Problem 50 (Banach). Prove that the integral of Denjoy is not a Baire functional in the space S (that is to say, in the space of measurable functions).

Commentary

The formulation of this problem is rather vague. However it was speculated that the results in the paper [1], could be construed as a solution to this problem; see, e.g., the review of H. Becker, MR0934228 (89g:03067) of [2] (which contains a summary of the results in [1]).

We present below a plausible precise interpretation of Problem 50 and explain how the results in the above paper provide a positive solution.

Denote by $\mathcal{S}$ the set of (Lebesgue) measurable functions $f : [0,1] \to \mathbb{R}$ (we use here the interval $[0,1]$ but of course everything below works for any interval $[a,b]$). We let for $f, g \in \mathcal{S}$, $f \sim g \iff f = g, a.e.$. Let $S = \mathcal{S}/\sim$ be the space of measurable functions (modulo equality a.e.). This is a topological vector space with the topology induced by the invariant (under translation), complete, separable metric

$$d([f]_\sim, [g]_\sim) = \int_0^1 \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} dx.$$

We refer to Chapter 7 of the book [3], for a detailed introduction to the Denjoy integral. We will only need the following facts.

The Denjoy integral is defined on a subset $\mathcal{DI}$ of $\mathcal{S}$ (the set of *Denjoy integrable functions*) such that $f \in \mathcal{DI}, g \sim f \implies g \in \mathcal{DI}$. Let $DI = \{[f]_\sim : f \in \mathcal{DI}\}$. For $f \in \mathcal{DI}$ the Denjoy integral of f is a continuous function F on $[0,1]$, uniquely determined up to a constant, and we denote by $\mathcal{I}(f) = F(1) - F(0)$ the corresponding definite integral. Moreover $f \sim g \implies \mathcal{I}(f) = \mathcal{I}(g)$, so $\mathcal{I}$ descends to a unique function $I : DI \to \mathbb{R}$, given by $I([f]_\sim) = \mathcal{I}(f)$. The crucial property of the Denjoy integral is now the following: If $F : [0,1] \to \mathbb{R}$ is a differentiable function with $F' = f$, then $f \in \mathcal{DI}$ and $\mathcal{I}(f) = F(1) - F(0)$, i.e, the Denjoy integral recovers the primitive of any derivative.

We can now formulate a precise version of Banach's problem: Prove that the function $I : DI \to \mathbb{R}$ is not in the Baire class of functions (from the separable metrizable space DI into $\mathbb{R}$), i.e., the smallest class of functions containing the continuous functions and closed under limits of pointwise convergent sequences of functions. Equivalently this means that I is not a Borel function (i.e., the preimage of some open set is not Borel in DI).

Under this interpretation, it is a corollary of the results in [1] that this is true. This can be seen as follows.

Let $C = C([0,1])$ be the Banach space of continuous functions on $[0,1]$ and consider the infinite product space $C^{\mathbb{N}}$, a Polish space. Let CN be the subset of $C^{\mathbb{N}}$ consisting of all pointwise convergent sequences of continuous functions and for $\bar{f} = (f_n) \in CN$, let $\lim \bar{f}$ be its pointwise limit, which is clearly in $\mathcal{S}$. It is easy to check that the function $L(\bar{f}) = [\lim\ \bar{f}]_\sim$ from CN into S is Borel. Let now D be the subset of CN consisting of all sequences $\bar{f}$ such that $\lim \bar{f}$ is a derivative (of some differentiable function on $[0,1]$). Then $L_D = L|D$ is a Borel function from D into DI. Thus if I was a Borel function, so would be the composition $I \circ L_D$ from D to $\mathbb{R}$. In particular, the set of all $\bar{f} \in D$ such that $\mathcal{I}(\lim \bar{f}) > 0$ would be a Borel subset of D, contradicting Theorem 4 of [1].

On the other hand it can be shown that the set DI is coanalytic in S and that the Denjoy integral $I : DI \to \mathbb{R}$ is $\mathbf{\Delta}^1_1$-measurable on DI, i.e., the preimage of any open set is both analytic and coanalytic in DI. This is due to Ajtai (unpublished). A proof can be also given using the techniques in [1].

References

[1] R. Dougherty and A.S. Kechris, The complexity of antidifferentiation, *Advances in Mathematics*, **(88)(2)** (1991), 145–169.

[2] A.S. Kechris, The complexity of antidifferentiation, Denjoy totalization, and hyperarithmetic reals, *Proceedings of the International Congress of Mathematicians, Vol. 1, 2* (Berkeley, Calif., 1986), 307–313, Amer. Math. Soc., Providence, RI, 1987.

[3] R.A. Gordon, *The Integrals of Lebesgue, Denjoy, Perron, and Henstock*, Graduate Studies in Mathematics, Volume 4, Amer. Math. Soc., 1994.

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