An effective version of Nadkarni's Theorem

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1 Introduction

The purpose of this note is to show that the notion of compressibility of a countable Borel equivalence relation (CBER) is effective, i.e., if a (lightface) Δ^1_1 CBER on the Baire space $\mathcal{N}$ is compressible, then it admits a (lightface) Δ^1_1 compression. This follows from an effective version of Nadkarni's Theorem that we state below.

First recall the following standard concepts. A **CBER** E on a standard Borel space X is a Borel equivalence relation all of whose classes are countable. A **compression** of E is an injective map $f : X \to X$ such that for each E-class C we have $f(C) \subsetneqq C$. We say that E is **compressible** if it admits a Borel compression. Finally a Borel probability measure μ on X is **invariant** for E if for any Borel bijection $f\colon X \to X$ with $f(x)Ex, \forall x$, we have that $f_*\mu = \mu$.

We now have:

Theorem 1.1 (Nadkarni's Theorem, see [N] and [KM]). *Let E be a* CBER *on the Baire space $\mathcal{N}$. Then exactly one of the following holds:*

(i) E is compressible;

(ii) E admits an invariant probability Borel measure.

We will sketch below a proof of the following effective version of Nadkarni's Theorem:

Theorem 1.2 (Effective Nadkarni's Theorem). *Let E be a (lightface) Δ^1_1* CBER *on the Baire space $\mathcal{N}$. Then exactly one of the following holds:*

(i) E has a (lightface) Δ^1_1 compression;
(ii) E admits an invariant probability Borel measure.

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2 Proof of Theorem 1.2

Definition 2.1. *A sequence (A_n) of Δ^1_1 subsets of $\mathcal{N}$ is* **uniformly** Δ^1_1 *if the relation $A \subseteq \mathbb{N} \times \mathcal{N}$ given by*

$$A(n.x) \iff x \in A_n,$$

is Δ^1_1. Similarly a sequence (f_n) of Δ^1_1 functions $f_n \colon \mathcal{N} \to \mathcal{N}$ is uniformly Δ^1_1 if the function $f \colon \mathbb{N} \times \mathcal{N} \to \mathcal{N}$ given by

$$f(n,x) = f_n(x),$$

is Δ^1_1.

We also say that a countable collection of subsets of $\mathcal{N}$ is uniformly Δ^1_1 if it admits a uniformly Δ^1_1 enumeration. Similarly for a countable set of functions.

The next result shows that the conclusion of the Effective Nadkarni's Theorem holds under certain assumptions on E.

Proposition 2.2. *Let E be a (lightface) Δ^1_1* CBER *on the Baire space $\mathcal{N}$. Assume the following:*

(1) E is induced by a Δ^1_1 action of the group $\Gamma = \mathbb{F}_\infty$ (the free group with $\aleph_0$ generators).

(2) There is a Polish 0-dimensional topology τ on $\mathcal{N}$, extending the standard topology, and a uniformly Δ^1_1 countable Boolean algebra $\mathcal{U}$ of clopen sets in τ, which is a basis for τ and is closed under the Γ-action.

(3) There is a complete compatible metric d for τ such that for every $U \in \mathcal{U}$ and $k > 0$, there is a uniformly Δ^1_1, pairwise disjoint, sequence (U_n) with $U_n \in \mathcal{U}$, $U = \bigcup_n U_n$ and $diam_d(U_n) < \frac{1}{k}$.

Then exactly one of the following holds:
(i) E has a (lightface) Δ^1_1 compression;
(ii) E admits an invariant Borel probability measure.

Proof. By a careful inspection of the proof of Nadkarni's Theorem in [M] or the original proof of Nadkarni (see, e.g., the exposition in [S]), one can see that there is a sequence (C_n) of E-invariant Δ^1_1 sets for which $E|C_n$ has a Δ^1_1 compression, such that every $x \notin \bigcup_n C_n$ defines an invariant probability Borel measure for E. We now have two cases:

Case 1: $\mathcal{N} \neq \bigcup_n C_n$.

Then clearly (ii) above holds.

Case 2: $\mathcal{N} = \bigcup_n C_n$. Fix a standard coding of the Δ^1_1 subsets of $\mathcal{N}$ given by a Π^1_1 subset $D \subseteq \mathbb{N}$ and a surjection $D \ni n \mapsto D_n$ from D to the set of Δ^1_1 subsets of $\mathcal{N}$, such that there are Σ^1_1, Π^1_1 relations $S, P \subseteq \mathbb{N} \times \mathcal{N}$ with

$$n \in D \implies [x \in D_n \iff S(n,x) \iff P(n,x)]$$

Let D' be the set of all $n \in D$ such that D_n is E-invariant and $E|D_n$ has a Δ^1_1 compression. Then by the Bounded Quantification Theorem for Π^1_1, $D' \in \Pi^1_1$. Also

$$\forall x \in \mathcal{N} \exists n (n \in D' \ \&\ x \in D_n).$$

So by the Number Uniformization Theorem for Π^1_1, there is a Δ^1_1 function $f \colon \mathcal{N} \to \mathbb{N}$ such that

$$\forall x \in \mathcal{N} (f(x) \in D' \ \&\ x \in D_{f(x)}).$$

Since $f(\mathcal{N})$ is a Σ^1_1 subset of D', let D'' be Δ^1_1 such that $f(\mathcal{N}) \subseteq D'' \subseteq D'$. Since

$$\mathcal{N} = \bigcup_{n \in D''} D_n,$$

we can write

$$\mathcal{N} = \bigcup_n K_n,$$

where (K_n) is a uniformly Δ^1_1 sequence of pairwise disjoint sets and each $E|K_n$ has a Δ^1_1 compression. Another application of the Number Uniformization Theorem for Π^1_1 shows that E admits a Δ^1_1 compression. □

To complete the proof of the Effective Nadkarni's Theorem it is thus enough to show the following:

Proposition 2.3. *Let E be a Δ^1_1* CBER *on $\mathcal{N}$, Then (1), (2), (3) of Proposition 2.2 hold.*

Proof. For (1): This follows for the proof of the Feldman-Moore Theorem (see, e.g., [S, Section 1.2]). So fix below a Δ^1_1 action of $\Gamma = \mathbb{F}_\infty$ that induces E.

For (2), (3): We will first find a topology τ as in (2), which has a uniformly Δ^1_1 countable basis $\mathcal{B}$ of clopen sets closed under the Γ-action, because we can then take $\mathcal{U}$ to be the Boolean algebra generated by $\mathcal{B}$. For (3) we will find a complete compatible Δ^1_1 metric d for τ (i.e., $d\colon \mathcal{N}^2 \to \mathbb{R}$ is Δ^1_1). Then if $(\mathcal{U}_n)$ is a uniformly Δ^1_1 enumeration of $\mathcal{U}$, we have that for each k

$$D_k = \{n\colon diam_d(\mathcal{U}_n) < \frac{1}{k}\}$$

is Π^1_1 and

$$\forall x \in \mathcal{N} \exists n (n \in D_k \ \&\ x \in \mathcal{U}_n).$$

So as before we can find, for each $k > 0$, a uniformly Δ^1_1 sequence (X_n) of sets in $\mathcal{U}$ of d-diameter less than $\frac{1}{k}$ that form a partition of $\mathcal{N}$. Finally given any $U \in \mathcal{U}$, let $U_n = X_n \cap U$.

It thus remains to find τ, d with these properties. We will need first a couple of lemmas.

Lemma 2.4. *Let $A \subseteq \mathcal{N}$ be Δ^1_1. Then there is a Polish 0-dimensional topology τ_A on $\mathcal{N}$, which extends the standard topology, has a uniformly Δ^1_1 countable basis consisting of clopen sets containing A, and has a complete compatible Δ^1_1 metric d_A.*

Proof. Let $f\colon \mathcal{N} \to \mathcal{N}$ be computable and let $B \subseteq \mathcal{N}$ be Π^0_1 and such that $f|B$ is injective and $f(B) = A$. Use f to move the (relative) topology of B to A and the standard metric of B to A. Do the same for $\mathcal{N} \setminus A$ and then take the direct sum of these topologies and metrics on $A, \mathcal{N} \setminus A$ to find τ_A, d_A. $\square$

Lemma 2.5. *Let $\mathcal{A} = (A_n)$ be a uniformly Δ^1_1 sequence of subsets of $\mathcal{N}$. Then there is a Polish 0-dimensional topology $\tau_{\mathcal{A}}$ on $\mathcal{N}$, which extends the standard topology, has a uniformly Δ^1_1 countable basis $\mathcal{B}_{\mathcal{A}}$ containing all the ssts in $\mathcal{A}$, and has a complete compatible Δ^1_1 metric $d_{\mathcal{A}}$.*

Proof. Consider $\tau_{\mathcal{A}_n}, d_{\mathcal{A}_n}$ as in Lemma 2.4. Then put

$$\tau_{\mathcal{A}} = \text{the topology generated by } \bigcup_n \tau_{\mathcal{A}_n}.$$

Then by [K, Lemma 13.3], $\tau_{\mathcal{A}}$ is Polish (and contains the standard topology). A basis for $\tau_{\mathcal{A}}$ consists of all sets of the form

$$U_1 \cap U_2 \cap \cdots \cap U_n,$$

where $U_i \in \mathcal{B}_{A_{j_i}}, 1 \leq I \leq n$, and so it is 0-dimensional with a uniformly Δ^1_1 basis $\mathcal{B}_{\mathcal{A}}$ containing all the sets in $\mathcal{A}$.

Finally, as in the proof of [K, Lemma 13.3] again, a complete compatible metric for $\tau_{\mathcal{A}}$ is

$$d_{\mathcal{A}}(x, y) = \sum_n 2^{-n-1} \cdot \frac{d_{A_n}(x, y)}{1 + d_{A_n}(x, y)}.$$

Because of the uniformity in the proof of Lemma 2.4 this metric is also Δ^1_1. □

We finally find τ, d. To do this we recursively define a sequence of Polish 0-dimensional topologies $\tau_0, \tau_1, \ldots$ on $\mathcal{N}$, extending the standard topology, and uniformly Δ^1_1 countable bases $\mathcal{B}_n$ for τ_n and complete compatible Δ^1_1 metrics d_n for τ_n, *all uniformly in* n *as well*, and such that $\Gamma \cdot \mathcal{B}_n \subseteq \mathcal{B}_{n+1}$.

For $n = 0$, let $\tau_0, d_0, \mathcal{B}_0$ be the standard topology, metric and basis for $\mathcal{N}$.

Given $\tau_n, d_n, \mathcal{B}_n$, consider $\Gamma \cdot \mathcal{B}_n$ and use Lemma 2.5 to define $\tau_{n+1}, \mathcal{B}_{n+1} \supseteq \Gamma \cdot \mathcal{B}_n, d_{n+1}$. The uniformity in n is clear from the construction.

Finally let τ be the topology generated by $\bigcup_n \tau_n$. It is Polish, 0-dimensional with basis the sets of the form

$$U_1 \cap U_2 \cap \cdots \cap U_n,$$

with $U_i \in \mathcal{B}_{j_i}, 1 \leq i \leq n$, so this is a uniformly Δ^1_1 countable basis $\mathcal{B}$ consisting of clopen sets. Also clearly for any $\gamma \in \Gamma$,

$$\gamma \cdot (U_1 \cap U_2 \cap \cdots \cap U_n) = \gamma \cdot U_1 \cap \gamma \cdot U_2 \cap \cdots \cap \gamma \cdot U_n,$$

where $\gamma \cdot U_i \in \mathcal{B}_{j_i+1}$, thus $\gamma \cdot (U_1 \cap U_2 \cap \cdots \cap U_n) \in \mathcal{B}$ as well. Finally, as before, a complete compatible Δ^1_1 metric for τ is

$$d(x, y) = \sum_n 2^{-n-1} \cdot \frac{d_n(x, y)}{1 + d_n(x, y)}$$

and the proof is complete. □

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