}, a \neq 0, b \in \mathbb{C}$) on the standard Borel space of discrete subsets of $\mathbb{C}$. Since this group is amenable, it follows that this isomorphism relation is essentially amenable. In [HK2, 5.C] it is shown that it is not smooth and is conjectured to be actually essentially hyperfinite.

Another interesting example of an amenable equivalence relation is given in [EH, Theorem 1.2, (i)]. Let G be a countable group with a finite set of generators S, and consider the action of G on the space of ends of this group (for this set of generators) and also its associated action on the space of 2-element subsets of the space of ends. Then if there are infinitely many ends, the equivalence relation generated by the action on the space of 2-element sets is amenable.

9.2   Fréchet amenability

We will now discuss a (possibly) wider notion of amenability, introduced in [JKL, Section 2.4].

Recall that a **free filter** on $\mathbb{N}$ is a filter containing the **Fréchet filter**

$$Fr = \{A \subseteq \mathbb{N} : A \text{ is cofinite}\}.$$

Definition 9.8. *Let E be a countable Borel equivalence relation on a standard Borel space X. Let $\mathcal{F}$ be a free filter on $\mathbb{N}$. We say that E is $\mathcal{F}$-**amenable** if there is a sequence (f_n) of Borel functions $f_n : E \to \mathbb{R}$, $f_n \geq 0$ such that letting $f_x^n(y) = f_n(x,y)$ we have: $\forall x(f_n^x \in \ell_1([x]_E), \|f_n^x\|_1 = 1)$ and $xEy \Rightarrow \|f_x^n - f_y^n\|_1 \to_{\mathcal{F}} 0$.*

As usual if $x_n, x \in \mathbb{R}$, then $x_n \to_{\mathcal{F}} x$ means that for every nbhd U of x there is $A \in \mathcal{F}$ such that $n \in A \Rightarrow x_n \in U$. Clearly $x_n \to_{Fr} x$ iff $x_n \to x$.

Define a quasi-order between filters on $\mathbb{N}$ by

$$\mathcal{F} \leq \mathcal{G} \iff \exists f : \mathbb{N} \to \mathbb{N}(f^{-1}(\mathcal{F}) \subseteq \mathcal{G})$$

and the corresponding equivalence relation

$$\mathcal{F} \equiv \mathcal{G} \iff \mathcal{F} \leq \mathcal{G} \text{ and } \mathcal{G} \leq \mathcal{F},$$

Then if E is $\mathcal{F}$-amenable and $\mathcal{F} \leq \mathcal{G}$, E is $\mathcal{G}$-amenable, so $\mathcal{F}$-amenability only depends on the $\equiv$-equivalence class of $\mathcal{F}$.

Next define a transfinite iteration of the Fréchet filter. For two filters $\mathcal{F}, \mathcal{G}$ on $\mathbb{N}$, define their (Fubini) **product** by

$$\mathcal{F} \otimes \mathcal{G} = \{A \subseteq \mathbb{N} : \{m : \{n : \langle m, n \rangle \in A\} \in \mathcal{G}\} \in \mathcal{F}\},$$

where $\langle m, n \rangle$ is a fixed bijection of $\mathbb{N}^2$ with $\mathbb{N}$. We also define for each sequence $(\mathcal{F}_n)$ of filters, the filter

$$\mathcal{F} \otimes (\mathcal{F}_n) = \{A \subseteq \mathbb{N} : \{m : \{n : \langle m, n \rangle \in A\} \in \mathcal{F}_m\} \in \mathcal{F}\}.$$

For each countable limit ordinal λ, fix an increasing sequence $\alpha_0 < \alpha_1 < \cdots < \lambda$ with limit λ and inductively define the αth **iterated Fréchet filter** Fr_α as follows:

$$
\begin{aligned}
Fr_1 &= Fr, \\
Fr_{\alpha+1} &= Fr \otimes Fr_\alpha, \\
Fr_\lambda &= Fr \otimes (Fr_{\alpha_n}).
\end{aligned}
$$

This definition depends on the choice of $\langle m, n \rangle$ and the sequences (α_n), but it can be shown that it is independent up to $\equiv$.

Definition 9.9. *Let E be a countable Borel equivalence relation and $1 \leq \alpha < \omega_1$ a countable ordinal. We say that E is $\boldsymbol{\alpha}$-**amenable** if E is Fr_α-amenable. It is* **Fréchet-amenable** *if it is α-amenable, for some $1 \leq \alpha < \omega_1$.*

Therefore for any countable Borel equivalence relation E:

$$E \text{ is amenable iff } E \text{ is 1-amenable.}$$

By a simple induction on β,

$$\alpha \leq \beta \Rightarrow Fr_\alpha \leq Fr_\beta$$

and so

$$\alpha \leq \beta \text{ and } E \text{ is } \alpha\text{-amenable} \implies E \text{ is } \beta\text{-amenable}.$$

We also have the analog of Proposition 9.4:

Proposition 9.10 ([JKL, 2.14]). *Let G be a countable group and let a be a free Borel action of G on a standard Borel space which admits an invariant probability measure. Then if E_a is Fréchet-amenable, G is amenable.*

We now have the following closure properties;

Proposition 9.11 ([JKL, 2.15] for (i)-(vi))**.** *Let E, F, E_n be countable Borel equivalence relations and $1 \leq \alpha < \omega_1$. Then we have:*

(i) If F is α-amenable and $E \leq_B^w F$, then E is α-amenable.

(ii) If $E \subseteq F$, E is α-amenable, and every F-equivalence class contains only finitely many E-classes, then F is α-amenable.

(iii) If each E_n is α-amenable, so is $\bigoplus_n E_n$.

(iv) If E, F are α-amenable, so is $E \times F$

(v) If (E_n) is an increasing sequence, and for each n, E_n is α_n-amenable for some $\alpha_n < \alpha$, then $E = \bigcup_n E_n$ is α-amenable.

(vi) If E is α-amenable and $T : E \leq_B^w E$, then $E_t(E, T)$ is $(\alpha + 1)$-amenable.

(vii) If E is α-amenable and a is a Borel action of an amenable countable group by automorphisms of E, then the expansion $E(a)$ is $(\alpha + 1)$-amenable.

In particular, the union of an increasing sequence of hyperfinite Borel equivalence relations is 2-amenable. It is not known if the union of an increasing sequence of α-amenable Borel equivalence relations is α-amenable.

The following are the two basic problems about Fréchet amenability, representing two opposite possibilities.

Problem 9.12. *Is Fréchet amenability equivalent to amenability? Even more, is Fréchet amenability equivalent to hyperfiniteness?*

Problem 9.13. *Is the transfinite hierarchy of Fréchet amenability proper, i.e., does $\alpha < \beta$ imply that there is a β-amenable Borel equivalence relation which is not α-amenable?*

An α-amenable Borel equivalence relation E is **invariantly universal α-amenable** if for every α-amenable Borel equivalence relation F, $F \sqsubseteq_B^i E$. As a special case of a general result, see [CK, 4.4], such a universal equivalence relation exists and it is of course unique up to Borel isomorphism. It will be denoted by $E_{\infty \alpha}$.

9.3   Amenable classes of structures

An important method used to generate Fréchet-amenable countable Borel equivalence relations proceeds through assigning to each equivalence class, in a uniform Borel way, a structure (in the sense of model theory) with

special properties. The more detailed study of structurability of countable Borel equivalence relation will be undertaken in Chapter 14.

Let $L = \{R_i : i \in I\}$ be a countable relational language, where R_i has arity n_i. Let $\mathcal{K}$ be a class of countable structures in L closed under isomorphism. Let E be a countable Borel equivalence relation on a standard Borel space X. An ***L*-structure on *E*** is a Borel structure $\mathbb{A} = \langle X, R_i^{\mathbb{A}} \rangle_{i \in I}$ of L with universe X, i.e., $R_i^{\mathbb{A}} \subseteq X^{n_i}$ is Borel, for each $i \in I$, such that $R_i^{\mathbb{A}}(x_1, x_2, \ldots, x_{n_i}) \implies x_1 E x_2 E \cdots E x_{n_i}$. Then for each E-class C, we have that $\mathbb{A} \restriction C = \langle C, R_i^{\mathbb{A}} \cap C^{n_i} \rangle_{i \in I}$ is an L-structure with universe C. If now $\mathbb{A} \restriction C \in \mathcal{K}$, for each E-class C, we say that $\mathbb{A}$ is a **$\mathcal{K}$-structure** on E. If E admits such a $\mathcal{K}$-structure, we say that E is **$\mathcal{K}$-structurable**.

For each (nonempty) countable set X, we denote by $\mathrm{Mod}_X(L)$ the space of L-structures with universe X. It can be identified with $\prod_i 2^{X^{n_i}}$, so it is a compact metrizable space. A class of countable structures $\mathcal{K}$ as above is **Borel** (resp., **analytic**, **coanalytic**) if for each countable X, $\mathcal{K} \cap \mathrm{Mod}_X(L)$ is Borel (resp., analytic, coanalytic) in $\mathrm{Mod}_X(L)$.

We now consider amenability for classes of structures.

Definition 9.14 ([JKL, Section 2.5]). *(i) An analytic class of L-structures $\mathcal{K}$ is **α-amenable**, where $\alpha \geq 1$ is a countable ordinal, if for each countable set X, there is a family of maps*

$$(f_n^{\mathbb{A}})_{n \in \mathbb{N}, \, \mathbb{A} \in \mathcal{K} \cap \mathrm{Mod}_X(L)},$$

such that

(a) $f_n^{\mathbb{A}} : X \to \mathbb{R}, f_n^{\mathbb{A}} \geq 0, f_n^{\mathbb{A}} \in \ell^1(X)$ and $\|f_n^{\mathbb{A}}\|_1 = 1$;

(b) The map $f_n : (\mathcal{K} \cap \mathrm{Mod}_X(L)) \times X \to \mathbb{R}$ defined by $f_n(\mathbb{A}, x) = f_n^{\mathbb{A}}(x)$ is Borel;

(c) If $\pi : \mathbb{A} \to \mathbb{B}$ is an isomorphism between $\mathbb{A}$ and $\mathbb{B}$, then

$$\|f_n^{\mathbb{A}} - f_n^{\mathbb{B}} \circ \pi\|_1 \to_{Fr_\alpha} 0.$$

*(ii) An arbitrary class $\mathcal{K}$ of L-structures is **Fréchet-amenable** if for any analytic class $\mathcal{K}' \subseteq \mathcal{K}$ there is a countable ordinal α (which may depend on $\mathcal{K}'$) such that $\mathcal{K}'$ is α-amenable. (So, in particular, if $\mathcal{K}$ is analytic, it is Fréchet-amenable iff it is α-amenable for some α.)*

*(iii) A countable L-structure $\mathbb{A}$, with universe A, is **α-amenable** if there is a sequence of maps $f_n : A \to \mathbb{R}$ such that $f_n \geq 0, f_n \in \ell^1(A), \|f_n\|_1 = 1$, and for every $\pi \in \mathrm{Aut}(\mathbb{A}), \|f_n - f_n \circ \pi\|_1 \to_{Fr_\alpha} 0$, where $\mathrm{Aut}(\mathbb{A})$ is the group of automorphisms of $\mathbb{A}$. (This is equivalent to saying that the isomorphism class of $\mathbb{A}$ is α-amenable.)*

Again in this definition, "1-amenable" will be simply called from now on "amenable".

We now have:

Proposition 9.15 ([JKL, 2.18]). *Let E be a countable Borel equivalence relation. If E is $\mathcal{K}$-structurable and $\mathcal{K}$ is Fréchet-amenable, then E is Fréchet-amenable. If moreover $\mathcal{K}$ is analytic and is α-amenable, then E is α-amenable.*

We proceed to describe various Fréchet-amenable classes of countable structures.

Recall that a linear order is **scattered** if it contains no copy of the rational order. The class of scattered linear orders is coanalytic but not Borel. We now have:

Theorem 9.16 ([Ke1], see also [JKL, 2.19]). *The class of countable scattered linear orders is Fréchet-amenable.*

This result was used in [Ke1] to show (assuming that sharps of reals exist) that if one assigns in a Borel way (in the sense described above) a linear order to each Turing degree, then on a cone of Turing degrees this linear order contains a copy of the rationals. Also if E is an aperiodic countable Borel equivalence relation on a Polish space, which is not Fréchet-amenable and admits an ergodic invariant probability measure μ, like for example $F(\mathbb{F}_2, 2)$, then, by Theorem 8.20, on a comeager invariant Borel set one can assign in a Borel way to each E-class a copy of the integer order but for any Borel assignment of a linear order to each E-class, there will be a μ-conull invariant Borel set on which this order contains a copy of the rationals.

Recall that a **tree** $\mathbb{T} = \langle T, R \rangle$ is a connected (undirected) graph with no cycles. Here T is the set of vertices and R is the edge relation. For the concept of the **branching number** of a locally finite $\mathbb{T}$, due to Lyons, see [JKL, 2.20] and references therein. Every locally finite tree of subexponential growth has branching number 1. We now have the following result which was proved in the measure theoretic context in [AL].

Theorem 9.17 ([AL], see also [JKL, 2.21]). *The class of infinite locally finite trees of branching number 1 is amenable.*

The same result also holds for the class of locally finite connected graphs of strongly subexponential growth, see [JKL, 2.22].

One can completely characterize when an infinite locally finite tree $\mathbb{T}$ is Fréchet-amenable.

Theorem 9.18 ([AL], [Ne], see also [JKL, 2.24]). *Let $\mathbb{T}$ be an infinite locally finite tree. Then the following are equivalent:*
(i) $\mathbb{T}$ is Fréchet-amenable;
(ii) $\mathbb{T}$ is amenable;
(iii) $\mathrm{Aut}(\mathbb{T})$ (which is a locally compact Polish group) is amenable.
(iv) One of the following is invariant under $\mathrm{Aut}(\mathbb{T})$; (a) a vertex; (b) the set of two vertices connected by an edge; (c) an end; (d) a line.

Finally we mention the following model-theoretic property of Fréchet-amenable structures. A countable structure $\mathbb{A} = \langle A, \dots \rangle$ has **trivial definable closure** if for every finite $F \subseteq A$ and formula $\phi(x)$ in $L_{\omega_1 \omega}$ with parameters in F, if there is a unique $a \in A$ such that $\mathbb{A} \models \phi(a)$, then $a \in F$.

Theorem 9.19 ([CK, 8.18]). *Let $\mathbb{A}$ be an infinite amenable countable structure. Then $\mathbb{A}$ has non-trivial definable closure.*

9.4   The Connes-Feldman-Weiss Theorem

Let E be a countable Borel equivalence relation on a standard Borel space X and μ a probability measure on X.

We say that E is **μ-amenable** if there is an E-invariant Borel set $A \subseteq X$ with $\mu(A) = 1$ such that $E|A$ is amenable. Similarly we define what it means to say that E is **μ-Fréchet-amenable**.

We will now formulate a number of conditions that turn out to be equivalent to μ-amenability. Then we state the Connes-Feldman-Weiss Theorem which identifies μ-amenability and μ-hyperfiniteness.

(1) First we discuss the following condition, due to Zimmer (see, [Z1, 3.1]), which was the original definition of the concept of μ-amenability. It is motivated by the formulation of amenability for countable groups in terms of a fixed point property of affine actions of the group; see [Z2, 4.1.4].

Let B be a separable real Banach space, $\mathrm{LI}(B)$ the group of its linear isometries, which is Polish under the strong operator topology. Let B_1^* be the closed unit ball of the dual B^*, with the weak*-topology. For $T \in \mathrm{LI}(B)$, denote by T^* the adjoint operator restricted to B_1^*, so that T^* is a homeomorphism of B_1^*. If $\alpha : E \to \mathrm{LI}(B)$ is a Borel cocycle, its adjoint

cocycle α^* (into the homeomorphism group $H(B_1^*)$ of B_1^*) is defined by $\alpha^*(x,y) = (\alpha(x,y)^{-1})^*$.

For each compact metrizable space C, let $K(C)$ be the compact metrizable space of closed subsets of C. A Borel map $x \mapsto K_x$ from X into $K(B_1^*)$ is a **Borel field**, if for all x, K_x is convex and nonempty. A Borel map $S : X \to B_1^*$ is a **section** of K_x, if $S(x) \in K_x$, μ-a.e. (x). A Borel field $(K_x)_{x \in X}$ is $\boldsymbol{\alpha}$**-invariant** if there is an E-invariant Borel set $A \subseteq X$ with $\mu(A) = 1$ such that $x, y \in A, xEy \implies \alpha^*(x,y)(K_x) = K_y$, and a section S is $\boldsymbol{\alpha}$**-invariant** if there is an E-invariant Borel set $A \subseteq X$ with $\mu(A) = 1$ such that $x, y \in A, xEy \implies \alpha^*(x,y)(S(x)) = S(y)$.

We say that E is $\boldsymbol{\mu}$**-Z-amenable** if for every separable real Banach space B and every Borel cocycle $\alpha : E \to \mathrm{LI}(B)$, every α-invariant Borel field $(K_x)_{x \in X}$ has an α-invariant section.

(2) The next condition comes from [CFW]. We say that E is $\boldsymbol{\mu}$**-CFW-amenable** if there is an assignment of means $[x]_E \mapsto \varphi_{[x]_E}$, which is μ-measurable (in the weak sense), i.e., for each Borel map $x \mapsto f_x \in \ell^\infty([x]_E)$, the function $x \mapsto \varphi_{[x]_E}(f_x)$ is μ-measurable. Compare this with the paragraph preceding Theorem 3.13.

Remark 9.20. In [CFW] another equivalent condition is also considered, which postulates the existence of a positive linear operator

$$P \colon L^\infty(E, M_l) \to L^\infty(X, \mu)$$

such that for every Borel map $f \colon A \to B$ in $[[E]]_B$, we have $P(F^f) = P(F)^f$, where for $F \in L^\infty(E, M_l)$, $F^f(x,y) = F(f^{-1}(x), y)$, if $x \in B$; 0, otherwise, while for $F \in L^\infty(X, \mu)$, $F^f(x) = F(f^{-1}(x))$, if $x \in B$; 0, otherwise.

(3) A further condition, formulated by Tucker-Drob and inspired by [Kai] and [El], is motivated by the following equivalent formulation of amenability for finitely generated groups (the **Følner condition**): Let G be a finitely generated group with finite symmetric set of generators S and let $\mathrm{Cay}(G,S)$ be the **Cayley graph** of (G,S). For finite $A \subseteq G$, let $\partial(A) = \{g \in G : g \in (G \setminus A) \ \& \ \exists s \in S \exists h \in A(hs = g)\}$ be the boundary of S in the Cayley graph. The **isoperimetric constant** of the Cayley graph is the infimum of the ratios $\frac{|\partial(A)|}{|A|}$ over all finite nonempty subsets A of G. Then G is amenable iff the isoperimetric constant of the Cayley graph is 0.

Analogously for each locally finite Borel graph $\mathbb{G} = \langle X, R \rangle$ with vertex set X and edge set R, we define for each Borel set $A \subseteq X$, the boundary

of A in $\mathbb{G}$ by $\partial_{\mathbb{G}}(A) = \{x \in X \colon x \in (X \setminus A) \ \& \ \exists y \in A(xRy)\}$. Then the **isoperimetric constant** of $\mathbb{G}$ is the infimum of the ratios $\frac{\mu(\partial(A))}{\mu(A)}$, over all Borel subsets $A \subseteq X$ of positive measure such that the induced subgraph $\mathbb{G} \restriction A = (A, R \cap A^2)$ has finite connected components.

We say that E is **μ-K-amenable** if for any locally finite Borel graph $\mathbb{G} = \langle X, R \rangle$ with $R \subseteq E$ and any Borel set A of positive measure the isoperimetric constant of $\mathbb{G} \restriction A = \langle A, R \cap A^2 \rangle$ (with the normalized probability measure $\mu \restriction A/\mu(A)$) is 0.

(4) The final condition originates in another characterization of the notion of amenability for countable groups: A countable group G is amenable iff every continuous action of G on a compact metrizable space admits an invariant probability measure.

The following condition, due to Furstenberg, see [H6], is as follows. We say that E is **μ-F-amenable** if for any Borel cocycle $\alpha \colon E \to H(K)$, where K is a compact metrizable space and $H(K)$ its homeomorphism group, there is a Borel map $x \mapsto \mu_x$ from X to $P(K)$ such that for some E-invariant Borel set $A \subseteq X$ with $\mu(A) = 1$ we have

$$x, y \in A, xEy \implies \mu_y = \alpha(x, y)_* \mu_x.$$

We now have the following main theorem concerning all these notions:

Theorem 9.21. *Let E be a countable Borel equivalence relation on the standard Borel space X and let μ be a probability measure on X. Then the following are equivalent:*

(i) E is μ-hyperfinite;
(ii) E is μ-amenable;
(iii) E is μ-Fréchet-amenable;
(iv) E is μ-Z-amenable;
(v) E is μ-CFW-amenable;
(vi) E is μ-K-amenable;
(vii) E is μ-F-amenable.

The equivalence of (i), (ii), (iv), (v) is due to [CFW]; see also [Kai]. The equivalence of (i) and (iii) follows from this and the proof of [JKL, 2.13, (ii)]. The equivalence of (i) and (vi) is shown in [Kai] and finally the equivalence of these conditions with (vii) is proved in [H6]. In [M4] there is a simpler proof of the equivalence of (i) and (ii). In [AL, Appendix 1] there

is an exposition of the proof of the equivalence of (iv) and (v) and in [KeL, Theorem 4.72] there is also an exposition of the proof of equivalence of the definition in Remark 9.20 with (i) and (ii), in the case of E-invariant μ. Finally, in the case that μ is E-quasi-invariant, in [Kai] another equivalent condition is formulated, denoted by (IS), which can be viewed as a "local" version of (vi) involving also the cocycle associated with μ. In the case that μ is actually E-invariant, it takes the following simpler form: For any Borel graph $\mathbb{G} = (X, R)$ with $R \subseteq E$ which has bounded degree, the isoperimetric ratio of $\mathbb{G} \upharpoonright [x]_E$ is 0, μ-a.e. (x).

Note that Theorem 9.21, together with Proposition 9.3, implies also Theorem 8.30.

We say that E is **measure-amenable** it is μ-amenable for every probability measure μ. Thus E is measure amenable iff it is measure hyperfinite. It turns out that, assuming the Continuum Hypothesis (CH), this is also equivalent to the condition (2) above but in which one requires that the map $x \mapsto \varphi_{[x]_E}(f_x)$ is universally measurable; see [Ke3] and [JKL, 2.8], where other equivalent conditions are also formulated. The role of CH here comes from the following result used in the proof of this equivalence, due independently to Christensen [Chr] and Mokobodzki (see [DM]).

Theorem 9.22. *Assume* CH. *Then there is a mean φ on $\mathbb{N}$ such that φ assigns the value 0 to every eventually 0 function, and it is universally measurable in the sense that $\varphi \upharpoonright [-1, 1]^{\mathbb{N}} \colon [1, -1]^{\mathbb{N}} \to [-1, 1]$ is universally measurable.*

It is now known, see [La], that this result cannot be proved in ZFC. On the other hand, Christensen and Mokobodzki have also shown, in ZFC alone, that for each probability measure μ on $[-1, 1]^{\mathbb{N}}$, one can find such φ for which $\varphi \upharpoonright [-1, 1]^{\mathbb{N}} \colon [-1, 1]^{\mathbb{N}} \to [-1, 1]$ is μ-measurable.

9.5 Free groups and failure of μ-amenability

A special case of Proposition 8.29 and Theorem 9.21 shows that if a is a free action of the free group $\mathbb{F}_2 = \langle g_1, g_2 \rangle$, where g_1, g_2 are free generators, on a standard Borel space which admits an invariant probability measure μ, then E_a is not μ-amenable. It is shown in [CG, Théoréme 1] that if a admits a quasi-invariant probability measure μ and each generator g_1, g_2 produces a non-smooth subequivalence relation of E_a, then E_a is not μ-amenable. Using this one can show that if $\mathbb{F}_2 \le G$, where G is a Lie group

and $\mathbb{F}_2$ is not discrete, the translation action of $\mathbb{F}_2$ on G is not μ-hyperfinite, where μ is (a probability measure in the measure class of) Haar measure; see [CG, Théoréme 2]. In [Mo, 4.8] the following generalization is proved:

Theorem 9.23 ([Mo, 4.8]). *Let X be a Polish space and μ a measure on X such that the open subsets of X of finite measure generate the topology of X. Suppose that a is a free action of $\mathbb{F}_2$ on X, which is continuous with respect to a metrizable non-discrete topology on $\mathbb{F}_2$. Then E_a is not μ-amenable.*

10. Treeability

10.1 Graphings and treeings

Let $\mathcal{G}$ be the class of all countable connected graphs and $\mathcal{T} \subseteq \mathcal{G}$ the class of all countable trees, i.e., connected acyclic graphs. Clearly these are both Borel classes.

Let E be a countable Borel equivalence relation on a standard Borel space X. A **Borel graphing** of E is a $\mathcal{G}$-structure on E, i.e., a Borel graph $\mathbb{G} = \langle X, R \rangle$ with $E = E_{\mathbb{G}}$. A **Borel treeing** of E is a $\mathcal{T}$-structure on E, i.e., a graphing $\mathbb{G}$ which is acyclic. Graphings and treeings of countable Borel equivalence relations play an important role in the study of countable Borel equivalence relations both in the measure theoretic context, e.g., in the Levitt-Gaboriau theory of costs (see [Le], [Ga1] and [KM1]), and in the Borel theoretic context. Graphings and treeings in the measure theoretic context were introduced in [A1], [A3].

Definition 10.1. *A countable Borel equivalence relation E is* **treeable** *if it admits a Borel treeing.*

Equivalently if $\mathcal{T}$ is the class of countable trees, then E is treeable iff E is $\mathcal{T}$-structurable. The most obvious examples of treeable countable Borel equivalence relations are generated by free actions of free groups. Let $\mathbb{F}_n$ be the countable free group with $n \leq \infty$ generators S and let a be a free Borel action of $\mathbb{F}_n$ on a standard Borel space X. Consider the Borel graph $\mathbb{G} = \langle X, R \rangle$, where $xRy \iff \exists s \in S(s \cdot x = y \text{ or } s \cdot y = x)$. Clearly this is a treeing of E_a. In particular, $F(\mathbb{F}_n, Y)$, for any standard Borel space Y, is treeable but we will see later that $E_\infty \sim_B E(\mathbb{F}_n, 2)$ is not treeable. Also every hyperfinite Borel equivalence relation is treeable but there are treeable countable Borel equivalence relations which are not hyperfinite, like, for example, $F(\mathbb{F}_2, 2)$.

10.2   Equivalent formulations and universality

For compressible countable Borel equivalence relations treeability coincides with generation by free actions of free groups.

Theorem 10.2 ([JKL, 3.16, 3.17]). *Let E be a compressible countable Borel equivalence relation. Then the following are equivalent:*

(i) E is treeable;

(ii) For every free group $\mathbb{F}_n, 2 \leq n \leq \infty$, there is a free Borel action a of $\mathbb{F}_n$ such that $E = E_a$.

(iii) For every countable group G such that $\mathbb{F}_2 \leq G$, there is a free Borel action a of G such that $E = E_a$.

It follows from the theory of cost that there are non-compressible treeable countable Borel equivalence relations which are not generated by free actions of free groups, see, e.g., [KM1, 36.4].

Below we will give a number of equivalent formulations of treeability. We need some definitions first.

Let G be a Polish group and let a be a Borel action of G on a standard Borel space X. We say that this action has the **cocycle property** if there is a Borel cocycle $\alpha\colon E_a \to G$ such that $\alpha(x, y) \cdot x = y$, see [HK1]. Clearly any free action has this property. We say that a countable Borel equivalence relation is **locally finite treeable** it admits a locally finite treeing. We now have:

Theorem 10.3 ([JKL, 3.7, 3.17, 3.12, page 45]). *Let E be a countable Borel equivalence relation.Then the following are equivalent:*

(i) E is treeable;

(ii) E is locally finite treeable;

(iii) For any free group $\mathbb{F}_n, 2 \leq n \leq \infty$, there is a free Borel action a of $\mathbb{F}_n$ such that $E \sim_B E_a$;

(iv) For every countable group G with $\mathbb{F}_2 \leq G$, there is a free Borel action a of G such that $E \sim_B E_a$;

(v) $E \sqsubseteq_B F(\mathbb{F}_2, 2)$;

(vi) $E \leq_B F(\mathbb{F}_2, 2)$;

(vii) For every countable group G and every Borel action a of G with $E = E_a$, the action a has the cocycle property;

(viii) For every countable group G and every Borel action a of G with $E = E_a$, there is a free Borel action b of G with $E \sim_B E_b$.

(ix) For every countable group G and every Borel action a of G with $E \subseteq E_a$, there is a free Borel action b of G such that $E \sqsubseteq_B E_b$.

Further equivalent conditions in the context of groupoids are contained in [L, 8.4.1], which studies functorial Borel complexity for Polish groupoids.

Remark 10.4. The proof of the equivalence of (i) and (ii) in Theorem 10.3 also shows that any countable Borel equivalence relation admits a locally finite graphing. For a stronger statement, see Section 8.7, 9)

From the theory of cost, it follows that there are non-compressible, treeable countable Borel equivalence relations that do not admit a bounded degree treeing. On the other hand, Theorem 10.2, (iii) for the group $G = \mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z}$ shows that every compressible treeable countable Borel equivalence relation has a Borel treeing in which every vertex has degree 3.

A treeable countable Borel equivalence E is **invariantly universal treeable** if for every treeable countable Borel equivalence relation F, $F \sqsubseteq_B^i E$. As a special case of a general result, see [CK, 4.4], such a universal equivalence relation exists and it is of course unique up to Borel isomorphism. It will be denoted by $E_{\infty\mathcal{T}}$. Clearly $E_{\infty\mathcal{T}} \sim_B F(\mathbb{F}_2, 2)$. On the other hand $E_{\infty\mathcal{T}}$ cannot be generated by a free Borel action of a countable group. To see this notice, following [A2], that $E_0 \oplus F(\mathbb{F}_2, 2)$ cannot be generated by any free Borel action of a countable group (because such a group would have to be amenable by Proposition 8.29). In fact from the theory of cost (see e.g., [KM1, 36.4]), it follows that $E_{\infty\mathcal{T}}$ cannot be written as $E_{\infty\mathcal{T}} = \bigoplus_n E_n$, where each E_n is generated by a free Borel action of a countable group G_n.

On the other hand we have the following:

Theorem 10.5 ([H7]). *Let E be a treeable countable Borel equivalence relation on a standard Borel space X and let μ be an E-ergodic, E-invariant probability measure. Then there is a countable group G, a Borel E-invariant set $Y \subseteq X$ with $\mu(Y) = 1$ and a free Borel action a of G on Y with $E_a = E \upharpoonright Y$.*

This fails for some non-treeable relations by a result of Furman [Fu].

It follows from the ergodic decomposition theorem and the proof of Theorem 10.5 that $E_{\infty\mathcal{T}} = \bigoplus_{x \in \mathbb{R}} E_x$, where each E_x is generated by a free action of a countable group G_x.

10.3 Closure properties

The following are the basic closure properties of treeability:

Theorem 10.6 ([JKL, Proposition 3.3]). *Let E, F, E_n be countable Borel equivalence relations. Then we have:*
    (i) If F is treeable and $E \leq_B^w F$, then E is treeable.
    (ii) If each E_n is treeable, so is $\bigoplus_n E_n$.

Although the product of a treeable countable Borel equivalence relation with a smooth one is treeable, it is not true that the product of a treeable countable Borel equivalence relation with a hyperfinite one is treeable, and also it is not true that the union of an increasing sequence of treeable countable Borel equivalence relations is treeable, see Theorem 10.12 below.

The following is a basic open problem:

Problem 10.7. *Let $E \subseteq F$ be countable Borel equivalence relations such that E is treeable and every F-class contains only finitely many E-classes. Is F treeable?*

For several results related to this problem, see [Ts3]

10.4 Treeable equivalence relations and actions of abelian groups

The following result in proved in [Al]:

Theorem 10.8 ([Al, Chapter 4]). *Let E be a treeable countable Borel equivalence relation. Then there is a Borel action a of an abelian Polish group such that $E \leq_B E_a$.*

This gives a negative answer to Problem 8.34. It is unknown if Theorem 10.8 holds for *all* countable Borel equivalence relations.

10.5 Essential and measure treeability

We say that a Borel equivalence relation E is **reducible to treeable** (resp., **essentially treeable**) if it is Borel reducible to a treeable countable Borel equivalence relation (resp., Borel bireducible with a treeable countable Borel equivalence relation). The following is a strengthening of Theorem 4.3.

Theorem 10.9 ([H4], [I2]). *There is a Borel equivalence relation which is reducible to treeable but not essentially treeable.*

The proof proceeds by using the results of [I2] to show that the family of equivalence relations E_S^2, in the notation of the paragraph before Theorem 7.8, satisfies all the conditions of [H4, 0.2] (Adrian Ioana pointed out that the proof of [I2, Corollary 4.4, (1)] can be used to show that this family satisfies condition (iii) in [H4, 0.2].)

On the other hand, by Theorem 4.6, if E is an idealistic Borel equivalence relation, then the following are equivalent:

(i) E is reducible to treeable;

(ii) E is essentially treeable;

(iii) E admits a countable complete Borel section A such that $E \upharpoonright A$ is treeable.

Let E be a countable Borel equivalence relation on a standard Borel space X and let μ be a probability measure on X. Then we say that E is **μ-treeable** if there is a Borel E-invariant set $A \subseteq X$ such that $\mu(A) = 1$ and $E \upharpoonright A$ is treeable. Finally E is **measure treeable** if it is μ-treeable for every probability measure μ. It is unknown if every measure treeable countable Borel equivalence relation is treeable. Problem 10.7 is also open in the case of μ-treeability, even for F-invariant μ.

10.6   Treeings and μ-hyperfiniteness

For each tree $\mathbb{T} = \langle T, R \rangle$, two infinite paths $(x_n), (y_n)$ (without backtracking) in T are **equivalent** if $\exists m \exists n \forall k (x_{m+k} = y_{n+k})$. The boundary $\partial \mathbb{T}$ of $\mathbb{T}$ is the set of equivalence classes of paths. The following result was proved in [A3] for locally finite treeings and in [JKL] in general.

Theorem 10.10 ([A3], [JKL, Section 3.6]). *Let $\mathbb{G}$ be a Borel treeing of a countable Borel equivalence relation E on a standard Borel space X. For each E-class C, let $\mathbb{T}_C = \mathbb{G} \upharpoonright C$. Let μ be an E-invariant probability measure. Then E is μ-hyperfinite iff* $\operatorname{card}(\partial \mathbb{T}_{[x]_E}) \leq 2$, *$\mu$-a.e.* (x).

In fact, it follows from Section 8.7, 9), that if $\mathbb{G}$ is a Borel treeing of a countable Borel equivalence relation E on a standard Borel space X and $\operatorname{card}(\partial \mathbb{T}_{[x]_E}) = 2$ for each $x \in X$, then E is hyperfinite, and from Section 8.7,

1), it follows that if $\mathrm{card}(\partial \mathbb{T}_{[x]_E}) = 1$, for each $x \in X$, then E is also hyper-finite. In particular if $\mathbb{G}$ is locally finite and $\mathrm{card}(\partial \mathbb{T}_{[x]_E}) \leq 2$, for all $x \in X$, then E is hyperfinite (since the set of points x for which $\mathrm{card}(\partial \mathbb{T}_{[x]_E}) = 1$ is Borel,). Also for *any* Borel treeing $\mathbb{G}$ of E and *any* probability measure μ on X, $\mathrm{card}(\partial \mathbb{T}_{[x]_E}) \leq 2$, μ-a.e.(x), implies that E is μ-hyperfinite. The converse is not necessarily true if μ is not E-invariant.

It is unknown whether every treeable countable Borel equivalence relation which is Fréchet amenable is hyperfinite. This is a special case of Problem 9.12.

Bowen and Tserunyan–Tucker-Drob have studied the structure of hyperfinite subequivalence relations of μ-treeable countable Borel equivalence relations. A survey of these results can be found in [Ts6].

10.7 Examples

1) A construction discussed in [JKL, Section 3.2] shows the following. Let G be a Polish group and $\mathbb{A}$ a countable structure with universe A for which there is a Borel action of G on A by automorphisms of $\mathbb{A}$ so that the stabilizer of every point in A in this action is compact. Then for every free Borel action a of G on a standard Borel space X, one can find an $\mathbb{A}$-**structurable** countable Borel equivalence relation E (i.e., $\mathcal{K}$-structurable, where $\mathcal{K}$ is the isomorphism class of $\mathbb{A}$) such that $E_a \sim_B E$.

We say that a Polish group G is **strongly Borel treeable** if for every free Borel action a of G on a standard Borel space, E_a is essentially treeable. Using the inducing construction, see Section 3.3, it follows that a closed subgroup of a strongly Borel treeable group is strongly Borel treeable. See also [CGMT, Appendix B] for more closure properties of the class of strongly Borel treeable countable groups. From [SeT, 1.1] (see also Theorem 11.2 below) it follows that a countable group G is strongly Borel treeable iff $F(G, 2)$ is treeable.

We now have:

Proposition 10.11 ([JKL, Proposition 3.4])**.** *Let G be a Polish group which has a Borel action on a countable tree with compact stabilizers. Then G is strongly Borel treeable.*

In particular, if a countable group G acts on a countable tree with finite stabilizers, then G is strongly Borel treeable. This includes, for example,

groups that contain a free subgroup with finite index, free products of finite cyclic groups, and in particular $\mathrm{PSL}_2(\mathbb{Z}), \mathrm{SL}_2(\mathbb{Z}), \mathrm{GL}_2(\mathbb{Z})$, see [JKL, page 41].

Other examples of strongly Borel treeable groups include $\mathrm{SL}_2(\mathbb{Q}_p)$ and the group of automorphisms $\mathrm{Aut}(\mathbb{T})$ of a locally finite tree $\mathbb{T}$; see [JKL, page 42].

The canonical action of $\mathrm{GL}_2(\mathbb{Z})$ on the two-dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$ is not free by it still generates a treeable (not hypefinite) countable Borel equivalence relation, see [JKL, page 42]. It was also shown in [T3, 5.11] that the action of $\mathrm{GL}_2(\mathbb{Z})$ on $\mathbb{Q}_p \cup \{\infty\}$ by fractional linear transformations generates a treeable but not hyperfinite equivalence relation. Compare this with Section 8.7, 3).

2) Let $\mathcal{K}$ be the class of all rigid locally finite trees and let $\sigma \in L_{\omega_1\omega}$ be a sentence such that $\mathcal{K} = \mathrm{Mod}(\sigma)$. Then $\cong_\sigma$ is essentially treeable, in fact $\cong_\sigma \sim_B E_{\infty\mathcal{T}}$, see [JKL, page 43] and [HK1, pages 241-242].

3) In [Ts5, Corollary 1.5] a sufficient condition for treeability of a countable Borel equivalence relation is given, in connection with graphings and Stallings' Theorem on ends of groups.

4) We call a Polish group G **strongly measure treeable** if for every free Borel action a of G on a standard Borel space X and any probability measure μ on X, there is an invariant Borel set $Y \subseteq X$ with $\mu(Y) = 1$ such that $E_a \upharpoonright Y$ is essentially treeable. It is shown in [CGMT] that the isometry group of the hyperbolic plane $\mathbb{H}^2, \mathrm{PSL}_2(\mathbb{R}), \mathrm{SL}_2(\mathbb{R})$ (and their closed subgroups, in particular surface groups), finitely generated groups with planar Cayley graphs, and elementarily free groups, are all strongly measure treeable.

10.8   Conditions implying non-treeability

1) Product indecomposabilty

The first obstruction to treeability has to do with indecomposability under products of treeable equivalence relations. The following was proved in [A1] in the locally finite case and in [JKL, 3.27] in general. Another proof can be given using the theory of cost, see, e.g., [KM1, 24.9].

Theorem 10.12 ([A3], [JKL, 3.27]). *Let E_1, E_2 be aperiodic countable Borel equivalence relations, let $E = E_1 \times E_2$ and let μ be an E-invariant probability measure. If E is treeable, then E is μ-hyperfinite.*

In particular it follows that $E_0 \times E_{\infty\mathcal{T}}, E_{\infty\mathcal{T}}^2$ are not treeable and thus in particular

$$E_{\infty\mathcal{T}} <_B E_\infty.$$

Also it follows that $\equiv_T, \equiv_A$ are not treeable.

Recall also here Theorem 7.12 that extends these results and determines the relation under Borel reducibility of $R_n = F(\mathbb{F}_2, 2)^n$ (product of the shifts) and $S_n = F(\mathbb{F}^n, 2)$ (shift of the products).

Product indecomposability results for countable Borel equivalence relations generated by free Borel actions of non-amenable hyperbolic groups are contained in [A5, Section 6].

2) Antitreeable groups

Let G be a Polish group. We say that G is **antitreeable** if for every free Borel action a of G on a standard Borel space X, which admits an invariant probability measure, E_a is not essentially treeable.

Remark 10.13. For a countable group G, being antitreeable means that for every free Borel action a of G, if E_a is not compressible, then it is not treeable. The requirement of non-compressibility is necessary, since every countable group G has a free Borel action a with E_a smooth, therefore treeable.

Theorem 10.14 ([AS]). *Let G be an infinite countable group G which has property* (T). *Then G is antitreeable.*

Thus groups such as $\mathrm{SL}_n(\mathbb{Z}), \mathrm{GL}_n(\mathbb{Z}), \mathrm{PSL}_n(\mathbb{Z}), n \geq 3$, are antitreeable. In particular, as opposed to the $n = 2$ case, see Section 10.7, 1), the equivalence relation induced by the canonical action of $\mathrm{GL}_n(\mathbb{Z})$ on the torus $\mathbb{R}^n/\mathbb{Z}^n$, for $n \geq 3$, is not treeable.

Actually one has the following strengthening of Theorem 10.14, as noted in [HK1, 10.5]:

Theorem 10.15 ([AS], [HK1, 10.5]). *Let G be an infinite countable group which has property* (T). *Let a be a Borel action of G on a standard Borel space X and let μ be an E-invariant, E-ergodic probability measure. If F is a treeable countable Borel equivalence relation, then E is μ, F-ergodic.*

Hjorth [H1] used the antitreeabilty of $\mathrm{PSL}_n(\mathbb{Z})$ to show that $\cong_n$, for $n \geq 3$, is not treeable and this was extended in [Ke7] to the case $n = 2$, see below. Thomas (unpublished) has shown that $\cong_n^*$, for $n \geq 3$, is not treeable but it seems to be unknown if this holds in the $n = 2$ case.

Certain products of groups are also antitreeable. Extending results of [Ga1] and [Ke7], the following was shown in [H9].

Theorem 10.16 ([H9, 0.6]). *Let $G = G_1 \times G_2$ be the product of two Polish locally compact, non-compact groups, and assume that G is not amenable. The G is antitreeable.*

Notice that, by the inducing construction, a lattice in a Polish locally compact antitreeable group is also antitreeable. In particular, $\mathrm{PSL}_2(\mathbb{Z}[1/2])$ is antitreeable, being a lattice in $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{Q}_2)$, and this was used in [Ke7] to show that $\cong_2$ is not treeable and in [T3, 5.3] to show that $\cong_2^p$ is not treeable. Hjorth [H1] had earlier shown that $\cong_n^p$ is not treeable for $n \geq 3$.

Also Calderoni in [C, Theorem 1.1] used Theorem 10.14 and [Ke7, Theorem 9], concerning the antitreeability of lattices in products as in the preceding paragraph, to show that for $n \geq 3$ the equivalence relation induced by the action of $\mathrm{SO}_n(\mathbb{Q})$ on S^{n-1} is not treeable.

Remark 10.17. A stronger ergodicity type result, related to Theorem 10.15, for actions of product groups is also proved in [Ke7, Theorem 10].

Remark 10.18. In [HK4, Chapters 6,7] various results are proved to the extent that for certain groups and free Borel actions with invariant probability measure, the associated countable Borel equivalence relation is not Borel reducible even to a finite product of treeable countable Borel equivalence relations.

10.9   Intermediate treeable relations

1) We have seen that there is a simplest non-smooth treeable countable Borel equivalence relation, namely E_0, and a most complex one, namely $E_{\infty \mathcal{T}}$. Thus all non-smooth treeable countable Borel equivalence relations are in the interval $[E_0, E_{\infty \mathcal{T}}]$ in the sense of $\leq_B$. The first problem, already raised in [JKL] was whether this interval is non-trivial, i.e., whether there are **intermediate treeable** countable Borel equivalence relations $E_0 <_B$

$E <_B E_{\infty \mathcal{T}}$. This was answered affirmatively by Hjorth in [H3]. To explain his result we need a definition first.

Let G be a countable group and let a be a Borel action of G on a standard Borel space X. The action a is **modular** if there is a sequence of countable Borel partitions $(\mathcal{P}_n)$ of X, which generates the Borel sets of X, and is such that each $\mathcal{P}_n$ is invariant under the action.

We now have the following result:

Theorem 10.19 ([H3]). *Consider the equivalence relation $F(\mathbb{F}_2, 2) \sim_B E_{\infty \mathcal{T}}$ and the usual product measure μ on $2^{\mathbb{F}_2}$. Then for any Borel invariant set $A \subseteq X$, with $\mu(A) = 1$, and any equivalence relation E_a generated by a modular Borel action a of a countable group G, $F(\mathbb{F}_2, 2) \restriction A \not\leq_B^w E_a$.*

There are free Borel actions a of $\mathbb{F}_2$ with invariant probability measure which are modular (see, e.g., [SlSt]). In fact a countable group G admits a free modular Borel action with invariant probability measure iff it is residually finite; see [Ke9, 1.4]. Thus we have:

Corollary 10.20. *There are intermediate treeable countable Borel equivalence relations $E_0 <_B E <_B E_{\infty \mathcal{T}}$.*

If a is a modular action of G on X and b is a modular action of H on Y, the action of $G \times H$ on $X \times Y$ is also modular. Using this and Theorem 10.12 it also follows that there are products of two treeable countable Borel equivalence relations which are incomparable in the sense of $\leq_B$ with $E_{\infty \mathcal{T}}$.

One can find in [H3] and [Ke9] several other characterizations of modularity as well various examples of modular actions, including the translation action of any countable subgroup $G \leq S_\infty$ on S_∞ and the translation action of $\mathrm{SL}_n(\mathbb{Z})$ on $\mathrm{SL}_n(\mathbb{Z}_p)$.

Let us a call a Borel action a of a countable group **antimodular** if for any modular Borel action b of a countable group, $E_a \not\leq_B^w E_b$. Thus Theorem 10.19 says that the shift action of $\mathbb{F}_2$ restricted to any invariant Borel set of measure 1 is antimodular. This was generalized in [Ke9] by extracting a representation theoretic condition from the proof of Theorem 10.19 that implies antimodularity.

Let a be a Borel action of a countable group G on a standard Borel space X with invariant probability measure μ. The **Koopman representation** associated to a, in symbols $\boldsymbol{\kappa^a}$, is the unitary representation of G on $L^2(X, \mu)$

defined by $g \cdot f(x) = f(g^{-1} \cdot x)$. Its restriction to the orthogonal of the constant functions $L_0^2(X, \mu) = (\mathbb{C}1)^\perp$ is denoted by κ_0^a. The **(left) regular representation** of G is the unitary representation $\boldsymbol{\lambda_G}$ of G on $\ell^2(G)$ defined by $g \cdot f(h) = f(g^{-1}h)$. Also $\boldsymbol{\pi \prec \rho}$ denotes **weak containment** of unitary representations of G, see, e.g., [Ke9, Section 2]. Finally we say that the action a is **tempered** if $\kappa_0^a \prec \lambda_G$. We now have:

Theorem 10.21 ([Ke9, 3.1]). *Let G be a countable group with $\mathbb{F}_2 \leq G$. If a is a Borel action of G on a standard Borel space which admits a nonatomic invariant probability measure, and a is tempered, then a is antimodular.*

Several examples of tempered actions are given in [Ke9, Sections 4, 5] These include the action of $\mathrm{SL}_2(\mathbb{Z})$ on the 2-dimensional torus, which in view of Section 10.7, 1) generates a treeable countable Borel equivalence relation R_2. It turns out also that $R_2 <_B E_{\infty\mathcal{T}}$ (see [CM2] and Theorem 10.26, (ii)). The action of $\mathrm{SL}_n(\mathbb{Z})$ on the n-dimensional torus is not tempered, if $n \geq 3$, but it is still antimodular (even when restricted to an invariant Borel set of measure 1). This is because there is a copy of $\mathbb{F}_2$ in $\mathrm{SL}_n(\mathbb{Z})$ such that its action on the n-dimensional torus is tempered.

These results were further generalized in [ET]. A unitary representation π of a countable group G on a Hilbert space H is called **amenable (in the sense of Bekka)** if there is a bounded linear functional Φ on the C^*-algebra $B(H)$ of bounded linear operators on H such that $\Phi \geq 0$, $\Phi(I) = 1$ and $\Phi(\pi(g)S\pi(g^{-1})) = \Phi(S)$, for every $g \in G, S \in B(H)$. The relevant point here is that for a non-amenable countable group G, λ_G is not amenable and that if $\pi \prec \rho$ and ρ is not amenable, so is π. Thus if G is not amenable and a is a tempered Borel action of G on a standard Borel space with invariant probability measure, then κ_0^a is not amenable. The following result, which is a corollary of a stronger result proved in [ET], generalizes Theorem 10.21:

Theorem 10.22 ([ET, 1.3]). *Let G be a countable group and let a be a Borel action of G on a standard Borel space which admits a nonatomic invariant probability measure. If κ_0^a is non-amenable, then a is antimodular. In particular, Theorem 10.21 is true for any non-amenable group G.*

As a consequence the following is also shown in [ET]:

Theorem 10.23 ([ET, 1.5, 1.6]). *(i) Let the countable group G have property* (T). *Then any weakly mixing action of G is antimodular.*

(ii) Let the countable group G fail the Haagerup Approximation Property (HAP). Then any mixing action of G is antimodular.

2) The next question is whether there are uncountably many incomparable, under Borel reducibility, treeable equivalence relations. Inspired by work in [I1], Hjorth provided a positive answer. In fact he showed the following:

Theorem 10.24 ([H12], see also [Mi11, 6.1]). *Let R_2 be the equivalence relation generated by the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{R}^2/\mathbb{Z}^2$ and let μ be the Lebesgue probability measure on $\mathbb{R}^2/\mathbb{Z}^2$. Then there is a family $(E_r)_{r \in \mathbb{R}}$ of countable Borel equivalence relations such that:*
   (i) $E_r \subseteq E_s \subseteq R_2$, if $r \le s$;
   (ii) E_r is induced by a free Borel action of $\mathbb{F}_2$;
   (iii) If $r \ne s$, then there is no μ-measurable reduction from E_r to E_s.

Recall from Section 10.7, 1) that the equivalence relation R_2 of Theorem 10.24 is treeable.

A streamlined version of Hjorth's work that isolated the key ideas was subsequently developed in [Mi11]. This eventually led to the work in [CM2], [CM3] (see also [Mi15]), in which the following concept was introduced.

Let F be a countable Borel equivalence relation on a standard Borel space Y. For any countable Borel equivalence relation E on a standard Borel space X and probability measure μ on X, consider the space of all (partial) weak Borel reductions $f \colon E \restriction A \le^w_B F$, where $A \subseteq X$ is a Borel set. This carries the pseudometric $d_\mu(f, g) = \mu(D(f, g))$, where if $f \colon A \to Y$ and $g \colon B \to Y$, then $D(f, g) = \{x \in A \cap B \colon f(x) \ne g(x)\} \cup (A \Delta B)$. We also say that E is **μ-nowhere hyperfinite** if there is no Borel set $A \subseteq X$ with $\mu(A) > 0$ and $E \restriction A$ hyperfinite.

Then the countable Borel equivalence relation F is called **projectively separable** if for every E, μ as above such that E is μ-nowhere hyperfinite, the pseudometric d_μ is separable.

The following is then shown in [CM2]:

Theorem 10.25 ([CM2, Theorem B, Proposition 2.3.4]). *(i) The equivalence relation R_2 induced by the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{R}^2/\mathbb{Z}^2$ is projectively separable.*
   (ii) If E, F are countable Borel equivalence relations, $E \le^w_B F$ and F is projectively separable, so is E.

In particular, since $\mathbb{R}R_2$ is not projectively separable, it follows that $R_2 <_B E_{\infty\mathcal{T}}$.

Note that every measure hyperfinite countable Borel equivalence relation is projectively separable but the equivalence relation R_2 is not measure hyperfinite. It is now shown in [CM2] that incomparability, and many other of the complexity phenomena for countable Borel equivalence relations that we have seen earlier, occur among subequivalence relations of *any* countable Borel equivalence relation, which is not measure hyperfinite and projectively separable and treeable, like, for example R_2. Here are some of the main results:

Theorem 10.26 ([CM3, Theorem G, Theorem E, Theorem F, Theorem H]). *Let E be a countable Borel equivalence relation on a standard Borel space X, which is not measure hyperfinite and is projectively separable and treeable. Then the following hold:*

(i) There is a family $(E_r)_{r\in\mathbb{R}}$ of pairwise incomparable under measure reducibility $\leq_M$ countable Borel subequivalence relations of E such that $r \leq s \implies E_r \subseteq E_s$;

(ii) $\mathbb{R}E \not\leq_M F$, for every Borel subequivalence relation $F \subseteq E$, and in particular $E <_M E_{\infty\mathcal{T}}$;

(iii) For some Borel set $A \subseteq X$, if $F = E \restriction A$, then for each $n \geq 1$, $nF <_M (n+1)F$;

(iv) If moreover E is aperiodic and the failure of measure hyperfiniteness for E is witnessed by an invariant probability measure, then there is an aperiodic Borel subequivalence relation $F \subseteq E$ such that for every $n \geq 1$, $F \times I_n \sqsubset_M F \times I_{n+1}$.

In Theorem 10.26, compare (i) with Theorem 10.24 and Theorem 3.38; (ii) with Corollary 10.20; (iii) with Theorem 7.20; (iv) with Theorem 3.33.

In [CM3] the authors study the situations under which certain such results hold for subequivalence relations induced by free actions of free groups.

3) The first explicit examples of uncountably many incomparable, under Borel reducibility, treeable countable Borel equivalence relations were constructed in [I2]. As usual, we view below $\mathrm{SL}_2(\mathbb{Z})$ as a dense subgroup of the compact group $H_S^2 = \prod_{p\in S} \mathrm{SL}_2(\mathbb{Z}_p)$, for any nonempty set of primes S. Also for any $G \leq \mathrm{SL}_2(\mathbb{Z})$, we let $K_{G,S}$ be the closure of G in H_S^2. Thus $K_{\mathrm{SL}_2(\mathbb{Z}),S} = H_S^2$. We denote by $E_{G,S}$ the equivalence relation induced by the translation action of G on $K_{G,S}$. Thus $E_{\mathrm{SL}_2(\mathbb{Z}),S} = E_S^2$, in the notation of the paragraph before Theorem 7.8.

Theorem 10.27 ([I2, Corollary C]). *Let $S \neq T$ be nonempty set of primes and $G, H \leq \mathrm{SL}_2(\mathbb{Z})$ be non-amenable. Then $E_{G,S}, E_{H,T}$ are treeable and incomparable in $\leq_B$.*

Recall from Section 10.7, 1), that the equivalence relation F_p induced by the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{Q}_p \cup \{\infty\}$ by fractional linear transformations is treeable. We now have:

Theorem 10.28 ([I2, Corollary D]). *Let p, q be primes. Then*

$$p = q \iff F_p \leq_B F_q.$$

10.10 Contractible simplicial complexes

We will discuss here a higher dimension generalization of treeability. For $n \geq 1$, a **simplicial complex** is a countable set X together with a collection S_k of subsets of X of cardinality $k + 1$, for each $k \in \mathbb{N}$, such that all singletons from X are in S_0 and every subset of an element of S_k of cardinality $m + 1 \leq k + 1$ belongs in S_m. (The elements of S_0 are called vertices, the elements of S_1 are called edges, etc.). We say that a simplicial complex $\mathbb{K}$ is **contractible** if its geometric realization is contractible (see, e.g., [Ch1, Section 2]). We can view each simplicial complex as a structure in some language L and then we let $\mathcal{C}$ be the class of all countable contractible simplicial complexes, which is a Borel class. A simplicial complex $\mathbb{K}$ is **n-dimensional** if $S_n \neq \emptyset$ but $S_m = \emptyset$ for $m > n$. We denote by $\mathcal{C}_n$ the class of all countable contractible n-dimensional simplicial complexes (which is again a Borel class).

For $n = 1$, $\mathcal{C}_1$ coincides with the class $\mathcal{T}$ of all countable trees. Thus $\mathcal{C}_1$-structurability coincides with treeability and $\mathcal{C}_n$-structurability, $n \geq 2$, can be considered as a higher dimensional analog of treeability. For example, any equivalence relation induced by a free Borel action of $(\mathbb{F}_2)^n$, for $n \geq 1$, is $\mathcal{C}_n$-structurable. The $\mathcal{C}_n$-structurable countable Borel equivalence relations, in a measure theoretic context, play a important role in Gaboriau's theory of ℓ^2 Betti numbers, see [Ga2]. From his work it follows that for each $n \geq 1$, if E_{n+1} is induced by a free Borel action of $(\mathbb{F}_2)^{n+1}$ with invariant probability measure, then E_{n+1} is $\mathcal{C}_{n+1}$-structurable but not $\mathcal{C}_n$-structurable, in fact E_{n+1} is not even Borel reducible to a $\mathcal{C}_n$-structurable

countable Borel equivalence relation (see also [HK4, Appendix D] and [Ke11, Section 7] here).

Again as a special case of a general result, see [CK, 4.4], there is an invariantly universal $\mathcal{C}_n$-structurable countable Borel equivalence relation, denoted by $E_{\infty \mathcal{C}_n}$ (so that $E_{\infty \mathcal{C}_1} = E_{\infty \mathcal{T}}$). Thus $E_{\infty \mathcal{C}_n} <_B E_{\infty \mathcal{C}_{n+1}}$, for each $n \geq 1$.

We have seen in Section 10.2 that every compressible treeable countable Borel equivalence relation admits a Borel treeing in which every vertex has degree 3. We now have the following higher dimensional analog:

Theorem 10.29 ([Ch1, Corollary 2]). *Let E be a compressible countable Borel equivalence relation which is $\mathcal{C}_n$-structurable. Then E admits a $\mathcal{C}_n$-structure in which every vertex belongs to exactly $2^{n-1}(n^2 + 3n + 2) - 2$ edges.*

There are many open problems concerning the class of $\mathcal{C}_n$-struturable Borel equivalence relations. For example, it is not known whether if F is $\mathcal{C}_n$-structurable and $E \sqsubseteq_B F$, then E is $\mathcal{C}_n$-structurable, when $n \geq 2$. For this and other open problems, see [Ch1, Section 4].

Call a simplicial complex **locally finite** if every vertex belongs to finitely many edges. Then we have:

Theorem 10.30 ([Ch2, Corollary 5]). *Every compressible countable Borel equivalence relation admits a $\mathcal{C}$-structure which is locally finite.*

11. Freeness

11.1 Free actions and equivalence relations

Definition 11.1. *A countable Borel equivalence relation E on a standard Borel space X is called* **free** *if there is a countable group G and a free Borel action a of G such that $E = E_a$.*

As in Section 6.3, for any infinite countable group G and any free Borel action a of G, we have that $E_a \sqsubseteq^i_B F(G, \mathbb{R})$. Actually by [JKL, 5.4], we in fact have that there is a Borel embedding of the action a into $s_{G,\mathbb{N}}$, so $E_a \sqsubseteq^i_B F(G, \mathbb{N})$ and therefore $F(G, \mathbb{R}) \cong_B F(G, \mathbb{N})$. For $G = \mathbb{Z}$ and more generally for any infinite countable group G for which all of its Borel actions generate a hyperfinite relation, we have, as it follows from Corollary 8.5 and Corollary 8.7, that actually $F(G, \mathbb{R}) \cong_B F(G, 2)$. On the other hand it was shown in [T14, 6.3] that for $G = \mathrm{SL}_3(\mathbb{Z})$ (and other groups) $F(G, 2) <_B F(G, 3) <_B \cdots <_B F(G, \mathbb{N})$. Apparently it is unknown if this also holds for $G = \mathbb{F}_2$.

Despite this, there is still a close relationship of E_a, for a free action a of G, with $F(G, 2)$ in view of the following theorem.

Theorem 11.2 ([SeT, 1.1]). *Let G be an infinite countable group and let a be a free Borel action of G on a standard Borel space X. Then there is a Borel homomorphism $f \colon X \to 2^G$ from a to $s_{G,2}$ such that $\overline{f(X)} \subseteq F(2^G)$.*

In particular, $f \colon E_a \to_B F(G, 2)$ and $f \restriction C$ is a bijection of every E-class C with the $F(G, 2)$-class $f(C)$.

Every aperiodic hyperfinite Borel equivalence relation is clearly free. Moreover, by [DJK, 11.2], for every for compressible hyperfinite Borel equivalence relation E and every infinite countable group G, there is a free Borel action a of G with $E = E_a$. In general if E is a compressible countable Borel

equivalence relation and $E = E_a$ for a free Borel action of a countable group G, then for any countable group $H \geq G$, there is a free Borel action b of H such that $E = E_b$, see [DJK, 11.1]. We have seen in Theorem 10.2 that in fact every compressible treeable countable Borel equivalence relation is free but we have also seen in Section 10.2 that there are treeable countable Borel equivalence relations that are not free.

We say that a countable Borel equivalence relation E is **reducible to free** if there is a free countable Borel equivalence relation F such that $E \leq_B F$ and **essentially free** if there is a free countable Borel equivalence relation F such that $E \sim_B F$. It is shown in [JKL, 5.13] that E is reducible to free iff E is essentially free. Also E is essentially free iff $E \times I_{\mathbb{N}}$ is free; see [JKL, 5.11]. Moreover the class of essentially free countable Borel equivalence relations has the following closure properties.

Proposition 11.3 ([JKL, 5.13]). *Let E, F, E_n be countable Borel equivalence relations. Then we have:*

(i) If F is essentially free and $E \leq_B^w F$, then E is essentially free.
(ii) If E, F, E_n are essentially free, so are $\bigoplus_n E_n$, $E \times F$;

The following is an open problem.

Problem 11.4. *Let $E \subseteq F$ be countable Borel equivalence relations such that each F-class contains only finitely many E-classes. If E is essentially free, is F also essentially free?*

From Theorem 10.3 it follows that every treeable countable Borel equivalence relation is essentially free (but there are essentially free countable Borel equivalence relations that are not treeable, see, e.g., Theorem 10.12). The question of whether *every* countable Borel equivalence relation is essentially free was raised in [DJK, Section 11]. It was shown in [T12], using the Popa cocycle superrigidity theory, that this is not the case.

Theorem 11.5 ([T12, 3.9]). *Let E be an essentially free countable Borel equivalence relation. Then there is a countable group G such that $F(G, 2) \not\leq_B E$.*

Below by a universal essentially free countable Borel equivalence relation we mean an essentially free countable Borel equivalence relation E such that for any essentially free countable Borel equivalence relation F, we have $F \leq_B E$.

Corollary 11.6. *There is no universal essentially free countable Borel equivalence relation. In particular, E_∞ and $\equiv_T$ are not essentially free.*

It follows, for example from Theorem 7.6, that there are continuum many pairwise incomparable under $\leq_B$ free countable Borel equivalence relations. It is also shown in [T12, 3.13] that there continuum many incomparable under $\leq_B$ countable Borel equivalence relations, which are not essentially free. In [T14, 5.2] it is shown that if $G = B(m,n)$ is the free m-generator Burnside group of exponent n, then, for sufficiently large odd n, $E(G, 2)$ is not essentially free.

Finally [T12, 6.3] raises the question of whether $\cong_n$, for $n \geq 2$, is essentially free.

If E is a countable Borel equivalence relation on a standard Borel space X and μ is a probability measure on X, we say that E is **μ-free,** resp., **μ-essentially free** if there is an E-invariant Borel set $A \subseteq X$ with $\mu(A) = 1$ such that $E \restriction A$ is free, resp. essentially free.

By Theorem 10.5 every treeable countable Borel equivalence relation E is μ-free for any E-invariant, E-ergodic probability measure μ. In [Fu] examples are given of countable Borel equivalence relations for which this fails. Finally in [H8] an example is constructed of a countable Borel equivalence relation E on a standard Borel space with E-invariant, E-ergodic probability measure μ such that for every E-invariant Borel set $A \subseteq X$ with $\mu(A) = 1$, $E \restriction A$ is not essentially free.

11.2   Everywhere faithfulness

We now consider a weakening of the notion of free action. Let a be an action of a group G on a space X. Then a is called **everywhere faithful** if the action of G on every orbit is faithful, i.e., for every $g \neq 1_G$ and every orbit C, there is $x \in C$ such that $g \cdot x \neq x$.

It is shown in [Mi6, Page 1] that for every group G, not isomorphic to $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z}$, of the form $G = H \star K$, with H, K countable non-trivial groups, every compressible countable Borel equivalence relation is generated by an everywhere faithful action of G. Moreover, as a special case of [Mi6, Theorem 20], it is shown that if G_n are non-trivial countable groups which are residually amenable, then every aperiodic countable Borel equivalence relation is generated by an everywhere faithful Borel action of $\star_n G_n$.

12. Universality

12.1 Structural results

Recall that a countable Borel equivalence relation E is universal if for every countable Borel equivalence relation F, $F \leq_B E$ or equivalently $E \sim_B E_\infty$. We will first discuss some structural properties of such equivalence relations.

The first result shows a stronger property enjoyed by all universal relations.

Theorem 12.1 ([MSS, 3.6]). *Let E be a universal countable Borel equivalence relation. Then for every countable Borel equivalence relation F, $F \sqsubseteq_B E$.*

In [JKL, 6.5, (C)] it was asked whether every universal countable Borel equivalence relation is indivisible, in the sense that in any partition of the space into two disjoint invariant Borel sets the restriction of the equivalence relation to one of these sets is still universal. In fact a much stronger statements turns out to be true.

Theorem 12.2 ([MSS, 3.1, 3.8]). *(i) Let E be a universal countable Borel equivalence relation on a standard Borel space and let $f \colon E \to_B \Delta_Y$, for some standard Borel space Y. Then for some $y \in Y$, $E \upharpoonright f^{-1}(y)$ is universal.*

(ii) Let E be a universal countable Borel equivalence relation on a standard Borel space X and let $X = \bigsqcup_n X_n$ be a Borel partition of X (where the X_n might not be E-invariant). Then for some n, $E \upharpoonright X_n$ is universal.

The next result, answering a question in [T12, 3.22], shows that universality is always present in null sets.

Theorem 12.3 ([MSS, 3.10]). *Let E be a universal countable Borel equivalence relation on a standard Borel space X and let μ be a probability measure on X.*

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Then there is an E-invariant Borel set $A \subseteq X$ with $\mu(A) = 0$ such that $E \upharpoonright A$ is still universal.

An alternative way to prove Theorem 12.2 was developed in [M3, Section 4]. In [M3, Section 4.2] a countably complete ultrafilter U on the σ-algebra of E_∞-invariant Borel sets is constructed, reminiscent of the Martin ultrafilter of invariant under $\equiv_T$ Borel sets (where such a Borel set is in the ultrafilter iff it contains a cone of Turing degrees). It is shown that for every set $A \in U$, we have $E_\infty \sim_B E \upharpoonright A$, i.e., $E \upharpoonright A$ is also universal. This implies immediately Theorem 12.2, (i). Using the definition of U, which involves infinite games, one can also prove Theorem 12.2, (ii).

12.2   Manifestations of universality

Below we call a Borel equivalence relation E **essentially universal countable** if $E \sim_B E_\infty$. We will next discuss universal and essential universal equivalence relations that occur in several different areas.

1) Computability theory

Slaman and Steel proved that the arithmetical equivalence relation is universal.

Theorem 12.4 (see [MSS]). *The equivalence relation $\equiv_A$ is universal.*

The universality of Turing equivalence is an open problem.

Problem 12.5. *Is $\equiv_T$ universal?*

Note that Martin's Conjecture implies a negative answer to this problem. In fact Martin's Conjecture easily implies that we cannot even have $\equiv_T \sim_B 2(\equiv_T)$.

We will next discuss some refinements of Turing equivalence that give universal relations. Consider the Polish space $k^{\mathbb{N}}$, where

$$k \in \{2, 3, \ldots n, \ldots, \} \cup \{\mathbb{N}\}.$$

Then the group S_∞ acts on $k^{\mathbb{N}}$ by shift $g \cdot x(n) = x(g^{-1}(n))$ and so does (by restriction) any countable subgroup $G \leq S_\infty$. We denote by $\cong_G^k$ the equivalence relation induced by the shift action of G on $k^{\mathbb{N}}$. Although the

results below hold for many other countable groups G, we are primarily interested here in the case where G is the group of all recursive permutations of $\mathbb{N}$, in which case we write $\cong_{rec}^k$ instead of $\cong_G^k$. In particular $\cong_{rec}^2$ is the usual notion of recursive isomorphism of subsets of $\mathbb{N}$.

It was shown in [DK] that $\cong_{rec}^{\mathbb{N}}$ is universal and in [ACH] that $\cong_{rec}^5$ is universal. Finally this was improved to the following:

Theorem 12.6 ([M3, 1.6]). *The equivalence relation $\cong_{rec}^3$ is universal.*

Surprisingly the following is still open.

Problem 12.7. *Is $\cong_{rec}^2$ universal?*

Call a countable Borel equivalence relation E **measure universal** if for every countable Borel equivalence relation F on a standard Borel space X and every probability measure on X, there is an F-invariant Borel set $A \subseteq X$ with $\mu(A) = 1$ such that $F \restriction A \leq_B E$. It is unknown if measure universality implies universality. It is shown in [M3, 1.7] that $\cong_{rec}^2$ is measure universal. Moreover it is also shown in that paper that the problem of the universality of $\cong_{rec}^2$ is related to problems in Borel graph combinatorics and leads the author to conjecture that the answer to Problem 12.7 is negative; see the discussion in [M3, pages 5-6]

In [M2] computational complexity refinements of Turing equivalence are shown to be universal, including the following:

Theorem 12.8 ([M2, 1.1]). *Let $\cong_T^P$ be polynomial time Turing equivalence. Then $\cong_T^P$ is universal.*

Consider now a Borel class $\mathcal{K}$ of countable structures, closed under isomorphism, in a countable relational language L. Then for some sentence σ in $L_{\omega_1\omega}$, we have $\mathcal{K} \cap \mathrm{Mod}_{\mathbb{N}}(L) = \mathrm{Mod}(\sigma)$. Let then $\cong_{\mathcal{K}} = \cong_\sigma$ be the isomorphism relation for the structures in $\mathcal{K}$. This is induced by the logic action of S_∞ on $\mathrm{Mod}(\sigma)$. Consider again the restriction of this action to the subgroup of recursive permutations of $\mathbb{N}$ and denote by $\cong_{\mathcal{K}}^{rec} = \cong_\sigma^{rec}$ the induced equivalence relation, i.e., the relation of **recursive isomorphism** for the structures in $\mathcal{K}$. Again, see Remark 4.22, in case we consider structures in a language with function symbols, we replace them by their graphs.

In [ACH] and [Ca] various recursive isomorphism relations are shown to be universal, including the following:

Theorem 12.9. *(i)* [ACH, 3.8] *Let $\mathcal{K}$ be the class of unary functions that are permutations. Then $\cong_{\mathcal{K}}^{rec}$ is universal. Similarly for the class of equivalence relations.*

(ii) [Ca] *Let $\mathcal{K}$ be one of the following classes: trees, groups, Boolean algebras, fields, linear orders. Then $\cong_{\mathcal{K}}^{rec}$ is universal.*

In particular, the first part of Theorem 12.9, (i) says that the equivalence relation induced by the conjugacy action of the group of recursive permutations on S_∞ is universal.

2) Isomorphism of countable structures

We will now consider the isomorphism relation $\cong_{\mathcal{K}}$ of various classes of structures $\mathcal{K}$.

Theorem 12.10. *The isomorphism relations of the following classes of structures are essentially universal countable:*

(i) ([TV1]) *Finitely generated groups;* ([H11]) *2-generated groups;*

(ii) ([TV2]) *Fields of finite transcendence degree over the rationals;*

(iii) ([JKL, 4.11]) *Locally finite trees.*

Two finitely generated groups G, H are **commensurable** if they have finite index subgroups which are isomorphic. This is a Borel equivalence relation, which is essentially universal countable, see [T10, Theorem 1.1].

3) Groups

For every countable group G, denote by $E_{conj}(G)$ the equivalence relation of conjugacy in the (compact metrizable) space of all subgroups of G. In [TV1] it was shown that $E_{conj}(\mathbb{F}_2)$ is universal. Later it was shown that this holds for all groups containing $\mathbb{F}_2$.

Theorem 12.11 ([ACH, 1.3]). *The equivalence relation $E_{conj}(G)$ is universal for all countable groups G such that $\mathbb{F}_2 \leq G$.*

In [G2] it was shown that for every countable group G and any nontrivial cyclic group H, $E(G, 2) \leq_B E_{conj}(G \star H)$ and this gives another proof of Theorem 12.11 for $\mathbb{F}_2$.

Various results about the equivalence relation $E_{conj}(G)$ are proved in [T14, Section 4], In particular it is shown in [T14, 4.7] that there are continuum many countable groups G for which the equivalence relations $E_{conj}(G)$ are essentially free and pairwise $\leq_B$-incomparable, thus non-universal.

In [TV1, Theorem 8] it is shown that the equivalence relation induced by the action of $\mathrm{Aut}(\mathbb{F}_5)$ on the space of subgroups of $\mathbb{F}_5$ is also universal.

For a left-orderable countable group G, let $\mathrm{LO}(G)$ be the compact metrizable space of all left-orderings of G. The group G acts continuously by conjugation on $\mathrm{LO}(G)$ and the corresponding equivalence relation $E_{lo}(G)$ is studied in [CC], where in particular it is shown that $E_{lo}(\mathbb{F}_n)$ is universal, for all $n \geq 2$.

Finally, in a different direction, consider the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{Z}^2$. This induces a shift action of $\mathrm{SL}_2(\mathbb{Z})$ on $2^{\mathbb{Z}^2}$. It is shown in [G1] that this shift action generates a universal relation.

4) Topological dynamics

For a countable group G consider now the isomorphism relation $E^k_{ssh}(G)$ of subshifts of k^G, $k \geq 2$, which is defined as in Section 8.7, 6). In [Cl2] it was shown that $E^k_{ssh}(\mathbb{Z}^n)$ is universal. This was extended in [GJS2, 9.4.9] and [Cl3, Section 2] to show the following:

Theorem 12.12 ([GJS2, 9.4.9], [Cl3, Section 2]). *Let G be a countable group which is not locally finite. Then $E^k_{ssh}(G)$ is universal. The same holds for the restriction of $E^k_{ssh}(G)$ to free subshifts.*

However, as shown in [GJS2, 9.4.3], $E^k_{ssh}(G) \sim_B E_0$, if G is an infinite countable group which is locally finite, so we have here a strong dichotomy.

5) Riemann surfaces and complex domains

Let $\cong_R$ be the isomorphism (conformal equivalence) relation of Riemann surfaces and let $\cong_D$ be its restriction to complex domains (open connected subsets of $\mathbb{C}$), in an appropriate standard Borel space of parameters for Riemann surfaces and domains, see [HK2, Section 3]. Then we have:

Theorem 12.13 ([HK2, 4.1]). *The equivalence relations $\cong_R$ and $\cong_D$ are essentially universal countable.*

In fact the same holds even if one restricts the isomorphism relation to complex domains of the form $\mathbb{H} \setminus S$, where S is a discrete subset of the upper half plane $\mathbb{H}$. On the other hand we have seen in Section 9.1 that the isomorphism relation on domains of the form $\mathbb{C} \setminus S$, where S is a discrete subset of $\mathbb{C}$, is essentially amenable, so it is not essentially universal countable. It is also shown in [HK2, 5.2] that the conjugacy equivalence relation

on the space of discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$ is also essentially universal countable.

6) Isometric classification

Recall here Section 4.6. Concerning Theorem 4.27 we actually have the following:

Theorem 12.14 (Hjorth; see [GK, 7.1]). *Let $\mathcal{M}$ be the class of proper Polish spaces. Then the relation $\cong_{\mathcal{M}}^{iso}$ is an essentially universal countable Borel equivalence relation. The same holds for any Borel class $\mathcal{M}$ of connected, locally compact Polish metric spaces that contains all connected, proper Polish metric spaces.*

7) Universal countable quasi-orders

Let Q be a Borel quasi-order on a standard Borel space X. We say that Q is **countable** if for each $x \in X$ the set $\{y \in X \colon yQx\}$ is countable. In particular a countable Borel equivalence relation is a countable quasi-order. For each countable Borel quasi-order we associate the countable Borel equivalence relation $xE_Qy \iff xQy \ \& \ yQx$. As with equivalence relations, if Q, Q' are quasi-orders on standard Borel spaces X, X', resp., then we let $f \colon Q \leq_B Q'$ denote that $f \colon X \to Y$ is a Borel function such that $xQy \iff f(x)Q'f(y)$ and we say that Q is **Borel reducible** to Q', in symbols $f \colon Q \leq_B Q'$, if such an f exists. Note that if $Q \leq_B Q'$, then $E_Q \leq_B E_{Q'}$. Borel reducibility for countable Borel quasi-orders was studied in [W1] where the following analogs of results for countable Borel equivalence relations were proved.

If a is a Borel action of a countable monoid S on a standard Borel space X such that for every $s \in S$, the map $x \mapsto s \cdot x$ is countable-to-1, we let Q_a be the countable Borel quasi-order defined by $x \ Q_a \ y \iff \exists s(s \cdot x = y)$. For a countable monoid S and standard Borel space X, we let $s_{S,X}$ be the **shift action** of S on X^S given by $(s \cdot p)_t = p_{ts}$. Let $Q(S, X) = Q_{s_{S,X}}$. We also let $E(S, X)$ be the associated equivalence relation.

We say that a countable Borel quasi-order Q is **universal** if for every countable Borel quasi-order R we have $R \leq_B Q$. Then E_Q is a universal countable Borel equivalence relation. Finally, let $\mathbb{S}_\infty$ is the free monoid with a countably infinite set of generators.

The following is an analog of Theorem 3.3.

Theorem 12.15 ([W1, 2.1]). *If Q is a countable Borel quasi-order on a standard Borel space X, then there is a countable monoid S and a Borel action A of S on X such that $Q = Q_a$.*

Next we have an analog of E_∞. Below let $Q_\infty = Q(\mathbb{S}_\infty, \mathbb{R})$.

Theorem 12.16 ([W1, 2.4]). *The quasi-order Q_∞ is a universal countable Borel quasi-order. In particular $E(\mathbb{S}_\infty, \mathbb{R})$ is a universal countable Borel equivalence relation.*

Finally we have the following, where we consider the relations as being defined in the space of finitely generated groups, defined for example in [T8, Section 2]; see also Remark 4.25.

Theorem 12.17 ([W1, 1.6]). *The embeddability quasi-order for finitely generated groups is a universal countable quasi-order. In particular, the bi-embeddability equivalence relation for finitely generated groups is a universal countable Borel equivalence relation.*

In view of Problem 12.5 it is a natural question to ask whether $\leq_T$ is a universal countable Borel quasi-order. However it was recently shown by P. Lutz and B. Siskind that this is not the case (unpublished but see [LS]).

8) Action universality

We have seen in Proposition 6.9 that $E(\mathbb{F}_2, 2)$ is universal and since for any $G \leq H$ we have that $E(G, X) \leq_B E(H, X)$, it follows that for any countable group G which contains a copy of $\mathbb{F}_2$, $E(G, 2)$ is universal. It is unknown if there are any other countable groups for which $E(G, 2)$ is universal. More generally, following [T14], call a countable group G **action universal** if there is a Borel action a of G with E_a universal. Then it is unknown if there are action universal groups that do not contain $\mathbb{F}_2$. Clearly no amenable group can be action universal. It is shown in [T14, 1.6] that there are countable non-amenable groups that are not action universal.

9) Generators and invariant universality

Given a Borel action a of a countable group G on a standard Borel space X and $n \in \{2, 3, \ldots, \mathbb{N}\}$ an **n-generator** is a Borel partition $X = \bigsqcup_{i<n} A_i$ of X such that $\{g \cdot A_i \colon g \in G, i < n\}$ generates the Borel sets in X. Equivalently such a generator exists iff the action a can be Borel embedded into the shift action of G on n^G. It is shown in [JKL, 5.4] that for every Borel action a of a countable group G for which E_a is aperiodic, there is an $\mathbb{N}$-generator. For every equivalence relation E on a set X, let E^{ap} be the **aperiodic part** of E, i.e., $E^{ap} = E \restriction X_E^{ap}$, where $X_E^{ap} = \{x \in X \colon [x]_E \text{ is infinite}\}$. If $E = E_a$ as above, then the aperiodic part of a is the action of G on $X_{E_a}^{ap}$.

Then E_a^{ap} is the associated equivalence relation. For the case of the shift action of G on X^G, denote by $E^{ap}(G, X)$ the aperiodic part of the associated equivalence relation.

Thus we have seen that for any Borel action a of a countable group G, the aperiodic part of a can be embedded in the aperiodic part of the shift action on $\mathbb{N}^G$. As a consequence, $E_a^{ap} \sqsubseteq_B^i E^{ap}(G, \mathbb{N})$, and therefore $E^{ap}(G, \mathbb{N})$ is invariantly universal for all aperiodic countable Borel equivalence relations. In particular, $E^{ap}(G, \mathbb{N}) \cong_B E^{ap}(G, \mathbb{R})$.

Because of entropy considerations, even for the group $G = \mathbb{Z}$ it is not the case that every Borel action of $\mathbb{Z}$ with an invariant probability measure admits a finite generator. The following open problem was raised in [We2]: Is it true that every Borel action a of $\mathbb{Z}$ with E_a compressible has a finite generator? In [JKL, 5.7] this question was extended to actions of arbitrary countable groups.

Recall that any Borel action of a countable group on a standard Borel space is Borel isomorphic to a continuous action of the group on a Polish space, so it is enough to consider this problem for continuous actions. In [Ts4] an affirmative answer (with a 32-generator) was obtained for any continuous action of a countable group on a σ-compact Polish space. Moreover it was shown that for any countable group and *any* continuous action of G on a Polish space with infinite orbits, there is a comeager invariant Borel set on which the action has a 4-generator. Later in [Ho], and by different methods, the original problem of Weiss, i.e., the case $G = \mathbb{Z}$, was shown to have a positive answer with a 2-generator. More recently Hochman-Seward (unpublished) have extended this to arbitrary countable groups and thus have solved Weiss' problem in complete generality for *all* countable groups..

In [FKSV] the question was considered of whether it is true that we have $E^{ap}(G, \mathbb{R}) \cong_B E^{ap}(G, 2)$. If this happens then the group G is called **2-adequate**.

Using the result of Hochman-Seward mentioned above, the following was shown:

Theorem 12.18 ([FKSV])**.** *Every infinite countable amenable group is 2-adequate.*

This in particular answers in the negative a question of Thomas [T14, Page 391], who asked whether there are infinite countable amenable groups G for which $E(G, \mathbb{R})$ is not Borel reducible to $E(G, 2)$.

Moreover we have:

Theorem 12.19 ([FKSV]). *(i) The free product of any countable group with a group that has an infinite amenable factor and thus, in particular, the free groups $\mathbb{F}_n, 1 \leq n \leq \infty$, are 2-adequate.*

(ii) Let Γ be n-generated, $1 \leq n \leq \infty$. Then $\Gamma \times \mathbb{F}_n$ is 2-adequate. In particular, all products $\mathbb{F}_m \times \mathbb{F}_n$, $1 \leq m, n \leq \infty$, are 2-adequate.

On the other hand there are groups which are not 2-adequate:

Theorem 12.20 ([FKSV]). *The group $\mathrm{SL}_3(\mathbb{Z})$ is not 2-adequate.*

It is not known if there is a characterization of 2-adequate groups.

12.3   Weak universality

Definition 12.21. *A countable Borel equivalence relation E is called* **weakly universal** *if for every countable Borel equivalence relation F, $F \leq_B^w E$.*

By Theorem 3.37 this is equivalent to stating that there is a universal countable Borel equivalence relation $E' \subseteq E$. In that form an old question of Hjorth, see [ACH, 1.4] or [JKL, 6.5, (A)], asks the following:

Problem 12.22. *Is every weakly universal countable Borel equivalence relation universal?*

A special case of this question is the following: If E is a countable Borel equivalence relation, $E' \subseteq E$ is a universal countable Borel equivalence relation and every E-class contains only finitely many E'-classes, is E universal?

We next discuss some examples of weakly universal countable Borel equivalence relations for which it is not known if there are universal:

(i) Since $E(\mathbb{F}_2, 2) \subseteq \cong_{rec}^2$, $\cong_{rec}^2$ and therefore also $\equiv_T$ are weakly universal.

(ii) ([TW2, 1.4, 1.5]) The isomorphism and biembeddability relations on Kazhdan groups are weakly universal.

(iii) ([W2, 1.4, 1.5] The isomorphism relation on finitely generated solvable groups of class 3 is weakly universal (thus the same holds for the isomorphism relation on finitely generated amenable groups).

Recall from the paragraph following Problem 12.5 that Martin's Conjecture implies that the weakly universal equivalence relation $\equiv_T$ is not

universal. In [T11] it was shown that Martin's Conjecture has several strong implications concerning weak universality, including the following:

Theorem 12.23 ([T11, 1.2]). *Assume Martin's Conjecture. Then there are continuum many weakly universal countable Borel equivalence relations which are pairwise incomparable in $\leq_B$.*

Theorem 12.24 ([T11, 1.4]). *Assume Martin's Conjecture. Let E be a countable Borel equivalence relation on a standard Borel space X. Then exactly one of the following hods:*
(i) E is weakly universal;
(ii) For every $f\colon \ \equiv_T \to_B E$, there is a cone of Turing degrees C such that $f(C)$ is contained in a single E-class.

It is not even known if (ii) in this result holds for $E = E_0$.

Also Theorem 12.24 has the following strong ergodicity consequence for weakly universal countable Borel equivalence relations.

Corollary 12.25 ([T11, 3.1]). *Assume Martin's Conjecture. Let E be a countable Borel equivalence relation on a standard Borel space X and F a countable Borel equivalence relation on a standard Borel space Y. Assume that E is weakly universal but F is not. Then for every $f\colon E \to_B F$, there is an E-invariant Borel set $A \subseteq X$ such that $E \restriction A$ is weakly universal and $f(A)$ is contained in a single F-class.*

Thomas [T12, 3.22] raised the question of whether there exists a countable Borel equivalence relation E and an E-invariant, E-ergodic measure such that the restriction of E to every E-invariant Borel set of measure 1 is universal. Such E are called **strongly universal**. Martin's Conjecture implies a negative answer.

Theorem 12.26 ([T11, 5.4]). *Assume Martin's Conjecture. For any countable Borel equivalence relation E and any probability measure μ, there is an E-invariant Borel set A with $\mu(A) = 1$ such that $E \restriction A$ is not weakly universal.*

Finally it is shown in [T14, 3.4] that, assuming Martin's Conjecture, a countable group G has a Borel action a with E_a weakly universal iff $E_{conj}(G)$ is weakly universal.

12.4   Uniform universality

The concept of uniform universality was introduce in unpublished work of Montalbán, Reimann and Slaman and extensively developed in [M3] (see also [MSS]).

Suppose X is a standard Borel space and $(\varphi_n)_{n \in \mathbb{N}}$ a sequence of partial Borel functions $\varphi_n \colon A_n \to X$, A_n a Borel subset of X, which contains the identity function and is closed under composition. Then $(\varphi_n)_{n \in \mathbb{N}}$ generates the countable Borel equivalence relation $R_{(\varphi_n)}$ defined by

$$x R_{(\varphi_n)} y \iff \exists m, n (\varphi_m(x) = y \ \& \ \varphi_n(y) = x).$$

In this definition we say that $x R_{(\varphi_n)} y$ via the pair (m, n).

For example, if G is a countable group, say $G = \{g_n \colon n \in \mathbb{N}\}$, a is a Borel action of G on a standard Borel space X and $\varphi_n(x) = g_n \cdot x$, then $E_a = R_{(\varphi_n)}$. Also on $2^{\mathbb{N}}$ if τ_n is the nth Turing functional (which is a Borel partial function on $2^{\mathbb{N}}$), then $\equiv_T = R_{(\tau_n)}$.

If now $E = R_{(\varphi_n)}, F = R_{(\psi_n)}$ are given and $f \colon E \to_B F$, then we say that f is a **uniform homomorphism**, with respect to $(\varphi_n), (\psi_n)$, if there is a function $u \colon \mathbb{N}^2 \to \mathbb{N}^2$ such that if $x E y$ via (m, n), then $f(x) F f(y)$ via $u(m, n)$.

For example, if a is a Borel action of a countable group G and b is a Borel free action of a countable group H, then a Borel homomorphism of E_a to E_b is uniform (with respect to the Borel functions given by the actions as above) iff the cocycle of the action a to H associated to this homomorphism (see Section 7.2) is simply a homomorphism from G to H.

Definition 12.27. *A countable Borel equivalence $E = R_{(\varphi_n)}$ is* **uniformly universal** *(with respect to (φ_n)) if for every countable Borel equivalence relation $F = R_{(\psi_n)}$, there is $f \colon E \leq_B F$ which is uniform (with respect to $(\varphi_n), (\psi_n)$).*

We now have the following result:

Theorem 12.28 ([M3, Proposition 3.3]). *(i) For every universal countable Borel equivalence relation E, there is a sequence of partial Borel functions (φ_n) such that $E = R_{(\varphi_n)}$ and E is uniformly universal with respect to (φ_n).*

Below we say that an equivalence relation $E = E_a$ generated by a Borel action a of a countable group G is uniformly universal if it is uniformly universal with respect to the Borel functions given by this action.

Theorem 12.29 ([M3, Theorem 1.5, Theorem 3.1]). *For any countable group G the following are equivalent:*

    (i) G contains a copy of $\mathbb{F}_2$;

    (i) There is a Borel action a of G such that E_a is uniformly universal;

    (ii) For every standard Borel space X, with more than one element, $E(G, X)$ is uniformly universal;

    (iii) $E_{conj}(G)$ is uniformly universal.

In fact as pointed out in [M3, page 20] every known proof that a countable Borel equivalence relation E is universal actually shows that E is uniformly universal for an appropriate (φ_n) such that $E = R_{(\varphi_n)}$. It is in fact conjectured in [M3, Conjecture 3.1] that for every universal countable Borel equivalence relation and every (φ_n) such that $E = R_{(\varphi_n)}$, E is uniformly universal with respect to (φ_n).

On the other hand, as pointed out in [M3, page 20], $\equiv_T = R_{(\tau_n)}$ is not uniformly universal. A much more general result on non-uniform universality is proved in [M3, Theorem 3.4], which includes many other equivalence relations in computability theory, including, for example, many-one equivalence on $2^{\mathbb{N}}$. However it is unknown if $\cong^2_{rec}$ is not uniformly universal.

Uniform universality is not preserved on measure theoretically large sets.

Theorem 12.30 ([M3, Theorem 3.7]). *If $E = R_{(\varphi_n)}$ is a uniformly universal countable Borel equivalence relation on a standard Borel space X and μ is a probability measure on X, then there is a Borel E-invariant set $A \subseteq X$ with $\mu(A) = 1$ such that $E \upharpoonright A$ is not uniformly universal.*

12.5   Inclusion universality

Recall from Section 8.1 the concept of the Borel inclusion order $\subseteq_B$, where for countable Borel equivalence relations E, F, we put $E \subseteq_B F$ iff there is $E' \cong_B E$ with $E' \subseteq E$. This quasi-order is studied in [FKSV]. We say that a countable Borel equivalence relation F is **inclusion universal** if for each countable Borel equivalence relation E on an uncountable standard Borel space we have that $E \subseteq_B F$.

Proposition 12.31 (Miller). *There exists an inclusion universal countable Borel equivalence relation.*

Proof. We will show that $F = E_\infty \times I_\mathbb{N}$ works. First notice that F contains a smooth aperiodic countable Borel equivalence relation, so if E is a smooth countable Borel equivalence relation, then there is a Borel isomorphic copy of E contained in F.

So let E be a non-smooth countable Borel equivalence relation. We can of course assume that $E = E_\infty \restriction Y$, where E_∞ is on the space X and Y is an uncountable Borel E_∞-invariant subset of X. Let $Z = (X \times \mathbb{N}) \setminus (Y \times \{0\})$. Then let $R \subseteq F \restriction Z$ be an aperiodic smooth Borel equivalence relation and put $E' = (F \restriction Y \times \{0\}) \cup R \subseteq F$. We will check that $E \cong_B E'$.

First we have that $E' \cong_B E \oplus R$. Let $A \subseteq Y$ be an E-invariant Borel set such that $E \restriction A \cong_B R$. Then $E' \cong_B E \oplus R \cong_B R \oplus E \restriction (Y \setminus A) \oplus R \cong_B R \oplus E \restriction (Y \setminus A) \cong_B E$. $\qquad\square$

13. The poset of bireducibility types

Let $\mathcal{E}$ denote the class of countable Borel equivalence relations equipped with the quasi-order $\leq_B$ and the associated equivalence relation $\sim_B$. In this section we also allow the empty equivalence relation $\emptyset$ on the empty space as a countable Borel equivalence relation with the convention that $\emptyset \sqsubseteq_B^i E$ for every countable Borel equivalence relation E. For each $E \in \mathcal{E}$ we denote by $e = [E]_{\sim_B} = \boldsymbol{E}$ the bireducibility class of E, usually called the **bireducibility type** of E. Let $\boldsymbol{\mathcal{E}}$ be the set of bireducibility types. Then $\leq_B$ descends to a partial order $\boldsymbol{E} \leq \boldsymbol{F} \iff E \leq_B F$ on $\boldsymbol{\mathcal{E}}$. It has a minimum element $\boldsymbol{e_0} = \boldsymbol{\emptyset}$ and a maximum element $\boldsymbol{e_\infty} = \boldsymbol{E_\infty}$. We call $\langle \boldsymbol{\mathcal{E}}, \leq \rangle$ the **poset of bireducibility types**. Notice that every bireducibility type $\boldsymbol{e}$ contains a unique Borel isomorphism class consisting of the compressible elements of $\boldsymbol{e}$.

It is clear from Theorem 7.1 that this poset is quite complex, since one can embed in it the poset of Borel subsets of $\mathbb{R}$ under inclusion. Until recently very little was known about the algebraic structure of this poset. Some progress has been now made by applying in this context Tarski's theory of cardinal algebras, see [Ta], which was originally developed as an algebraic approach to the theory of cardinal addition devoid of the use of the Axiom of Choice.

A **cardinal algebra**, see [Ta], is a system $\mathbb{A} = \langle A, +, \sum \rangle$, where $\langle A, + \rangle$ is an abelian semigroup with identity, which will be denoted by 0, and $\sum \colon A^\mathbb{N} \to A$ is an infinitary operation, satisfying the following axioms, where we put $\sum_{n < \infty} a_n = \sum((a_n)_{n \in \mathbb{N}})$:

    (i) $\sum_{n < \infty} a_n = a_0 + \sum_{n < \infty} a_{n+1}$.
    (ii) $\sum_{n < \infty} (a_n + b_n) = \sum_{n < \infty} a_n + \sum_{n < \infty} b_n$.
    (iii) If $a + b = \sum_{n < \infty} c_n$, then there are $(a_n), (b_n)$ with $a = \sum_{n < \infty} a_n, b =$

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$\sum_{n<\infty} b_n$, $c_n = a_n + b_n$.

(iv) If $(a_n), (b_n)$ are such that $a_n = b_n + a_{n+1}$, then there is c such that for each n, $a_n = c + \sum_{i<\infty} b_{n+i}$.

Let also

$$a \leq b \iff \exists c (a + c = b).$$

It turns out that this is a partial ordering. All the expected commutativity, associativity laws for $+, \sum$ and monotonicity with respect to $\leq$ hold (see [Ta, Section 1]).

We can define on $\mathcal{E}$ the operations

$$\boldsymbol{E} + \boldsymbol{F} = \text{the bireducibility class of } E \oplus F;$$

and

$$\sum_n \boldsymbol{E_n} = \text{the bireducibility class of } \bigoplus_n E_n.$$

and, as a special case of a more general result, we now have the following:

Theorem 13.1 ([KMa, 3.3]). $\langle \mathcal{E}, +, \sum \rangle$ *is a cardinal algebra. Moreover, for* $E, F \in \mathcal{E}$, $E \leq_B F \iff \boldsymbol{E} \leq \boldsymbol{F}$.

Also clearly $\boldsymbol{e_0}$ is the additive identity of this cardinal algebra. One can now apply the algebraic laws of cardinal algebras established in [Ta] to immediately derive such laws for the poset of bireducibilty types and thus for the quasi-order $\leq_B$, including the following:

Theorem 13.2 ([KMa, 1.1]). *(i) (Existence of least upper bounds) Any increasing sequence* $F_0 \leq_B F_1 \leq_B \dots$ *of countable Borel equivalence relations has a least upper bound (in the quasi-order $\leq_B$).*

(ii) (Interpolation) If $\mathcal{S}, \mathcal{T}$ *are countable sets of countable Borel equivalence relations and* $\forall E \in \mathcal{S} \forall F \in \mathcal{T} (E \leq_B F)$, *then there is a countable Borel equivalence relation G such that* $\forall E \in \mathcal{S} \forall F \in \mathcal{T} (E \leq_B G \leq_B F)$.

(iii) (Cancellation) If $n > 0$ *and* E, F *are countable Borel equivalence relations, then*

$$nE \leq_B nF \implies E \leq_B F$$

and therefore

$$nE \sim_B nF \implies E \sim_B F.$$

(iv) (Dichotomy for integer multiples) For any countable Borel equivalence relation E, exactly one of the following holds:

$$(a)\ E <_B 2E <_B 3E <_B \ldots,$$

$$(b)\ E \sim_B 2E \sim_B 3E \sim_B \ldots.$$

References to the parts of [Ta], where the relevant laws that are used in Theorem 13.2 are proved, can be found in [KMa, 2.2]. Another result proved in [Ta, 3.4] is that in any cardinal algebra, if the infimum (meet) $a \wedge b$ of two elements exists, then the supremum (join) $a \vee b$ exists (and $(a \wedge b) + (a \vee b) = a + b$). It is unknown if the poset of bireducibilty types is a lattice and this can be stated equivalently as follows:

Problem 13.3 ([KMa, 3.12]). *Is it true that any two bireducibility types have an infimum?*

In fact until very recently it was even unknown if there exist two incomparable under $\leq$ bireducibility types that have an infimum. A positive answer is given in Theorem 14.7.

Finally, concerning the cancellation law Theorem 13.2, (iii), for sums, it is natural to ask if there is a similar cancellation law for products. Using methods of ergodic theory it can be shown that this is not the case.

Theorem 13.4 ([KMa, 4.1]). *There are countable Borel equivalence relations $E <_B F$ such that $E^2 \sim_B F^2$.*

Remark 13.5. Cardinal algebras also occur in another context in the theory of countable Borel equivalence relations. Let E be a compressible countable Borel equivalence relation on a standard Borel space X. Consider the space of all Borel subsets of X modulo $\sim_E$ (see the paragraph following Definition 3.21). It was shown in [Ch2] that, with some natural operations, this becomes a cardinal algebra, which exhibits interesting properties and is studied in detail in [Ch2].

14. Structurability

14.1 Universal structurability

We will consider in this section $\mathcal{K}$-structurable countable Borel equivalence relations for Borel classes $\mathcal{K}$ of countable structures in some language L (see Section 9.3). We denote by $\mathcal{E}_\mathcal{K}$ the class of countable Borel equivalence relations that are $\mathcal{K}$-structurable. Examples of such classes, for various $\mathcal{K}$, include the following: aperiodic, smooth, compressible, hyperfinite, treeable, α-amenable, the equivalence relations induced by a free Borel action of a fixed countable group G, all countable Borel equivalence relations.

The next result shows that $\mathcal{E}_\mathcal{K}$ contains an invariantly universal element. It was proved in [KST, 7.1] for classes of graphs and generalized by Miller. Below for any class $\mathcal{C}$ of countable Borel equivalence relations a relation $E \in \mathcal{C}$ is called **invariantly universal** for $\mathcal{C}$ if for $F \in \mathcal{C}$, $F \sqsubseteq_B^i E$.

Theorem 14.1 (see [M3, Theorem 4.3], [CK, 1.1]). *For each Borel class $\mathcal{K}$ of countable structures, there is a (unique up to Borel isomorphism) invariantly universal equivalence relation in $\mathcal{E}_\mathcal{K}$.*

This invariantly universal relation will be denoted by $E_{\infty\mathcal{K}}$. For example, for the classes $\mathcal{K}$ of hyperfinite (resp., treeable, induced by a free Borel action of a countable group G, all countable Borel equivalence relations), $E_{\infty\mathcal{K}}$ is Borel isomorphic to $E_{\infty h}$ (resp., $E_{\infty\mathcal{T}}$, $F(G, \mathbb{R})$, E_∞).

14.2 Elementary classes of relations

By the result of Lopez-Escobar in [LE] a class $\mathcal{K}$ of countable structures is Borel iff there is a countable $L_{\omega_1\omega}$ theory, i.e., an $L_{\omega_1\omega}$ sentence σ, such that

$\mathcal{K}$ is exactly the class of countable models of σ. We thus often write

$$\mathcal{E}_\sigma = \mathcal{E}_\mathcal{K}, E_{\infty\sigma} = E_{\infty\mathcal{K}}$$

and for a countable Borel equivalence relation E, we put

$$E \models \sigma \iff E \in \mathcal{E}_\mathcal{K}.$$

We then say that a class $\mathcal{C}$ of countable Borel equivalence relations is **elementary** if $\mathcal{C} = \mathcal{E}_\sigma$ for some σ. Thus all the examples we mentioned in Section 14.1 are elementary. The elementary classes can be characterized as follows.

A Borel homomorphism $f\colon E \to_B F$ between countable Borel equivalence relations E, F on standard Borel spaces X, Y, resp., is called **class bijective** if for each $x \in X$, $f \restriction [x]_E$ is a bijection of $[x]_E$ onto $[f(x)]_F$. We write in this case $f\colon E \to^{cb}_B F$ and if such f exists we put $E \to^{cb}_B F$. We now have:

Theorem 14.2 ([CK, 1.2]). *Let $\mathcal{C}$ be a class of countable Borel equivalence relations. Then $\mathcal{C}$ is an elementary class iff it is closed downwards under $\to^{cb}_B$ and contains an invariantly universal element.*

The following classes are not elementary (see [CK, Section 3.1]): non-smooth, non-compressible, free, essentially free.

Every countable Borel equivalence relation is contained in a smallest, under inclusion, elementary class.

Theorem 14.3 ([CK, 1.3]). *Let E be a countable Borel equivalence relation. Then $\mathcal{E}_E = \{F \in \mathcal{E}\colon F \to^{cb}_B E\}$ is the smallest elementary class containing E.*

We next consider elementary classes closed downwards under Borel reductions, like, e.g., hyperfinite or treeable. These are called **elementary reducibility classes**.

Theorem 14.4 ([CK, 1.4]). *Let $\mathcal{C}$ be an elementary class of countable Borel equivalence relations. Then $\mathcal{C}^r = \{F \in \mathcal{E}\colon \exists E \in \mathcal{C}(F \leq_B E)\}$ is the smallest elementary class containing $\mathcal{C}$.*

Elementary reducibility classes can be characterized as follows. A Borel homomorphism $f\colon E \to_B F$ between countable Borel equivalence relations E, F on standard Borel spaces X, Y, resp., is called **smooth** if for each $y \in Y$, $E \restriction f^{-1}(y)$ is smooth. This notion was considered in [CCM]. We write in this case $f\colon E \to^{sm}_B F$ and if such f exists we put $E \to^{sm}_B F$. We now have:

Theorem 14.5 ([CK, 1.5]). *Let $\mathcal{C}$ be a class of countable Borel equivalence relations. Then $\mathcal{C}$ is an elementary reducibility class iff it is closed downwards under $\rightarrow_B^{sm}$ and contains an invariantly universal element.*

There is an interesting connection between these concepts and amenability of groups. For each infinite countable group G, let $\mathcal{E}_G^*$ be the elementary class of all countable Borel equivalence relations whose aperiodic part is generated by a free Borel action of G.

Theorem 14.6 ([CK, 1.6]). *Let G be an infinite countable group. Then the following are equivalent:*
    (i) G is amenable;
    (ii) $\mathcal{E}_G^*$ is an elementary reducibility class.

We call any countable Borel equivalence relation Borel isomorphic to one of the form $E_{\infty\sigma}$, for a $L_{\omega_1\omega}$ theory σ, **universally structurable**. Let $\mathcal{E}_\infty$ be the class of these equivalence relations and let $\boldsymbol{\mathcal{E}}_\infty = \{\boldsymbol{E} \colon E \in \mathcal{E}_\infty\}$. Then $\langle \boldsymbol{\mathcal{E}}_\infty, \leq \rangle$ is a subposet of $\langle \boldsymbol{\mathcal{E}}, \leq \rangle$. It is quite rich since it can be shown that the poset of Borel subsets of $\mathbb{R}$ under inclusion can be embedded into it, see [CK, 1.9]. However this subposet is now known to have desirable algebraic properties.

Theorem 14.7 ([CK, 1.8]). *The poset $\langle \boldsymbol{\mathcal{E}}_\infty, \leq \rangle$ is a countably complete, distributive lattice. Moreover the countable meets and joins in this lattice are also meets and joins in the poset $\langle \boldsymbol{\mathcal{E}}, \leq \rangle$.*

Remark 14.8. Notice that if $E \in \mathcal{E}_\infty$, then $\mathbb{R}E \cong_B E$. It follows that $\boldsymbol{\mathcal{E}}_\infty$ is a proper subset of $\boldsymbol{\mathcal{E}}$, even when restricted to non-smooth countable Borel equivalence relations, see Theorem 7.20. It is unknown if there is $E \notin \mathcal{E}_\infty$ with $\mathbb{R}E \cong_B E$.

It is an interesting problem to understand the connection between the model theoretic properties of a theory σ and the Borel theoretic properties of the class $\mathcal{E}_\sigma$. The following result, answering a question of Marks [M3, end of Section 4.3], is a step in that direction.

Theorem 14.9 ([CK, 1.10]). *Let σ be a theory in $L_{\omega_1\omega}$. Then the following are equivalent:*
    (i) Every equivalence relation in $\mathcal{E}_\sigma$ is smooth;
    (ii) There is a formula $\phi(x)$ in $L_{\omega_1\omega}$ which defines a finite nonempty set in every countable model of σ.

The next step would be to characterize the σ for which every equivalence relation in $\mathcal{E}_\sigma$ is hyperfinite. This is however an open problem.

It is also of interest to find out which theories σ have the property that every aperiodic countable Borel equivalence relation is in $\mathcal{E}_\sigma$. For example, in Section 8.7, 9) we have seen that this is the case for the σ whose countable models are the connected locally finite graphs with one end.

In the case where σ is the Scott sentence of a countable structure $\mathbb{A}$, the following result was proved by Marks as a consequence of the work in [AFP]. Recall that a countable Borel equivalence relation is $\mathbb{A}$-stucturable iff it is in $\mathcal{E}_\sigma$ for the Scott sentence σ of $\mathbb{A}$.

Theorem 14.10 (Marks, see [CK, 1.11]). *Let $\mathbb{A}$ be a countable structure with trivial definable closure. Then every aperiodic countable Borel equivalence relation is $\mathbb{A}$-structurable.*

Such structures $\mathbb{A}$ include, for example, many Fraïssé structures such as the rational order and the random graph, see [CK, 8.17].

14.3   About universally structurable relations

Results that are analogous to those for E_∞ in Section 12.1 have been proved in [M3, Sections 4.3, 4.4] for certain $E_{\infty\sigma}$.

The following is an analog of Theorem 12.3.

Theorem 14.11 ([M3, Theorem 4.4]). *Let σ be a theory in $L_{\omega_1\omega}$. Let $E_{\infty\sigma}$ be on the space $X_{\infty\sigma}$ and let μ be a probability measure on $X_{\infty\sigma}$. Then there is an $E_{\infty\sigma}$-invariant Borel set $A \subseteq X_{\infty\sigma}$ with $\mu(A) = 0$ such that $E_{\infty\sigma} \cong_B E_{\infty\sigma} \restriction A$.*

Recall that a family $(E_i)_{i\in I}$ of equivalence relations on a set X is **independent** if for any sequence $x_0, x_1, \ldots, x_n = x_0$, with $n > 1$, if

$$x_0 E_{i_0} x_1 E_{i_1} x_2 \ldots x_{n-1} E_{i_{n-1}} x_0,$$

where $i_k \neq i_{k+1}$, if $k < n - 2$, and $i_{n-1} \neq i_0$, there is $j < n$, with $x_j = x_{j+1}$. In this case we call $\bigvee_i E_i$ an **independent join**.

Note, for example, that the classes $\mathcal{C}_n$, $n \geq 1$, defined in Section 10.10 (which are of the form $\mathcal{E}_{\sigma_n}$ for an appropriate σ_n) are closed under countable independent joins. For σ such that $\mathcal{E}_\sigma$ is closed under under independent joins of two relations, it was shown in [M3, Theorem 4.5], generalizing the result mentioned in Section 12.1, that there is a countably

complete ultrafilter U on the $E_{\infty\sigma}$-invariant Borel sets such that if $A \in U$, $E_{\infty\sigma} \sim_B E_{\infty\sigma} \restriction A$. From this, and the definition of U, we have the analog of Theorem 12.2:

Theorem 14.12 ([M3, Theorem 4.5, Theorem 4.6]). *Let σ be a theory in $L_{\omega_1\omega}$.*

(i) Assume that $\mathcal{E}_\sigma$ is closed under independent joins of two relations. Then if $f : E \to_B \Delta_Y$, for some standard Borel space Y, there is $y \in Y$ such that $E_{\infty\sigma} \restriction f^{-1}(y) \sim_B E_{\infty\sigma}$.

(ii) If moreover $\mathcal{E}_\sigma$ is closed under countable independent joins, $E_{\infty\sigma}$ is on the space $X_{\infty\sigma}$ and $X_{\infty\sigma} = \bigsqcup_n X_n$ is a Borel partition (where the X_n might not be $E_{\infty\sigma}$-invariant), then for some n, $E_{\infty\sigma} \sqsubseteq_B E_{\infty\sigma} \restriction X_n$. In particular, if $E_{\infty\sigma} \leq_B F$, for some countable Borel equivalence relation F, then $E_{\infty\sigma} \sqsubseteq_B F$.

15. Topological realizations

The main concern of [FKSV] is the subject of well-behaved, in some sense, realizations of countable Borel equivalence relations. Generally speaking a realization of a countable Borel equivalence relation E is a countable Borel equivalence relation $F \cong_B E$ with desirable properties.

To start with, a **topological realization** of E on a standard Borel space is an equivalence relation F on a *Polish space Y* such that $E \cong_B F$, in which case we say that F is a topological realization of E in the space Y. It is clear that every E admits a topological realization in some Polish space but we will look at topological realizations that have additional properties.

Also by the Feldman-Moore Theorem 3.3, it is clear that every countable Borel equivalence relation E admits a topological realization, in some Polish space Y, which is induced by a *continuous action* of some countable (discrete) group G on Y. We will look again at such **continuous action realizations** for which the space and the action have additional properties.

To avoid uninteresting situations, unless it is otherwise explicitly stated or clear from the context, all the standard Borel or Polish spaces below will be uncountable and all countable Borel equivalence relations *will be aperiodic. We will denote by $\mathcal{AE}$ the class of all aperiodic countable Borel equivalence relations on uncountable standard Borel spaces.*

Concerning topological realizations, we first have the following:

Theorem 15.1 ([FKSV, Theorem 1.1.1]). *For every equivalence relation $E \in \mathcal{AE}$ and every perfect Polish space Y, there is a topological realization of E in Y in which every equivalence class is dense.*

This has in particular as a consequence a stronger new version of a *marker lemma*. Let E be a CBER on a standard Borel space X. A **Lusin marker scheme** for E is a family $\{A_s\}_{s \in \mathbb{N}^{<\mathbb{N}}}$ of Borel sets such that
   (i) $A_\emptyset = X$;

(ii) $\{A_{sn}\}_n$ are pairwise disjoint and $\bigsqcup_n A_{sn} \subseteq A_s$;

(iii) Each A_s is a complete section for E.

We have two types of Lusin marker schemes:

(1) The Lusin marker scheme $\{A_s\}_{s \in \mathbb{N}^{<\mathbb{N}}}$ for E is of **type I** if in (ii) above we actually have that $\bigsqcup_n A_{sn} = A_s$ and moreover the following holds:

(iv) For each $x \in \mathcal{N} = \mathbb{N}^{\mathbb{N}}$, $\bigcap_n A_{x|n}$ is a singleton.

(Then in this case, for each $x \in \mathcal{N}$, $A_n^x = A_{x|n} \setminus \bigcap_n A_{x|n}$ is a vanishing sequence of markers (i.e., $\bigcap_n A_n^x = \emptyset$).)

(2) The Lusin marker scheme $\{A_s\}_{s \in \mathbb{N}^{<\mathbb{N}}}$ for E is of **type II** if it satisfies the following:

(v) If for each n, $B_n = \bigsqcup\{A_s : s \in \mathbb{N}^n\}$, then $\{B_n\}$ is a vanishing sequence of markers.

We now have:

Theorem 15.2 ([FKSV, Theorem 1.1.2]). *Every $E \in \mathcal{AE}$ admits a Lusin marker scheme of type* I *and a Lusin marker scheme of type* II.

We next look at continuous action realizations. An important case of such a realization of $E \in \mathcal{AE}$ would be a continuous action realization F on a compact Polish space, called a **compact action realization**. If in addition the action is minimal, we call it a **minimal, compact action realizations**. Excluding the case of smooth relations, for which compact action realizations are impossible, we have the following result. Below for each countable group G and topological space X, a **subshift** of X^G is the restriction of the shift action of G on X^G to a nonempty shift-invariant closed set.

Theorem 15.3 ([FKSV, Theorem 1.1.3]). *Every non-smooth hyperfinite equivalence relation in $\mathcal{AE}$ has a minimal, compact action realization. In fact this realization can be taken to be a subshift of $2^{\mathbb{F}_2}$ if the equivalence relation is compressible and a subshift of $2^{\mathbb{Z}}$ otherwise.*

We next discuss other cases of countable Borel equivalence relations which admit such realizations. For each infinite countable group G, let $F(G, 2^{\mathbb{N}})$ be the equivalence relation induced by the shift action of G on $(2^{\mathbb{N}})^G$ restricted to its free part. Every equivalence relation induced by a free Borel action of G is Borel isomorphic to the restriction of $F(G, 2^{\mathbb{N}})$ on an invariant Borel set. As opposed to Theorem 15.3, the next results show that some very complex countable Borel equivalence relations have compact action realizations.

Theorem 15.4 ([FKSV, Theorem 1.1.4]). *(i) For every infinite countable group G, $F(G, 2^{\mathbb{N}})$ admits a compact action realization.*

(ii) Every compressible, universal countable Borel equivalence relation admits a compact action realization. In fact such a realization can be taken to be a subshift of $2^{\mathbb{F}_4}$.

In particular, it follows that arithmetical equivalence $\equiv_A$ on $2^{\mathbb{N}}$ has a compact action realization but it is unknown if Turing equivalence $\equiv_T$ has such a realization. More generally, we do not know whether *every* non-smooth countable Borel equivalence relation has a compact action realization. We also do not know if *every* non-smooth countable Borel equivalence relation even admits some other kinds of realizations, for example transitive (i.e., having at least one dense orbit) continuous action realizations on arbitrary or special types of Polish spaces.

In relation to Theorem 15.4, call a countable group G **minimal subshift universal** if there is a minimal subshift of 2^G on which the restriction of the shift equivalence relation is universal. Then we have the following result:

Theorem 15.5 ([FKSV, Theorem 1.1.5]). *Let G and H be infinite groups, where H admits a Borel action on a standard Borel space whose induced equivalence relation is universal (e.g., any group containing $\mathbb{F}_2$). Then we have that the wreath product $G \wr H$ is minimal subshift universal. In particular, $\mathbb{F}_3$ is minimal subshift universal.*

We do not know if $\mathbb{F}_2$ is minimal subshift universal.

The preceding considerations have some connections to the concept of tests of amenability. It is well known that a countable group G is amenable iff every continuous action of G on a compact space admits an invariant Borel probability measure. Call a class $\mathcal{F}$ of such actions a **test for amenability** for G if G is amenable provided that every action in $\mathcal{F}$ admits an invariant Borel probability measure. In [GdlH] it is shown that the class of actions on $2^{\mathbb{N}}$ is a test for amenability for any group. Equivalently this says that the class of all subshifts of $(2^{\mathbb{N}})^G$ is a test of amenability for G. It turns out that the strongest result along these lines is actually true, namely that the class of all subshifts of 2^G is a test of amenability for G. This gives a new characterization of amenability.

Theorem 15.6 ([FKSV, Theorem 1.1.6]). *Let G be a countable group. Then G is amenable iff every subshift of 2^G admits an invariant Borel probability measure.*

Finally we mention that, in response to a question raised by Clinton Conley, it is shown in [FKSV] that every every $E \in \mathcal{AE}$ admits a K_σ realization on $2^{\mathbb{N}}$ with dense classes.

16. A universal space for actions and equivalence relations

In [FKSV] the authors also consider a universal space for actions and equivalence relations and study the descriptive or topological properties of various subclasses.

For a countable group G and Polish space X, define the standard Borel space of subshifts of X^G by:

$$\mathrm{Sh}(G, X) = \{F \in F(X^G) : F \text{ is } G\text{-invariant}\}$$

Here $F(Y)$ is the standard Borel space of nonempty closed subsets of a Polish space Y. If X is compact, we view this as a compact Polish space with the Vietoris topology.

Every compact Polish space is (up to homeomorphism) a closed subspace of the Hilbert cube $\mathbb{I}^{\mathbb{N}}$, and thus every continuous action of a countable group G on a compact Polish space is (topologically) isomorphic to a subshift of $(\mathbb{I}^{\mathbb{N}})^G$. We can thus consider the compact Polish space $\mathrm{Sh}(G, \mathbb{I}^{\mathbb{N}})$ as the universal space of such actions.

Similarly consider the product space $\mathbb{R}^{\mathbb{N}}$. Every Polish space is (up to homeomorphism) a closed subspace of $\mathbb{R}^{\mathbb{N}}$, and thus every continuous action of G on a Polish space is (topologically) isomorphic to a subshift of $(\mathbb{R}^{\mathbb{N}})^G$. We can thus consider the standard Borel space $\mathrm{Sh}(G, \mathbb{R}^{\mathbb{N}})$ as the universal space of such actions.

In particular taking $G = \mathbb{F}_{\infty}$, we see that every countable Borel equivalence relation is Borel isomorphic to the equivalence relation E_F induced on some subshift F of $(\mathbb{R}^{\mathbb{N}})^{\mathbb{F}_{\infty}}$ and so we can view $\mathrm{Sh}(\mathbb{F}_{\infty}, \mathbb{R}^{\mathbb{N}})$ also as the universal space of countable Borel equivalence relations and study the complexity of various classes of countable Borel equivalence relations (like, e.g., smooth, aperiodic, hyperfinite, etc.) as subsets of this universal

space. Similarly we can view $\mathrm{Sh}(\mathbb{F}_\infty, \mathbb{I}^{\mathbb{N}})$ as the universal space of countable Borel equivalence relations that admit a compact action realization. In this case one can also consider complexity questions as well as genericity questions of various classes. Below are some of the main results proved in [FKSV]. Below G is a countable infinite group. A countable group G is called **exact** if it admits a topologically amenable action on a compact Polish space (see [BO]). For example, free groups are exact.

Theorem 16.1 ([FKSV, Section 3.6 (B)]). *(i) In the space of subshifts of $(\mathbb{I}^{\mathbb{N}})^G$, finiteness and smoothness are meager properties, while freeness (of the action), aperiodicity, compressibility (for non-amenable G) are comeager. Also amenability and measure hyperfiniteness are comeager (for exact groups G).*

(ii) In the space of subshifts of $(\mathbb{I}^{\mathbb{N}})^G$, finiteness and smoothness (for residually finite G) are $\mathbf{\Pi}^1_1$-complete, while freeness (of the action) and aperiodicity are G_δ and compressibility is open. Also (for residually finite and non-amenable G) measure hyperfiniteness is $\mathbf{\Pi}^1_1$-complete and hyperfiniteness is $\mathbf{\Pi}^1_1$-hard.

(iii) In the space of subshifts of $(\mathbb{R}^{\mathbb{N}})^G$, finiteness, smoothness (for residually finite G) are $\mathbf{\Pi}^1_1$-complete, freeness (of the action), aperiodicity and compressibility are $\mathbf{\Pi}^1_1$-complete, while (for residually finite and non-amenable G) hyperfiniteness and amenability are $\mathbf{\Pi}^1_1$-hard and measure hyoperfiniteness is $\mathbf{\Pi}^1_1$-complete.

We do not know if hyperfiniteness is generic in $\mathrm{Sh}(G, \mathbb{I}^{\mathbb{N}})$ for every non-amenable G (or just $\mathbb{F}_\infty$) and we do not know the exact descriptive complexity of hyperfiniteness.

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