Problem 50 (Banach). Prove that the integral of Denjoy is not a Baire functional in the space $S$ (that is to say, in the space of measurable functions).

Commentary

The formulation of this problem is rather vague. However it was speculated that the results in the paper [1], could be construed as a solution to this problem; see, e.g., the review of H. Becker, MR0934228 (89g:03067) of [2] (which contains a summary of the results in [1]).

We present below a plausible precise interpretation of Problem 50 and explain how the results in the above paper provide a positive solution.

Denote by $S$ the set of (Lebesgue) measurable functions $f : [0, 1] \to \mathbb{R}$ (we use here the interval $[0, 1]$ but of course everything below works for any interval $[a, b]$). We let for $f, g \in S$, $f \sim g \iff f = g, a.e.$ Let $S = S/\sim$ be the space of measurable functions (modulo equality a.e.). This is a topological vector space with the topology induced by the invariant (under translation), complete, separable metric

$$d([f]_\sim, [g]_\sim) = \int_0^1 \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} dx.$$  

We refer to Chapter 7 of the book [3], for a detailed introduction to the Denjoy integral. We will only need the following facts.

The Denjoy integral is defined on a subset $DI$ of $S$ (the set of Denjoy integrable functions) such that $f \in DI, g \sim f \implies g \in DI$. Let $DI = \{[f]_\sim : f \in DI\}$. For $f \in DI$ the Denjoy integral of $f$ is a continuous function $F$ on $[0, 1]$, uniquely determined up to a constant, and we denote by $I(f) = F(1) - F(0)$ the corresponding definite integral. Moreover $f \sim g \implies I(f) = I(g)$, so $I$ descends to a unique function $I : DI \to \mathbb{R}$, given by $I([f]_\sim) = I(f)$. The crucial property of the Denjoy integral is now the following: If $F : [0, 1] \to \mathbb{R}$ is a differentiable function with $F' = f$, then $f \in DI$ and $I(f) = F(1) - F(0)$, i.e., the Denjoy integral recovers the primitive of any derivative.

We can now formulate a precise version of Banach’s problem: Prove that the function $I : DI \to \mathbb{R}$ is not in the Baire class of functions (from the separable metrizable space $DI$ into $\mathbb{R}$), i.e., the smallest class of functions containing the continuous functions and closed under limits of pointwise convergent sequences of functions. Equivalently this means that $I$ is not a Borel function (i.e., the preimage of some open set is not Borel in $DI$).
Under this interpretation, it is a corollary of the results in [1] that this is true. This can be seen as follows.

Let $C = C([0, 1])$ be the Banach space of continuous functions on $[0, 1]$ and consider the infinite product space $C^\mathbb{N}$, a Polish space. Let $CN$ be the subset of $C^\mathbb{N}$ consisting of all pointwise convergent sequences of continuous functions and for $\bar{f} = (f_n) \in CN$, let $\lim \bar{f}$ be its pointwise limit, which is clearly in $S$. It is easy to check that the function $L(\bar{f}) = [\lim \bar{f}]_\sim$ from $CN$ into $S$ is Borel. Let now $D$ be the subset of $CN$ consisting of all sequences $\bar{f}$ such that $\lim \bar{f}$ is a derivative (of some differentiable function on $[0, 1]$). Then $L_D = L|D$ is a Borel function from $D$ into $DI$. Thus if $I$ was a Borel function, so would be the composition $I \circ L_D$ from $D$ to $\mathbb{R}$. In particular, the set of all $\bar{f} \in D$ such that $I(\lim \bar{f}) > 0$ would be a Borel subset of $D$, contradicting Theorem 4 of [1].

On the other hand it can be shown that the set $DI$ is coanalytic in $S$ and that the Denjoy integral $I : DI \to \mathbb{R}$ is $\Delta^1_2$-measurable on $DI$, i.e., the preimage of any open set is both analytic and coanalytic in $DI$. This is due to Ajtai (unpublished). A proof can be also given using the techniques in [1].

References


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