Remarks on invariant uniformization and reducibility

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1 Introduction

Given sets X, Y and $P \subseteq X \times Y$ with $\text{proj}_X(P) = X$, a **uniformization** of P is a function $f: X \to Y$ such that $\forall x \in X((x, f(x)) \in P)$. If now E is an equivalence relation on X, we say that P is ***E*-invariant** if $x_1 E x_2 \implies P_{x_1} = P_{x_2}$, where $P_x = \{y: (x, y) \in P\}$ is the x-**section** of P. Equivalently this means that P is invariant under the equivalence relation $E \times \Delta_Y$ on $X \times Y$, where Δ_Y is the equality relation on Y. In this case an ***E*-invariant uniformization** is a uniformization f such that $x_1 E x_2 \implies f(x_1) = f(x_2)$.

Also if E, F are equivalence relations on sets X, Y, resp., a **homomorphism** of E to F is a function $f: X \to Y$ such that $x_1 E x_2 \implies f(x_1) F f(x_2)$. Thus an invariant uniformization is a uniformization that is a homomorphism of E to Δ_Y.

Consider now the situation where X, Y are Polish spaces and P is a Borel subset of $X \times Y$. In this case standard results in descriptive set theory provide conditions which imply the existence of Borel uniformizations. These fall mainly into two categories, see [K1, Section 18]: "small section" and "large section" uniformization results. We will concentrate here on the following standard instances of these results:

Theorem 1.1 (Measure uniformization)**.** *Let X, Y be Polish spaces, μ a probability Borel measure on Y and $P \subseteq X \times Y$ a Borel set such that $\forall x \in X(\mu(P_x) > 0)$. Then P admits a Borel uniformization.*

Theorem 1.2 (Category uniformization)**.** *Let X, Y be Polish spaces and $P \subseteq X \times Y$ a Borel set such that $\forall x \in X(P_x$ is non-meager$)$. Then P admits a Borel uniformization.*

Theorem 1.3 (K_σ uniformization)**.** *Let X, Y be Polish spaces and $P \subseteq X \times Y$ a Borel set such that $\forall x \in X(P_x$ is non-empty and $K_\sigma)$. Then P admits a Borel uniformization.*

A special case of Theorem 1.3 is the following:

Theorem 1.4 (Countable uniformization)**.** *Let X, Y be Polish spaces and $P \subseteq X \times Y$ a Borel set such that $\forall x \in X(P_x$ is non empty and countable). Then P admits a Borel uniformization.*

Suppose now that E is a Borel equivalence relation on X and P in any one of these results is E-invariant. When does there exist a Borel E-invariant uniformization, i.e., a Borel uniformization that is also a homomorphism of E to Δ_Y? We say that E satisfies **measure (resp., category, $\boldsymbol{K_\sigma}$, countable) invariant uniformization** if for every Y, μ, P as in the corresponding uniformization theorem above, if P is moreover E-invariant, then it admits a Borel E-invariant uniformization.

The following gives a complete answer to this question. Recall that a Borel equivalence relation E on X is **smooth** if there is a Polish space Z and a Borel function $S\colon X \to Z$ such that $x_1 E x_2 \iff S(x_1) = S(x_2)$.

Theorem 1.5. *Let E be a Borel equivalence relation on a Polish space X. Then the following are equivalent:*
(i) E is smooth;
(ii) E satisfies measure invariant uniformization;
(iii) E satisfies category invariant uniformization;
(iv) E satisfies K_σ invariant uniformization.
(v) E satisfies countable invariant uniformization.

The equivalence of (i) and (v) in Theorem 1.5 essentially reduces to the fact that if E is a countable Borel equivalence relation (i.e., one all of which equivalence classes are countable) which is not smooth, then the relation

$$(x, y) \in P \iff xEy,$$

is clearly E-invariant with countable nonempty sections but has no E-invariant uniformization. Recently Miller [Mi] proved the following dichotomy that shows that this is essentially the only obstruction to (v). Below E_0 is the non-smooth Borel equivalence relation on $2^{\mathbb{N}}$ given by $xE_0y \iff \exists m \forall n \geq m(x_n = y_n)$ and $E_0 \times I_{\mathbb{N}}$ is the equivalence relation on $2^{\mathbb{N}} \times \mathbb{N}$ given by

$(x, m) E_0 \times I_{\mathbb{N}} (y, n) \iff x E_0 y$. Also if E, F are equivalence relations on spaces X, Y, resp., an **embedding** of E into F is an injection $\pi \colon X \to Y$ such that $x_2 E x_2 \iff \pi(x_1) F \pi(x_2)$.

Theorem 1.6 ([Mi, Theorem 2]). *Let X, Y be Polish spaces, E a Borel equivalence relation on X and $P \subseteq X \times Y$ an E-invariant Borel relation with countable non-empty sections. Then exactly of the following holds:*

(1) There is a Borel E-invariant uniformization,

(2) There is a continuous embedding $\pi_X \colon 2^{\mathbb{N}} \times \mathbb{N} \to X$ of $E_0 \times I_{\mathbb{N}}$ into E and a continuous injection $\pi_Y \colon 2^{\mathbb{N}} \times \mathbb{N} \to Y$ such that $P \cap (\pi_X(2^{\mathbb{N}} \times \mathbb{N}) \times Y) = (\pi_X \times \pi_Y)(E_0 \times I_{\mathbb{N}})$.

It would be interesting to find a dichotomy that determines the obstructions to (ii)-(iv).

We next consider a somewhat less strict notion of invariant uniformization, where instead of selecting a single point in each section we select a countable nonempty subset. More precisely, given Polish spaces X, Y, a Borel equivalence relation E on X and an E-invariant Borel set $P \subseteq X \times Y$, with $\mathrm{proj}_X(P) = X$, a Borel ***E*-invariant countable uniformization** is a Borel function $f \colon X \to Y^{\mathbb{N}}$ such that $\forall x \in X \forall n \in \mathbb{N} ((x, f(x)_n) \in P)$ and $x_1 E x_2 \implies \{f(x_1)_n \colon n \in \mathbb{N}\} = \{f(x_2)_n \colon n \in \mathbb{N}\}$. Equivalently, if for each Polish space Y, we denote by E^Y_{ctble} the equivalence relation on $Y^{\mathbb{N}}$ given by

$$(x_n) E^Y_{ctble} (y_n) \iff \{x_n \colon n \in \mathbb{N}\} = \{y_n \colon n \in \mathbb{N}\},$$

then an E-invariant countable uniformization is a Borel homomorphism f of E to E^Y_{ctble} such that for each x, n, we have that $(x, f(x)_n) \in P$..

We say that E satisfies **measure (resp., category, $\boldsymbol{K_\sigma}$) countable invariant uniformization** if for every Y, μ, P as in the corresponding uniformization theorem above, if P is moreover E-invariant, then it admits a Borel E-invariant countable uniformization.

Recall that a Borel equivalence relation E on X is **reducible to countable** if there is a Polish space Z, a countable (i.e., having all classes countable) Borel equivalence relation F on Y and a Borel function $S \colon X \to Z$ such that $x_1 E x_2 \iff S(x_1) F S(x_2)$.

As in the proof of Theorem 1.5, one can see that if a Borel equivalence relation E on X is reducible to countable, then E satisfies measure (resp. category, K_σ) countable invariant uniformization. We conjecture the following:

Conjecture 1.7. *Let E be a Borel equivalence relation on a Polish space X. Then the following are equivalent:*

(a) E is reducible to countable;

(b) E satisfies measure countable invariant uniformization;

(c) E satisfies category countable invariant uniformization;

(d) E satisfies K_σ countable invariant uniformization.

We discuss some partial results in Section 3.

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2 Proof of Theorem 1.5

(A) We first show that (i) implies (ii), the proof that (i) implies (iii) being similar. Fix a Polish space Z and a Borel function $S\colon X \to Z$ such that $x_1 E x_2 \iff S(x_1) = S(x_2)$. Fix also Y, μ, P as in the definition of measure invariant uniformization. Define $P^* \subseteq Z \times Y$ as follows:

$$(z, y) \in P^* \iff \forall x \in X\big(S(x) = z \implies (x, y) \in P\big).$$

Then P^* is $\mathbf{\Pi^1_1}$ and we have that

$$S(x) = z \implies P^*_z = P_x,$$
$$z \notin S(X) \implies P^*_z = Y.$$

Thus $\forall z \in Z(\mu(P^*_z) > 0)$. Then, by [K1, 36.24], there is a Borel function $f^*\colon Z \to Y$ such that $\forall z \in Z((z, f^*(z)) \in P^*)$. Put

$$f(x) = f^*(S(x)).$$

Then f is an E-invariant uniformization of P.

We next prove that (i) implies (iv) (and therefore (v)). Fix Z, S as in the previous case and Y, P as in the definition of K_σ invariant uniformization. Define P^* as before. Let $\varphi\colon P^* \to \omega_1$ be a $\mathbf{\Pi^1_1}$-rank. Then $A = \{(z, y)\colon z \in S(X), (z, y) \in P\}$ is a $\mathbf{\Sigma^1_1}$ subset of P^*, so, by boundedness, there is a countable ordinal α such that $\forall (z, y) \in A(\varphi(z, y) < \alpha)$. Therefore there is a Borel subset P^{**} of P^* such that $A \subseteq P^{**}$. By [K1, 35.47], the set C of all $z \in Z$ such that P^{**}_z is K_σ is $\mathbf{\Pi^1_1}$ and contains the $\mathbf{\Sigma^1_1}$ set $S(X)$, so by separation there is a Borel set B with $A \subseteq B \subseteq C$. Then if $Q \subseteq Z \times Y$ is defined by

$$(z, y) \in Q \iff z \in B \;\&\; (z, y) \in P^{**},$$

we have that

$$S(x) = z \implies Q_z = P_x,$$

and every Q_z is K_σ. It follows, by [K1, 35.46], that $D = \mathrm{proj}_Z(Q)$ is Borel and there is a Borel function $g \colon D \to Y$ such that $\forall z \in D(z, g(z)) \in Q$. Since $f(X) \subseteq D$, the function

$$f(x) = g(S(x)).$$

is an E-invariant uniformization of P.

(B) We will next show that $\neg$(i) implies $\neg$(ii), $\neg$(iii) and $\neg$(v) (and thus also $\neg$(iv)). We will use the following lemma. Below for Borel equivalence relations E, E' on Polish spaces X, X', resp., we write $E \leq_B E'$ iff there is a Borel map $f \colon X \to X'$ such that $x_1 E x_2 \iff f(x_1) E' f(x_2)$, i.e., E can be **Borel reduced** to E' (via the reduction f).

Lemma 2.1. *Let E, F be Borel equivalence relations on Polish spaces X, X', resp., such that $E \leq_B E'$. If E fails (ii) (resp., (iii), (iv), (v)), so does E'.*

Proof. Let $f \colon X \to X'$ be a Borel reduction of E into E'. Assume first that E fails (ii) with witness Y, μ, P. Define $P' \subseteq X' \times Y$ by

$$(x', y) \in P' \iff \forall x \in X\Big(f(x) E' x' \implies (x, y) \in P\Big).$$

Then note that

$$f(x) E' x' \implies P'_{x'} = P_x,$$
$$x' \notin [f(X)]_{E'} \implies P'_{x'} = Y.$$

Now clearly P' is $\mathbf{\Pi^1_1}$ and invariant under the Borel equivalence relation $E' \times \Delta_Y$. Then by a result of Solovay (see [K1, 34.6]), there is a $\mathbf{\Pi^1_1}$-rank $\varphi \colon P' \to \omega_1$ which is $E' \times \Delta_Y$-invariant. Consider then the $\mathbf{\Sigma^1_1}$ subset P'' of P' defined by:

$$(x', y) \in P'' \iff \exists x \in X\Big(f(x) E' x' \;\&\; (x, y) \in P\Big).$$

By boundedness there is a Borel $E' \times \Delta_Y$-invariant set P''' with $P'' \subseteq P''' \subseteq P'$. Let now $Z \subseteq X'$ be defined by

$$x' \in Z \iff \mu(P'''_{x'}) > 0.$$

Then Z is Borel and E'-invariant and contains $[f(X)]_{E'}$. Finally define $Q \subseteq X' \times Y$ by

$$(x', y) \in Q \iff \big(x' \in Z \ \& \ (x', y) \in P'''\big) \text{ or } x' \notin Z.$$

Then $f(x) = x' \implies Q_{x'} = P_x$, so Y, μ, Q witnesses the failure of (ii) for E'.

The case of (iii) is similar and we next consider the case of (iv). Repeat then the previous argument for case (ii) until the definition of P'''. Then define $Z' \subseteq X'$ by

$$x' \in Z' \iff P'''_{x'} \text{ is } K_\sigma \text{ and nonempty.}$$

Then Z' is $\mathbf{\Pi^1_1}$, by [K1, 35.47] and the relativization of the fact that every nonempty $\Delta^1_!$ K_σ set contains a $\Delta^1_!$ member, see [M, 4F.15]. It is also E'-invariant and contains $[f(X)]_{E'}$. Let then Z be E'-invariant Borel with $[f(X)]_{E'} \subseteq Z \subseteq Z'$ and define Q as before but replacing "$x' \notin Z$" by "$(x \notin Z$ and $y = y_0)$", for some fixed $y_0 \in Y$. Then Y, Q witnesses the failure of (iv) for E'.

Finally, the case of (v) is similar to (iv) by now defining

$$x' \in Z' \iff P'''_{x'} \text{ is countable and nonempty.}$$

and using that Z' is $\mathbf{\Pi^1_1}$ by [K1, 35.38] (and [M, 4F.15] again). □

Assume now that E is not smooth. Then by [HKL] we have $E_0 \leq_B E$. Thus by Lemma 2.1 it is enough to show that E_0 fails (ii), (iii), and (v) (thus also (iv)).

We first prove that E_0 fails (ii). We view here $2^{\mathbb{N}}$ as the Cantor group $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ with pointwise addition $+$ and we let μ be the Haar measure, i.e., the usual product measure. Let then $A \subseteq (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ be an F_σ set which has μ-measure 1 but is meager. Let $X = Y = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ and define $P \subseteq X \times Y$ as follows:

$$(x, y) \in P \iff \exists x' E_0 x (x' + y \in A).$$

Clearly P is F_σ and, since $P_x = \bigcup_{x' E_0 x} (A - x')$, clearly $\mu(P_x) = 1$. Moreover P is E_0-invariant. Assume then, towards a contradiction that f is a Borel

E_0-invariant uniformization. Since $xE_0x' \implies f(x) = f(x')$, by generic ergodicity of E_0 there is a comeager Borel E_0-invariant set $C \subseteq X$ and y_0 such that $\forall x \in C(f(x) = y_0)$, thus $\forall x \in C(x, y_0) \in P$, so $\forall x \in C \exists x' E_0 x(x' \in A - y_0)$. If $G \subseteq (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ is the subgroup consisting of the eventually 0 sequences, then $xE_0y \iff \exists g \in G(g+x = y)$, thus $C = \bigcup_{g \in G}(g+(A-y_0))$, so C is meager, a contradiction.

To show that E_0 fails (v), define

$$(x, y) \in P \iff xE_0y.$$

Then any Borel E_0-invariant uniformization of P gives a Borel selector for E_0, a contradiction.

Finally to see that E_0 fails (iii), use above $B = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}} \setminus A$, instead of A, to produce a G_δ set Q as follows:

$$(x, y) \in Q \iff \forall x' E_0 x(x' + y \in B).$$

Then Q is E_0-invariant and has comeager sections. If g is a Borel E_0-invariant uniformization, then by the ergodicity of E_0, there is a μ-measure 1 set D and y_0 such that $\forall x \in D \forall x' E_0 x(x' \in B - y_0)$, so $D \subseteq B - y_0$, thus $\mu(D) = 0$, a contradiction.

This completes the proof of Theorem 1.5.

(C) The following is an open problem:

Problem 2.2. *Is there a Polish space Y, probability Borel measure μ on Y and a G_δ set $P \subseteq 2^{\mathbb{N}} \times Y$ (or even a Borel set $P \subseteq 2^{\mathbb{N}} \times Y$ with G_δ sections) with $\mu(P_x) > 0$, which is E_0-invariant but admits no Borel E_0-invariant uniformization? Similarly, is there a Polish space Y and an F_σ set $Q \subseteq 2^{\mathbb{N}} \times Y$ (or even a Borel set $Q \subseteq 2^{\mathbb{N}} \times Y$ with F_σ sections) with $\mu(Q_x)$ non-meager, which is E_0-invariant but admits no Borel E_0-invariant uniformization?*

3 On Conjecture 1.7

Concerning Conjecture 1.7, we first note the following analog of Lemma 2.1.

Lemma 3.1. *Let E, F be Borel equivalence relations on Polish spaces X, X', resp., such that $E \leq_B E'$. If E fails (b) (resp., (c), (d)), so does E'.*

The proof is identical to that of Lemma 2.1. Note now that any countable Borel equivalence relation E trivially satisfies (b), (c), and (d), so by Lemma 3.1, in Conjecture 1.7, (a) implies (b), (c) and (d).

To verify then Conjecture 1.7, one needs to show that if E is not reducible to countable, then (b), (c) and (d) fail. It is an open problem (see [HK, end of Section 6]) whether the following holds:

Problem 3.2. *Let E be a Borel equivalence relation which is not reducible to countable. Then one of the following holds:*

(1) $E_1 \leq_B E$, where E_1 is the following equivalence relation on $(2^{\mathbb{N}})^{\mathbb{N}}$:

$$xE_1y \iff \exists m \forall n \geq m(x_n = y_n);$$

(2) There is a Borel equivalence relation F induced by a turbulent Borel action of a Polish group such that $F \leq_B E$;

(3) $E_0^{\mathbb{N}} \leq_B E$, where $E_0^{\mathbb{N}}$ is the following equivalence relation on $(2^{\mathbb{N}})^{\mathbb{N}}$:

$$xE_0^{\mathbb{N}}y \iff \forall n(x_n E_0 y_n).$$

It is therefore interesting to show that (b), (c) and (d) fail for E_1, F as in (2) above, and $E_0^{\mathbb{N}}$. Here are then some partial results.

Proposition 3.3. *Let E be a Borel equivalence relation which is not reducible to countable but is Borel reducible to a Borel equivalence relation F with K_σ classes. Then E fails (d).*

Proof. Suppose E, F live on the Polish spaces X, Y, resp., and let $g \colon X \to Y$ be a Borel reduction of E to F. Define $P \subseteq X \times X$ as follows:

$$(x, y) \in P \iff g(x)Fy.$$

Clearly P is E-invariant and has K_σ sections. Suppose then that P admitted a Borel E-invariant countable uniformization $f \colon X \to Y^{\mathbb{N}}$. Then define $h \colon X \to X$ by $g(x) = f(x)_0$. Then by [K2, Proposition 3.7], h shows that E is reducible to countable, a contradiction. □

In particular, E_1 and E_2 (as in [HK]) fail (d). Concerning (b) and (c) for E_1, the following is a possible example for their failure.

Problem 3.4. *Let $X = (2^{\mathbb{N}})^{\mathbb{N}}, Y = 2^{\mathbb{N}}$ and define $P \subseteq X \times Y$ as follows:*

$$(x, y) \in P \iff \exists m \forall n \geq m(x_n \neq y),$$

so that P is E_1-invariant and each section P_x is co-countable, so has μ-measure 1 (for μ the product measure on Y) and is comeager. Is there a Borel E_1-invariant countable uniformization of P?

One can show the following weaker result, which provides a Borel anti-diagonalization theorem for E_1.

Proposition 3.5. *Let $f\colon (2^{\mathbb{N}})^{\mathbb{N}} \to 2^{\mathbb{N}}$ be a Borel function such that $xE_1y \implies f(x) = f(y)$. Then there is $x \in (2^{\mathbb{N}})^{\mathbb{N}}$ such that for infinitely many n, $f(x) = x_n$.*

Thus if X, Y, P are as in Problem 3.4, P does not admit a Borel E_1-invariant uniformization.

Proof. For any nonempty countable set $S \subseteq 2^{\mathbb{N}}$ consider the product space $S^{\mathbb{N}}$ with the product topology, where S is taken to be discrete. Denote by $E_0(S)$ the equivalence relation on $S^{\mathbb{N}}$ given by $xE_0(S)y \iff \exists m \forall n \geq m(x_n = y_n)$. This is generically ergodic and for $x, y \in S^{\mathbb{N}}$ we have that $xE_0(S)y \implies f(x) = f(y)$, so there is (unique) $x_S \in 2^{\mathbb{N}}$ such that $f(x) = x_S$, for comeager many $x \in S^{\mathbb{N}}$. Clearly x_S can be computed in a Borel way given any $x \in (2^{\mathbb{N}})^{\mathbb{N}}$ with $S = \{x_n : n \in \mathbb{N}\}$, i.e., we have a Borel function $F\colon (2^{\mathbb{N}})^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that

$$\{x_n : n \in \mathbb{N}\} = \{y_n : n \in \mathbb{N}\} = S \implies F((x_n)) = F((y_n)) = x_S.$$

We now use the following Borel anti-diagonalization theorem of H. Friedman, see [S, Theorem 2, page 23]:

Theorem 3.6 (H. Friedman)**.** *Let E be a Borel (even analytic) equivalence relation on a Polish space X. Let $F\colon X^{\mathbb{N}} \to X$ be a Borel function such that*

$$\{[x_n]_E : n \in \mathbb{N}\} = \{[y_n]_E : n \in \mathbb{N}\} \implies F((x_n)) = F((y_n)).$$

Then there is $x \in X^{\mathbb{N}}$ and $i \in \mathbb{N}$ such that $F(x)Ex_i$.

Applying this to E being the equality relation on $2^{\mathbb{N}}$ and F as above, we conclude that for some S, we have that $x_S \in S$. Then for comeager many $x \in S^{\mathbb{N}}$ we have that $x_n = x_S$, for infinitely many n, and also $(x, x_S) \in P$, a contradiction. $\square$

In response to a question by Andrew Marks, we note the following version of Proposition 3.5 for E_1 restricted to injective sequences. Below $[2^{\mathbb{N}}]^{\mathbb{N}}$ is the Borel subset of $(2^{\mathbb{N}})^{\mathbb{N}}$ consisting of injective sequences and $x \leq_T y$ means that x is recursive in y.

Proposition 3.7. *Let $g\colon [2^{\mathbb{N}}]^{\mathbb{N}} \to 2^{\mathbb{N}}$ be a Borel function such that $xE_1y \implies g(x) = g(y)$. Then there is $y \in [2^{\mathbb{N}}]^{\mathbb{N}}$ such that for all n, $g(y) \leq_T y_n$.*

Proof. Fix a recursive bijection $x \mapsto \langle x \rangle$ from $(2^{\mathbb{N}})^{\mathbb{N}}$ to $2^{\mathbb{N}}$ and for each $i \in \mathbb{N}$ let $\bar{i} \in 2^{\mathbb{N}}$ be the characteristic function of $\{i\}$. Then for each $x \in (2^{\mathbb{N}})^{\mathbb{N}}$ and $i \in \mathbb{N}$, put

$$\bar{x}^i = \langle \bar{i}, x_i, x_{i+1}, \dots \rangle \in 2^{\mathbb{N}}.$$

and

$$x' = \langle \bar{x}^0, \bar{x}^1, \dots \rangle \in [2^{\mathbb{N}}]^{\mathbb{N}}.$$

Note that $xE_1y \implies x'E_1y'$. Finally define $f\colon (2^{\mathbb{N}})^{\mathbb{N}} \to 2^{\mathbb{N}}$ by $f(x) = g(x')$. Then by Proposition 3.5, there is $x \in (2^{\mathbb{N}})^{\mathbb{N}}$ such that for infinitely many n we have that $f(x) = x_n$. Let $y = x'$.

If n is such that $f(x) = g(y) = x_n$, then as $x_n \leq_T \bar{x}^k = y_k, \forall k \leq n$, we have that $g(y) \leq_T y_k, \forall k \leq n$. Since this happens for infinitely many n, we have that $g(y) \leq_T y_n$, for all n. $\square$

We do not know anything about $E_0^{\mathbb{N}}$ but if we let E_{ctble} be the equivalence relation $E_{ctble}^{2^{\mathbb{N}}}$ (so that $E_0^{\mathbb{N}} <_B E_{ctble}$), we have:

Proposition 3.8. *E_{ctble} fails (b) and (c).*

Proof. We will prove that E_{ctble} fails (b), the proof that it also fails (c) being similar. Let $X = (2^{\mathbb{N}})^{\mathbb{N}}, Y = 2^{\mathbb{N}}$, let μ be the usual product measure on Y and put $E = E_{ctble}$. Define $P \subseteq X \times Y$ by

$$(x, y) \in P \iff y \notin \{x_n \colon n \in \mathbb{N}\}.$$

Clearly $\mu(P_x) = 1$ and P is E-invariant. Assume now, towards a contradiction, that there is a Borel function $f\colon X \to Y^{\mathbb{N}}$ such that $\forall x \in X \forall n \in \mathbb{N}((x, f(x)_n) \in P)$ and $x_1Ex_2 \implies \{f(x_1)_n \colon n \in \mathbb{N}\} = \{f(x_2)_n \colon n \in \mathbb{N}\}$. Then

$$\forall x \in X\big(\{f(x)_n \colon n \in \mathbb{N}\} \cap \{x_n \colon n \in \mathbb{N}\} = \emptyset\big).$$

Define $F\colon X^{\mathbb{N}} \to X$ as follows: Fix a bijection $(i,j) \mapsto \langle i,j\rangle$ from $\mathbb{N}^2$ to $\mathbb{N}$ and for $n \in \mathbb{N}$ put $n = \langle n_0, n_1\rangle$. Given $x \in X^{\mathbb{N}}$, define $x' \in X$ by $x'_n = (x_{n_0})_{n_1}$. Then let $F(x) = f(x')$. First notice that

$$\{[x_n]_E\colon n \in \mathbb{N}\} = \{[y_n]_E\colon n \in \mathbb{N}\} \implies F((x_n)) = F((y_n)).$$

Thus by Theorem 3.6, there is some $x \in X^{\mathbb{N}}$ and $i \in \mathbb{N}$ such that $F(x)Ex_i$, i.e., $f(x')Ex_i$ or $\{f(x')_n\colon n \in \mathbb{N}\} = \{(x_i)_n\colon n \in \mathbb{N}\} = \{x'_{\langle i,n\rangle}\colon n \in \mathbb{N}\}$. Thus $\{f(x')_n\colon n \in \mathbb{N}\} \cap \{x'_n\colon n \in \mathbb{N}\} \neq \emptyset$), a contradiction. $\square$

Finally, we note that by the dichotomy theorem of Hjorth concerning reducibility to countable (see [H] or [K2, Theorem 3.8]), in order to prove Conjecture 1.7 for Borel equivalence relations induced by Borel actions of Polish groups, it would be sufficient to prove it for Borel equivalence relations induced by stormy such actions.

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