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Ramsey properties of finite measure algebras and topological dynamics of the group of measure preserving automorphisms: some results and an open problem

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Dedicated to Hugh Woodin on his 60th birthday

Abstract. We study in this paper ordered finite measure algebras from the point of view of Fraïssé and Ramsey theory. We also propose an open problem, which is a homogeneous version of the Dual Ramsey Theorem of Graham-Rothschild, and derive consequences of a positive answer to the study of the topological dynamics of the automorphism group of a standard probability space and also the group of measure preserving homeomorphisms of the Cantor space.

1. Introduction

In this paper we continue the theme developed in the paper Kechris-Pestov-Todorcevic [**KPT**] of exploring the connections between structural Ramsey theory and topological dynamics of Polish groups, especially the study of extreme amenability and the calculation of universal minimal flows.

A topological group G is called *extremely amenable* (or said to have the *fixed point on compacta property*) if every continuous action of G on a (non-empty, Hausdorff) compact space has a fixed point. There is now a plethora of Polish groups known to have this very strong fixed point property, e.g., the unitary group of a separable infinite-dimensional Hilbert space (Gromov-Milman [**GM**]), the automorphism group of the ordered rationals (Pestov [**P1**]), the isometry group of the Urysohn space (Pestov [**P2**]), the automorphism group of a standard measure space (Giordano-Pestov [**GP**]), etc. An excellent exposition of the theory of such groups and its connections with other areas, such as asymptotic geometric analysis, in particular concentration of measure phenomena, and Ramsey theory, can be found in

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the recent book Pestov [**P3**]. As pointed out in this monograph there are primarily two main known general techniques for establishing extreme amenability: one analytical, via the concept of a Lévy group and concentration of measure, and the other combinatorial, via finite Ramsey theory.

In [**KPT**] structural Ramsey theory has been used to prove extreme amenability of many automorphism groups of countable structures, where concentration of measure techniques are not directly applicable. It is interesting to investigate to what extent Ramsey methods can be used to establish extreme amenability for other types of groups of a more analytic or topological nature and a first step in that direction was taken in [**KPT**], where a new proof of the extreme amenability of the isometry group of the Urysohn space, $\mathrm{Iso}(\mathbf{U})$, was established, a result originally proved in Pestov [**P2**] using concentration of measure (see also Pestov [**P3**]). An important dividend of this approach is that it motivates the discovery of new classes that satisfy the Ramsey Property. For example, the Ramsey theory approach to the extreme amenability of $\mathrm{Iso}(\mathbf{U})$ led [**KPT**] to conjecture that the class of finite metric spaces has the Ramsey Property, which was indeed established in Neśetřil [**N**].

Until now this example is the only one in which combinatorial Ramsey techniques are used to replace analytical arguments in establishing extreme amenability. We propose in this paper to apply such techniques to another important example of an extremely amenable group, namely the automorphism group $\mathrm{Aut}(X, \mu)$ of a standard probability measure space (X, μ). This result was originally established in Giordano-Pestov [**GP**] by concentration of measure arguments. Again our approach leads us to propose a new Ramsey theorem, a homogeneous version of the Dual Ramsey Theorem. A proof of this Ramsey theorem would also allow us to compute the universal minimal flow of another interesting group, namely the group of measure preserving homeomorphisms of the Cantor space (with the topology it inherits as a closed subgroup of the homeomorphism group).

We now proceed to describe more precisely the contents of this paper. First we recall some basic notions that are discussed in detail in Section 2.

A class of finite structures in a given countable language is said to be a *Fraïssé class* if it is unbounded (i.e., contains structures of arbitrarily large finite cardinality), has only countably many members, up to isomorphism, is hereditary (under embeddability) and satisfies the joint embedding and amalgamation properties. For such a class $\mathcal{K}$ there is a unique countably infinite structure, its *Fraïssé limit*, $\mathrm{Flim}(\mathcal{K})$, which is locally finite (finitely generated substructures are finite), ultrahomogeneous (isomorphisms between finite substructures extend to automorphisms) and whose class of finite substructures, up to isomorphism, coincides with $\mathcal{K}$.

In this paper we study $\mathcal{K} = \mathcal{OMBA}_{\mathbb{Q}_2}$, the class of naturally ordered finite measure algebras with measure taking values in the dyadic rationals. These are structures of the form $\boldsymbol{A} = \langle A, \wedge, \vee, 0, 1, \mu, < \rangle$, where $\langle A, \wedge, \vee, 0, 1\rangle$ is a finite Boolean algebra, $\mu : A \to [0, 1]$ a positive measure on A with values in the set $\mathbb{Q}_2$ of dyadic rationals, and $<$ an ordering induced antilexicographically by an ordering of the atoms. We first show the following result.

Theorem 1.1. *The class $\mathcal{K} = \mathcal{OMBA}_{\mathbb{Q}_2}$ is a Fraïssé class.*

It is not hard then to compute that the Fraïssé limit of this class is

$$\langle \boldsymbol{B}_\infty, \lambda, \prec \rangle,$$

where $\boldsymbol{B}_\infty$ is the algebra of clopen subsets of the Cantor space $2^{\mathbb{N}}$ (i.e., the countable atomless Boolean algebra), λ is the usual product measure on $2^{\mathbb{N}}$ and $\prec$ the so-called *canonical ordering* on $\boldsymbol{B}_\infty$, defined by

$$\langle \boldsymbol{B}_\infty, \prec \rangle \cong \mathrm{Flim}(\mathcal{OBA}),$$

where $\mathcal{OBA}$ is the class of naturally ordered finite Boolean algebras (see [**KPT**], 6,**(D)**).

Next recall that a class $\mathcal{K}$ has the *Ramsey Property* (RP) if for any two structures $\boldsymbol{A} \leq \boldsymbol{B}$ in $\mathcal{K}$ (where $\boldsymbol{A} \leq \boldsymbol{B}$ means that $\boldsymbol{A}$ can be embedded in $\boldsymbol{B}$) and $N = 1, 2, \ldots$, there is a $\boldsymbol{C} \in \mathcal{K}$, with $\boldsymbol{B} \leq \boldsymbol{C}$, such that for any coloring with N colors of the set of isomorphic copies of $\boldsymbol{A}$ in $\boldsymbol{C}$, there is a copy $\boldsymbol{B}'$ of $\boldsymbol{B}$ in $\boldsymbol{C}$ so that all the copies of $\boldsymbol{A}$ contained in $\boldsymbol{B}'$ have the same color. We now propose the following problem:

Problem 1.2. Is it true that the class $\mathcal{K} = \mathcal{OMBA}_{\mathbb{Q}_2}$ has the Ramsey Property?

In fact this statement can be viewed as a homogeneous version of the Dual Ramsey Theorem of Graham-Rothschild [**GR**] which asserts the following: Given $k < \ell$ and N there is $m > \ell$ so that for any coloring with N colors of the set of equivalence relations on $\{1, \ldots, m\}$ with k many classes, there is an equivalence relation F with ℓ many classes, such that any coarser than F equivalence relation with k many classes has the same color.

It is easy to show that the Dual Ramsey Theorem is equivalent to the assertion that the class of finite Boolean algebras $\mathcal{BA}$ has the Ramsey Property. One can now see that the class $\mathcal{K} = \mathcal{OMBA}_{\mathbb{Q}_2}$ has the Ramsey Property iff the class of homogeneous finite measure Boolean algebras with measure taking values in $\mathbb{Q}_2$ has the Ramsey Property. Here *homogeneous* means that all atoms have the same measure. This in turn translates to the following homogeneous version of the Dual Ramsey Theorem. Call an equivalence relation *homogeneous* if all its classes have the same cardinality. Then we have that for each $k < \ell, N$, there is $m > \ell$ such that for any coloring with N colors of the set of homogeneous equivalence relations on $\{1, \ldots, 2^m\}$ with 2^k many classes, there is a homogeneous equivalence relation F on $\{1, 2, \ldots, 2^m\}$ with 2^ℓ many classes such that all coarser homogeneous equivalence relations with 2^k many classes have the same color.

In fact there is no reason to restrict ourselves to numbers that are powers of two, except for some obvious restrictions: Let below $k \ll \ell$ mean that $k < \ell$ and k divides ℓ. Then we have the following problem.

Problem 1.3 (Homogeneous Dual Ramsey). Is it true that given N and $k \ll \ell$, there is $m \gg \ell$ such that for any coloring with N colors of the homogeneous equivalence relations on $\{1, \ldots, m\}$ with k many classes, there is a homogeneous equivalence relation F with ℓ many classes, so that all homogeneous equivalence relations coarser than F with k many classes have the same color?

Assuming now a positive answer to Problem 1.2, we can apply the general theory of [**KPT**] to give a combinatorial proof of the extreme amenability for the automorphism group $\mathrm{Aut}(X, \mu)$ of a standard measure space (X, μ). Clearly we can take $X = 2^{\mathbb{N}}, \mu = \lambda$. By Theorem 1.1, a positive answer to Problem 1.2 and the results in [**KPT**], $\mathrm{Aut}(\boldsymbol{B}_\infty, \lambda, \prec)$ is extremely amenable. Now (the Polish group) $\mathrm{Aut}(\boldsymbol{B}_\infty, \lambda, \prec)$, with the pointwise convergence topology, can be continuously embedded in a natural way in $\mathrm{Aut}(2^{\mathbb{N}}, \lambda)$, where, as usual, $\mathrm{Aut}(2^{\mathbb{N}}, \lambda)$ has the standard (weak) topology.

THEOREM 1.4. *The image of the natural embedding of* $\mathrm{Aut}(\boldsymbol{B}_\infty, \lambda, \prec)$ *in the group* $\mathrm{Aut}(2^{\mathbb{N}}, \lambda)$ *is dense in* $\mathrm{Aut}(2^{\mathbb{N}}, \lambda)$.

It is clear that if an extremely amenable group G embeds densely in a group H, then H is also extremely amenable. Thus, assuming a positive answer to Problem 2, one obtains a new, combinatorial proof of the Giordano-Pestov [**GP**] result that the automorphism group of a standard measure space, $\mathrm{Aut}(X, \mu)$, is extremely amenable.

One can also use Theorem 1.1, a positive answer to Problem 1.2 (plus a bit more), as well as the results in [**KPT**], to calculate the universal minimal flow of another interesting group. Recall that the *universal minimal flow* of a topological group G is the (unique up to isomorphism) compact G-flow X (i.e., compact space with a continuous action of G) which is minimal (orbits are dense) and has the property that for any minimal G-flow Y there is a continuous epimorphism from X to Y preserving the actions.

THEOREM 1.5. *Assuming a positive answer to Problem 1.2, the universal minimal flow of the group of measure preserving homeomorphisms of the Cantor space* $2^{\mathbb{N}}$ *(with the usual product measure* λ*) is the (canonical action of this group on the) space of all orderings on the Boolean algebra* $\boldsymbol{B}_\infty$ *of clopen sets which have the property that their restrictions to finite subalgebras are natural.*

We note here that in [**KPT**] it is shown that the universal minimal flow of the group of homeomorphisms of $2^{\mathbb{N}}$ is (its canonical action on) the same space of orderings.

2. Preliminaries

(A) We will first review some standard concepts concerning Fraïssé classes.

Fix a countable signature L. A *Fraïssé class* in L is a class of finite structures in L which contains structures of arbitrarily large (finite) cardinality, contains only countably many structures, up to isomorphism, and satisfies the following properties:

(i) (*Hereditary Property* – HP) If $\boldsymbol{B} \in \mathcal{K}$ and $\boldsymbol{A} \leq \boldsymbol{B}$, then $\boldsymbol{A} \in \mathcal{K}$. (Here $\boldsymbol{A} \leq \boldsymbol{B}$ means that $\boldsymbol{A}$ can be embedded in $\boldsymbol{B}$.)

(ii) (*Joint Embedding Property* – JEP) If $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{K}$, then there is $\boldsymbol{C} \in \mathcal{K}$ with $\boldsymbol{A} \leq \boldsymbol{C}, \boldsymbol{B} \leq \boldsymbol{C}$.

(iii) (*Amalgamation Property* – AP) If $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathcal{K}$ and $f : \boldsymbol{A} \to \boldsymbol{B}, g : \boldsymbol{A} \to \boldsymbol{C}$ are embeddings, then there is $\boldsymbol{D} \in \mathcal{K}$ and embeddings $r : \boldsymbol{B} \to \boldsymbol{D}, s : \boldsymbol{C} \to \boldsymbol{D}$ such that $r \circ f = s \circ g$.

A structure $\boldsymbol{F}$ in L is called a *Fraïssé structure* if it is countably infinite, locally finite (i.e., finite generated substructures are finite) and *ultrahomogeneous*, i.e., every isomorphism between finite substructures of $\boldsymbol{F}$ can be extended to an automorphism of $\boldsymbol{F}$.

For a Fraïssé structure $\boldsymbol{F}$, we denote by $\mathrm{Age}(\boldsymbol{F})$ the class of all finite structures that can be embedded in $\boldsymbol{F}$.

A basic theorem of Fraïssé associates to each Fraïssé class $\mathcal{K}$ a unique (up to isomorphism) Fraïssé structure $\boldsymbol{F}$ such that $\mathcal{K} = \mathrm{Age}(\boldsymbol{F})$. This is denoted by

$$\mathrm{Flim}(\mathcal{K})$$

and called the *Fraïssé limit* of $\mathcal{K}$. The map $\mathcal{K} \mapsto \mathrm{Flim}(\mathcal{K})$ is a bijection between Fraïssé classes and Fraïssé structures (up to isomorphism) with inverse $\boldsymbol{F} \mapsto \mathrm{Age}(\boldsymbol{F})$.

An *order Fraïssé class* in a signature $L \supseteq \{<\}$ is a Fraïssé class $\mathcal{K}$ such that for each $\boldsymbol{A} \in \mathcal{K}, <^{\boldsymbol{A}}$ is a linear ordering.

Let $L \supseteq L_0$ be two signatures. For a structure $\boldsymbol{A}$ in $L, \boldsymbol{A}|L_0$ is its *reduct* to L_0. We will also call $\boldsymbol{A}$ an expansion of $\boldsymbol{A}|L_0$. If $\mathcal{K}$ is a class of structures in L we denote by

$$\mathcal{K}|L_0 = \{\boldsymbol{A}|L_0 : \boldsymbol{A} \in \mathcal{K}\}$$

the class of reducts to L_0 of structures in $\mathcal{K}$. We also call $\mathcal{K}|L_0$ the *reduct* of $\mathcal{K}$ to L_0 and call $\mathcal{K}$ an *expansion* of $\mathcal{K}|L_0$ to L.

We say that $\mathcal{K}$ is *reasonable* (with respect to (L, L_0)) if for every $\boldsymbol{A}_0, \boldsymbol{B}_0 \in \mathcal{K}|L_0$, embedding $\pi : \boldsymbol{A}_0 \to \boldsymbol{B}_0$ and expansion $\boldsymbol{A}$ of $\boldsymbol{A}_0$ such that $\boldsymbol{A} \in \mathcal{K}$ there is an expansion $\boldsymbol{B}$ of $\boldsymbol{B}_0$ such that $\boldsymbol{B} \in \mathcal{K}$, so that $\pi : \boldsymbol{A} \to \boldsymbol{B}$ is also an embedding. The proof of Proposition 5.2 in [**KPT**] can be trivially adapted to show that if $L \supseteq L_0, \mathcal{K}$ is a Fraïssé class in $L, \mathcal{K}_0 = \mathcal{K}|L_0$ and $\boldsymbol{F} = \mathrm{Flim}(\mathcal{K}), \boldsymbol{F}_0 = \mathrm{Flim}(\mathcal{K}_0)$, then the following are equivalent:

(i) $\mathcal{K}_0$ is a Fraïssé class and $\boldsymbol{F}_0 = \mathrm{Flim}(\mathcal{K}_0)$

(ii) $\mathcal{K}$ is reasonable.

Given a signature $L \supseteq \{<\}$, let $L_0 = L \setminus \{<\}$. If $\mathcal{K}$ is a class of structures in L, let $\mathcal{K}_0 = \mathcal{K}|L_0$. We say that $\mathcal{K}$ satisfies the *ordering property* if for every $\boldsymbol{A}_0 \in \mathcal{K}_0$, there is $\boldsymbol{B}_0 \in \mathcal{K}_0$ such that for every linear ordering $\prec$ on A_0 (the universe of $\boldsymbol{A}_0$) and linear ordering $\prec'$ on B_0, if $\boldsymbol{A} = \langle \boldsymbol{A}_0, \prec \rangle \in \mathcal{K}$ and $\boldsymbol{B} = \langle \boldsymbol{B}, \prec' \rangle \in \mathcal{K}$, then $\boldsymbol{A} \leq \boldsymbol{B}$.

Finally, for any countable structure $\boldsymbol{A}$ we denote by $\mathrm{Aut}(\boldsymbol{A})$ its automorphism group. It is a closed subgroup of the group of all permutations of A under the pointwise convergence topology, thus a Polish group.

(B) We next recall some basic concepts of topological dynamics. A topological group is always assumed to be Hausdorff and a compact space non-empty and Hausdorff.

Given a topological group G a *G-flow* is a continuous action of G on a compact space X. If no confusion arises, we simply identify X with the flow. A G-flow is *minimal* if all orbits are dense. A minimal G-flow X is *universal* if for every minimal G-flow Y, there is a continuous surjection $\pi : X \to Y$ preserving the actions. A basic fact of topological dynamics asserts the existence of a universal minimal flow for each G, which is unique up to isomorphism (i.e., homeomorphism preserving the actions). It is denoted by $M(G)$.

We call G *extremely amenable* if $M(G)$ is a single point. Since every G-flow contains a minimal subflow, this is equivalent to the statement that every G-flow has a fixed point.

(C) We now review some concepts of Ramsey theory.

Given structures $\boldsymbol{A}, \boldsymbol{B}$ in a signature L with $\boldsymbol{A} \leq \boldsymbol{B}$ we denote by $\binom{\boldsymbol{B}}{\boldsymbol{A}}$ the set of all substructures of $\boldsymbol{B}$ which are isomorphic to $\boldsymbol{A}$, i.e.,

$$\binom{\boldsymbol{B}}{\boldsymbol{A}} = \{\boldsymbol{A}' \subseteq \boldsymbol{B} : \boldsymbol{A}' \cong \boldsymbol{A}\},$$

where $\boldsymbol{A}' \subseteq \boldsymbol{B}$ means that $\boldsymbol{A}'$ is a substructure of $\boldsymbol{B}$. Given $N = 1, 2, \ldots$ and $\boldsymbol{A} \leq \boldsymbol{B} \leq \boldsymbol{C}$ (all structures in L) the notation

$$\boldsymbol{C} \to (\boldsymbol{B})^{\boldsymbol{A}}_N$$

signifies that for every coloring $c : \binom{\boldsymbol{C}}{\boldsymbol{A}} \to \{1, \ldots, N\}$, there is $\boldsymbol{B}' \in \binom{\boldsymbol{C}}{\boldsymbol{B}}$ such that c is constant on $\binom{\boldsymbol{B}'}{\boldsymbol{A}}$. We say that a class $\mathcal{K}$ of structures in L, closed under isomorphism, satisfies the *Ramsey Property* (RP) is for every $N \geq 1$ and $\boldsymbol{A} \leq \boldsymbol{B}$ in $\mathcal{K}$ there is $\boldsymbol{C} \in \mathcal{K}, \boldsymbol{C} \geq \boldsymbol{B}$ such that $\boldsymbol{C} \to (\boldsymbol{B})^{\boldsymbol{A}}_N$.

3. Finite ordered measure algebras

We denote by $\mathcal{BA}$ the class of all finite Boolean algebras (viewed as first order structures in the language $\{\wedge, \vee, 0, 1\}$). A *natural ordering* on such a Boolean algebra is one induced antilexicographically by an ordering of the atoms (see [**KPT**, 6, **(D)**]). We denote by $\mathcal{OBA}$ the class of all finite Boolean algebras with natural orderings (in the language $\{\wedge, \vee, 0, 1, <\}$). If $\mathbb{Q}_2$ is the ring of dyadic rationals, let $\mathcal{MBA}_{\mathbb{Q}_2}$ be the class of all finite measure algebras taking values in $\mathbb{Q}_2$. These can be viewed as first-order structures in the language $\{\wedge, \vee, 0, 1, M_r\}_{r \in \mathbb{Q}_2}$, where M_r are unary relations with the intended meaning: $M_r(a) \Leftrightarrow \mu(a) = r$, where μ is the measure in the Boolean algebra. Recall that measures in Boolean algebras are always *positive*: $\mu(a) > 0$, if $a \neq 0$. Finally let

$$\mathcal{K} = \mathcal{OMBA}_{\mathbb{Q}_2}$$

be the class of all finite naturally ordered Boolean algebras with measure taking values in $\mathbb{Q}_2$. Thus every $\boldsymbol{A} \in \mathcal{K}$ is of the form

$$\boldsymbol{A} = \langle A, \wedge, \vee, 0, 1, <, \mu \rangle,$$

where $\langle A, \wedge, \vee, 0, 1, < \rangle$ is a naturally ordered Boolean algebra and $\mu : A \to [0, 1]$ is a measure on A with $\mu(a) \in \mathbb{Q}_2$. (Again these can be viewed as structures in the language $L = \{\wedge, \vee, 0, 1, <, M_r\}_{r \in \mathbb{Q}_2}$.) Let $L_0 = L \setminus \{<\}$.

THEOREM 3.1. *The class $\mathcal{K} = \mathcal{OMBA}_{\mathbb{Q}_2}$ is a reasonable, with respect to (L, L_0), Fraïssé order class.*

PROOF. The proof that $\mathcal{K}$ is reasonable is as in [**KPT**, 6.13] and we will not repeat it here.

Since $\mathcal{MBA}_{\mathbb{Q}_2}$ and $\mathcal{OBA}$ are hereditary (for the second, see [**KPT**, 6.13]), so is $\mathcal{K}$.

JEP follows from AP by considering the 2-element Boolean algebra. Finally we verify that $\mathcal{K}$ satisfies AP.

Fix $\boldsymbol{B} \in \mathcal{K}$, say with atoms

$$b_1 <_{\boldsymbol{B}} \cdots <_{\boldsymbol{B}} b_n$$

(where $<_{\boldsymbol{B}}$ is the ordering in $\boldsymbol{B}$$-$the case $n = 1$ corresponds to the 2-element Boolean algebra). Fix also embeddings $f : \boldsymbol{B} \to \boldsymbol{C} \in \mathcal{K}, g : \boldsymbol{B} \to \boldsymbol{D} \in \mathcal{K}$. We will find $\boldsymbol{E} \in \mathcal{K}$ and embeddings $r : \boldsymbol{C} \to \boldsymbol{E}, s : \boldsymbol{D} \to \boldsymbol{E}$ which satisfy $r \circ f = s \circ g$.

The embedding f sends b_i to (the join of) a set of atoms C_i of C (i.e., $f(b_i) = \bigvee C_i$) and $C_1, \ldots, C_n$ is a partition of the atoms of $\boldsymbol{C}$. Similarly g sends b_i to D_i

and $D_1, \dots, D_n$ is a partition of the atoms of $\boldsymbol{D}$. Take $\boldsymbol{E}$ to be the Boolean algebra with set of atoms:

$$\bigcup_{i=1}^{n} (C_i \times D_i).$$

Below we will write $c \otimes d$ instead of (c, d), where c is an atom of $\boldsymbol{C}$ and d is an atom of $\boldsymbol{D}$. We define $r : \boldsymbol{C} \to \boldsymbol{E}$ by

$$r(c) = c \otimes D_i, \text{ if } c \in C_i,$$

(where we literally mean $r(c) = \bigvee c \otimes D_i = \bigvee \{c \otimes d : d \in D_i\}$) and $s : \boldsymbol{D} \to \boldsymbol{E}$ by

$$s(d) = C_i \otimes d, \text{ if } d \in D_i.$$

Clearly

$$\begin{aligned} r(f(b_i)) &= r(\bigvee C_i) = \bigvee (C_i \otimes D_i), \\ s(g(b_i)) &= s(\bigvee D_i) = \bigvee (C_i \otimes D_i), \end{aligned}$$

so $r \circ f = s \circ g$.

We now need to define the measure and the ordering of $\boldsymbol{E}$.

(1) *Measure*

If for $c \otimes d \in C_i \times D_i$ we put

$$\mu'(c \otimes d) = \frac{\mu_{\boldsymbol{C}}(c)\mu_{\boldsymbol{D}}(d)}{\mu_{\boldsymbol{B}}(b_i)},$$

this would work, except that μ' may not take values in $\mathbb{Q}_2$. We therefore need to be more careful.

Fix $1 \le k \le n$ and put $b = b_k, P = C_k, Q = D_k$. Say P has atoms $c_1, \dots, c_p$ and Q has atoms $d_1, \dots, d_q$. We need to define $\mu_{\boldsymbol{E}}(c_i \otimes d_j) = x_{ij}$ $(1 \le i \le p, 1 \le j \le q)$. They have to satisfy:

$$0 < x_{ij}, x_{ij} \in \mathbb{Q}_2,$$

$$(*) \begin{cases} \sum_{j=1}^{q} x_{ij} = a_i \overset{\text{def}}{=} \mu_{\boldsymbol{C}}(c_i), & 1 \le i \le p, \\ \sum_{i=1}^{p} x_{ij} = b_j \overset{\text{def}}{=} \mu_{\boldsymbol{D}}(d_j), & 1 \le j \le q. \end{cases}$$

Note here that $\sum_{i=1}^{p} a_i = \sum_{j=1}^{q} b_j = \mu_{\boldsymbol{B}}(b)$.

First assume $p, q \ge 2$. Then $(*)$ can be rewritten as

$$(**) \begin{cases} x_{i1} = a_i - \sum_{j=2}^{q} x_{ij}, & 1 < i \le p, \\ x_{1j} = b_j - \sum_{i=2}^{p} x_{ij}, & 1 < j \le q, \\ x_{11} = a_1 - \sum_{j=2}^{q} x_{1j}, & \\ x_{11} = b_1 - \sum_{i=2}^{p} x_{i1}, & \end{cases}$$

and the last equation is redundant as

$$\begin{aligned} a_1 - \textstyle\sum_{j=2}^{q} x_{ij} &= a_1 - \textstyle\sum_{j=2}^{q} 1(b_j - \sum_{i=2}^{p} x_{ij}) \\ &= a_1 - \textstyle\sum_{j=2}^{q} b_j + \sum_{j=2}^{q}\sum_{i=2}^{p} x_{ij}, \\ b_1 - \textstyle\sum_{i=2}^{p} x_{i1} &= b_1 - \textstyle\sum_{i=2}^{p}(a_i - \sum_{j=2}^{q} x_{ij}) \\ &= b_1 - \textstyle\sum_{i=2}^{p} a_i + \sum_{i=2}^{p}\sum_{j=2}^{q} x_{ij} \\ &= a_1 - \textstyle\sum_{j=2}^{q} b_j + \sum_{j=2}^{q}\sum_{i=2}^{p} x_{ij}, \end{aligned}$$

so the system is equivalent to the system $(***)$ consisting of the first 3 rows of $(**)$, which express x_{k1}, x_{1l}, for $1 \le k \le p, 1 \le l \le q$, in terms of $x_{ij}, i \ge 2, j \ge 2$.

Since $a_i, b_j \in \mathbb{Q}_2$, if all $x_{ij}, i \geq 2, j \geq 2$, are in $\mathbb{Q}_2$, so are all x_{ij} satisfying $(***)$. This system has a solution in positive numbers in $\mathbb{Q}$, namely

$$x^*_{ij} = \mu'(c_i \otimes d_j).$$

It is clear then that if we choose positive $x_{ij} \in \mathbb{Q}_2, i \geq 2, j \geq 2$, very close to x^*_{ij} and determine x_{i1}, x_{1j} using $(***)$, then all x_{ij} will be positive and in $\mathbb{Q}_2$. So the system $(*)$ admits a positive solution in $\mathbb{Q}_2$, say x_{ij}, and we define

$$\mu_{\boldsymbol{E}}(c_i \otimes d_j) = x_{ij}.$$

This clearly works, i.e., r, s are also measure preserving.

If now $p = q = 1$, we can simply take $\mu_{\boldsymbol{E}}(c_i \otimes d_j) = \mu_{\boldsymbol{E}}(c_1 \otimes d_1) = \mu'(c_1 \otimes d_1) = \frac{\mu_{\boldsymbol{C}}(c_1)\mu_{\boldsymbol{D}}(d_1)}{\mu_{\boldsymbol{B}}(b)} \in \mathbb{Q}_2$, as $\mu_{\boldsymbol{B}}(b) = \mu_{\boldsymbol{C}}(c_1) = \mu_{\boldsymbol{D}}(d_1)$. This definition also works if say $p = 1, q = 2$, since then $\mu_{\boldsymbol{C}}(c_1) = \mu_{\boldsymbol{B}}(b)$, so $\mu_{\boldsymbol{E}}(c_1 \otimes d_j) = \frac{\mu_{\boldsymbol{C}}(c_1)\mu_{\boldsymbol{D}}(d_j)}{\mu_{\boldsymbol{B}}(b)} \in \mathbb{Q}_2$.

(2) *Order*

Let the atoms of $\boldsymbol{B}$ be $b_1 <_{\boldsymbol{B}} \cdots <_{\boldsymbol{B}} b_n$, the atoms of $\boldsymbol{C}, \gamma_1 <_{\boldsymbol{C}} \cdots <_{\boldsymbol{C}} \gamma_m$, and of $\boldsymbol{D}, \delta_1 <_{\boldsymbol{D}} \cdots <_{\boldsymbol{D}} \delta_k$. We will define the order $<_{\boldsymbol{E}}$ of the atoms of $\boldsymbol{E}$. Denote by $\gamma_{i,1}$ the $<_{\boldsymbol{C}}$-largest element of C_i and similarly define $\delta_{i,1}$ for D_i. We have

$$C_1 < C_2 \cdots <_{\boldsymbol{C}} C_n$$

(which abbreviates $\bigvee C_1 <_{\boldsymbol{C}} \bigvee C_2 < \cdots <_{\boldsymbol{C}} \bigvee C_n$) and thus

$$\gamma_{i,1} <_{\boldsymbol{C}} \cdots <_{\boldsymbol{C}} \gamma_{n,1},$$

and similarly

$$D_1 <_{\boldsymbol{D}} \cdots <_{\boldsymbol{D}} D_n,$$

therefore

$$\delta_{i,1} <_{\boldsymbol{D}} \cdots <_{\boldsymbol{D}} \delta_{n,1}.$$

(In particular, $\gamma_{n,1} = \gamma_m, \delta_{n,1} = \delta_k$.)

Our goal is to find an order $\prec$ for all the atoms of $\boldsymbol{E}$ of the form

$$\begin{gathered} \gamma \otimes \delta_{i,1}, \gamma \in C_i, \\ \gamma_{i,1} \otimes \delta, \delta \in D_i, \end{gathered}$$

so that

(i) $\gamma \otimes \delta_{i,1} \prec \gamma' \otimes \delta_{i',1} \Leftrightarrow \gamma <_{\boldsymbol{C}} \gamma'$,

(ii) $\gamma_{i,1} \otimes \delta \prec \gamma_{i',1} \otimes \delta' \Leftrightarrow \delta <_{\boldsymbol{D}} \delta'$.

Then we let $<_{\boldsymbol{E}}$ agree with $\prec$ on these atoms and let all the other atoms of $\boldsymbol{E}$ be smaller in $<_{\boldsymbol{E}}$ than these atoms and otherwise ordered arbitrarily. It is clear then that r, s are also order preserving and completes the proof.

To find $\prec$ consider

$$X = \{\gamma_{1,1} \otimes \delta_{1,1}, \gamma_{2,1} \otimes \delta_{2,1}, \ldots, \gamma_{n,1} \otimes \delta_{n,1}\}$$

and put

$$\begin{gathered} \gamma_{i,1} \otimes \delta_{i,1} <_X \gamma_{i',1} \otimes \delta_{i',1} \Leftrightarrow i < i' \\ (\Leftrightarrow \gamma_{i,1} <_{\boldsymbol{C}} \gamma_{i',1} \Leftrightarrow \delta_{i,1} <_{\boldsymbol{D}} \delta_{i',1}). \end{gathered}$$

Let also

$$Y = \{\gamma \otimes \delta_{i,1} : \gamma \in C_i, 1 \le i \le n\}$$

and put

$$\gamma \otimes \delta_{i,1} <_Y \gamma' \otimes \delta_{i',1} \Leftrightarrow \gamma <_{\boldsymbol{C}} \gamma'.$$

Finally, let

$$Z = \{\gamma_{i,1} \otimes \delta : \delta \in D_i, 1 \le i \le n\},$$

and put

$$\gamma_{i,1} \otimes \delta <_Z \gamma_{i',1} \otimes \delta' \Leftrightarrow \delta <_{\boldsymbol{D}} \delta'.$$

Then $X = Y \cap Z, <_X = <_Y |X = <_Z |X$. So by the amalgamation property for linear orderings, there is an order $\prec$ on $Y \cup Z$ such that $\prec |Y = <_Y, \prec |Z = <_Z$ and we are done. $\square$

If $\mathcal{K}_0 = \mathcal{MBA}_{\mathbb{Q}_2}$, then, as verified in Kechris-Rosendal [**KR**], $\mathcal{MBA}_{\mathbb{Q}_2}$ is a Fraïssé class with Fraïssé limit

$$\mathrm{Flim}(\mathcal{K}_0) \cong \langle \boldsymbol{B}_\infty, \lambda \rangle,$$

where $\boldsymbol{B}_\infty$ = the algebra of clopen sets in $2^{\mathbb{N}}$ and λ the usual product measure. Since $\mathcal{K}$ is a reasonable expansion of $\mathcal{K}_0$, we have, by [**KPT**, 5.2],

$$\mathrm{Flim}(\mathcal{K}) = \langle \boldsymbol{B}_\infty, \lambda, \prec \rangle$$

for some appropriate ordering $\prec$. We will next identify $\prec$.

If $\mathcal{K}' = \mathcal{OBA}$, then $\mathcal{K}'$ is a reasonable order Fraïssé class, with respect to (L', L'_0), where $L' = \{\wedge, \vee, 0, 1, <\}, L'_0 = L' \setminus \{<\})$ (see [**KPT**, 6, **(D)**]) and

$$\mathrm{Flim}(\mathcal{K}') = \langle \boldsymbol{B}_\infty, \prec' \rangle,$$

with $\prec'$ the canonical ordering on $\boldsymbol{B}_\infty$ (see [**KPT**], 6,**(D)**). We will verify that $\prec = \prec'$. To see this note that $\mathcal{K} = \mathcal{OMBA}_{\mathbb{Q}_2}$ is reasonable with respect to (L, L'), where $L' = L \setminus \{M_r\}_{r \in \mathbb{Q}_2}$. This means that for any $\boldsymbol{A}_0, \boldsymbol{B}_0 \in \mathcal{K}'$, embedding $\pi : \boldsymbol{A}_0 \to \boldsymbol{B}_0$ and measure μ on $\boldsymbol{A}_0$, there is a measure μ' on $\boldsymbol{B}_0$ so that $\pi : \langle \boldsymbol{A}_0, \mu \rangle \to \langle \boldsymbol{B}_0, \mu' \rangle$ is also an embedding. (Measures here take values in $\mathbb{Q}_2$.) To see this, let $a_1, \dots, a_k$ be the atoms of $\boldsymbol{A}_0$ and let the set of atoms in $\pi(a_i)$ be B_i, so that $B_1, \dots, B_k$ is a partition of the atoms of $\boldsymbol{B}_0$. Say $\mu(a_i) = x_i$, so that $x_1 + \cdots + x_k = 1$. Since $x_i \in \mathbb{Q}_2$, write $x_i = \frac{m_i}{2^n}$, so that $\sum_{i=1}^k m_i = 2^n$. Say B_i has k_i atoms, $b_{i1}, \dots, b_{ik_i}$. We need to define $\mu'(b_{ij})$ so that $\sum_{j=1}^{k_i} \mu'(b_{ij}) = x_i$. We can clearly assume that $m_i \ge k_i$. Take then $\mu'(b_{i1}) = \cdots = \mu'(b_{i(k_i-1)}) = \frac{1}{2^n}, \mu'(b_{ik_i}) = \frac{m_i-(k_i-1)}{2^n} > 0$. Thus $\mathcal{K}$ is a reasonable expansion of $\mathcal{K}'$, so we have

Theorem 3.2. *The Fraïssé limit of $\mathcal{K} = \mathcal{OMBA}_{\mathbb{Q}_2}$ is*

$$\mathrm{Flim}(\mathcal{K}) = \langle \boldsymbol{B}_\infty, \lambda, \prec \rangle,$$

where $\boldsymbol{B}_\infty$ is the algebra of clopen sets of $2^{\mathbb{N}}, \lambda$ the usual product measure, and $\prec$ the canonical ordering on $\boldsymbol{B}_\infty$ defined by:

$$\mathrm{Flim}(\mathcal{OBA}) \cong \langle \boldsymbol{B}_\infty, \prec \rangle.$$

4. A dense subgroup of the group of measure preserving automorphisms

Clearly we can view an element of $\mathrm{Aut}(\boldsymbol{B}_\infty, \lambda, \prec)$ as a measure preserving homeomorphism of $2^{\mathbb{N}}$, which has the additional property that when viewed as an automorphism of $\boldsymbol{B}_\infty$ it also preserves the canonical ordering $\prec$. As usual we denote by $\mathrm{Aut}(2^{\mathbb{N}}, \lambda)$ the Polish group of measure-preserving automorphisms of $(2^{\mathbb{N}}, \lambda)$ with the weak topology. Thus there is a canonical continuous embedding of the group $\mathrm{Aut}(\boldsymbol{B}_\infty, \lambda, \prec)$ into $\mathrm{Aut}(2^{\mathbb{N}}, \lambda)$.

THEOREM 4.1. *The image of the canonical embedding of* $\mathrm{Aut}(\boldsymbol{B}_\infty, \lambda, \prec)$ *is dense in* $\mathrm{Aut}(2^{\mathbb{N}}, \lambda)$.

PROOF. Since (see Halmos [**H**]) the dyadic permutations, i.e., the maps of the form

$$s\hat{\ }x \mapsto \alpha(s)\hat{\ }x,$$

where $\alpha \in S_{2^n}$ = the symmetric group of 2^n (viewed as the set of all binary sequences of length n), are dense in $\mathrm{Aut}(2^{\mathbb{N}}, \lambda)$, it is enough to show that for every such dyadic permutation (also denoted by) α and every nbhd V of α in $\mathrm{Aut}(2^{\mathbb{N}}, \lambda), V \cap \mathrm{Aut}(\boldsymbol{B}_\infty, \lambda, \prec) \neq \emptyset$ (where we identify $\mathrm{Aut}(\boldsymbol{B}_\infty, \lambda, \prec)$ with its image here). We can assume that V has the following form:

For some $m > n, \epsilon > 0, V$ consists of all $T \in \mathrm{Aut}(2^{\mathbb{N}}, \lambda)$ such that

$$\lambda(T(N_t)\Delta\alpha(N_t)) < \epsilon, \forall t \in 2^m,$$

where

$$N_t = \{x \in 2^{\mathbb{N}} : t \subseteq x\}.$$

Note that $\alpha(N_t) = N_{\overline{\alpha}(t)}$, for some $\overline{\alpha} \in S_{2^m}$. So it is enough to show that for any given $m > 0, \epsilon > 0, \beta \in S_{2^m}$, there is $T \in \mathrm{Aut}(\boldsymbol{B}_\infty, \lambda, \prec)$ such that

$$\lambda(T(N_t)\Delta N_{\beta(t)}) < \epsilon, \forall t \in 2^m.$$

First choose $M > 2m$, with $\frac{2^m+1}{2^M} < \epsilon$. Next let $\boldsymbol{A}$ be the substructure of $\langle \boldsymbol{B}_\infty, \lambda, \prec\rangle$, whose underlying Boolean algebra $\boldsymbol{A}_0$ has atoms $\{N_t : t \in 2^m\}$. Consider also the substructure $\boldsymbol{C}$ of $\langle \boldsymbol{B}_\infty, \lambda, \prec\rangle$, whose underlying Boolean algebra $\boldsymbol{C}_0$ has atoms $\{N_S : S \in 2^M\}$.

For $t \in 2^m$, let

$$N_t^* = \{N_S : S \in 2^M, S \supseteq t\}.$$

Clearly $N_t = \bigcup N_t^*$ and $\{N_t^*\}_{t\in 2^m}$ is a partition of the atoms of $\boldsymbol{C}_0$. We will find a new partition $\{\overline{N_t}\}_{t\in 2^m}$ of the atoms of $\boldsymbol{C}_0$ such that

(i) $\lambda(\bigcup N_t^* \Delta \bigcup \overline{N_t}) < \epsilon, \forall t \in 2^m$,

(ii) $\lambda(\bigcup \overline{N_t}) = \lambda(\bigcup N_t^*) = 2^{-m}, \forall t \in 2^m$,

(iii) $N_t \mapsto \bigcup \overline{N_{\beta(t)}}, t \in 2^m$, preserves $\prec$ i.e.,

$$N_t \prec N_{t'} \Rightarrow \bigcup \overline{N_{\beta(t)}} \prec \bigcup \overline{N_{\beta(t')}}.$$

Let then $\overline{\boldsymbol{C}_0}$ be the Boolean algebra with atoms $\{\bigcup \overline{N_t}\}_{t\in 2^m}$. Consider the map $N_t \mapsto \bigcup \overline{N_{\beta(t)}}$ from the atoms of $\boldsymbol{A}_0$ to the atoms of $\overline{\boldsymbol{C}_0}$. Then (ii), (iii) above imply that this map extends to an isomorphism of $\boldsymbol{A}$ with $\langle \overline{\boldsymbol{C}_0}, \lambda, \prec\rangle$. By the ultrahomogeneity of $\langle \boldsymbol{B}_\infty, \lambda, \prec\rangle$ this extends to $T \in \mathrm{Aut}(\boldsymbol{B}_\infty, \lambda, \prec)$. Then $T(N_t) = \bigcup \overline{N_{\beta(t)}}, t \in 2^m$, so

$$\lambda(T(N_t)\Delta N_{\beta(t)}) = \lambda(\bigcup \overline{N_{\beta(t)}}\Delta \bigcup N^*_{\beta(t)}) < \epsilon,$$

and we are done.

Construction of $\{\overline{N_t}\}_{t\in 2^m}$:

Say $N_{t_1} \prec N_{t_2} \prec \cdots \prec N_{t_{2^m}}$. Let $a_1 \prec a_2 \prec \cdots \prec a_{2^m}$ be the last 2^m atoms of $\boldsymbol{C}_0$ in the order $\prec$. Let

$$k_i = |N^*_{t_i} \cap \{a_1, \dots, a_{2^m}\}|,$$

so that $0 \le k_i \le 2^m$. Put

$$N'_t = N^*_t \setminus \{a_1, \dots, a_{2^m}\},$$

so that $|N'_t| = |N^*_t| - k_i$. Note that $|N^*_t| = 2^{M-m} > 2^m \ge k_i$, thus $|N'_t| > 0$. Let

$$N''_{\beta(t_i)} = N'_{\beta(t_i)} \cup \{a_i\},$$

so a_i is the $\prec$-last element of $N''_{\beta(t_i)}$, therefore

$$i < j \Rightarrow \bigcup N''_{\beta(t_i)} \prec \bigcup N''_{\beta(t_j)},$$

i.e.,

$$N_t \prec N_{t'} \Rightarrow \bigcup N''_{\beta(t)} \prec \bigcup N''_{\beta(t')}.$$

However $|N''_t|$ may not equal 2^{M-m}, thus $\lambda(\bigcup N''_t)$ may not equal $\lambda(\bigcup N^*_t)$. Note though that the $\{N''_t\}_{t\in 2^m}$ are pairwise disjoint and

$$\sum_{i=1}^{2^m} |N''_{t_i} \setminus \{a_j : j \le 2^m\}| = 2^M - 2^m,$$

so we can redistribute the $2^M - 2^m$ atoms of $\boldsymbol{C}_0$, which are not in $\{a_1, \dots, a_{2^m}\}$, among the N''_{t_i} to define $\overline{N_{t_i}}$ so that $|\overline{N_{t_i}}| = 2^{M-m}$, a_i is still the $\prec$-last element of $\overline{N_{\beta(t_i)}}$, and in this redistribution we do not move more than 2^m elements in or out of each N''_{t_i}. Thus

$$\lambda(\bigcup N_t \Delta \bigcup N^*_t) \le \frac{(2^m+1)}{2^M} < \epsilon,$$

and we are done. □

5. Homogeneous measure algebras and the Ramsey Property

We call $\boldsymbol{A} \in \mathcal{OMBA}_{\mathbb{Q}_2}$ *homogeneous* if every atom has the same measure. Thus a homogeneous $\boldsymbol{A}$ has 2^n atoms, for some n, and each atom has measure 2^{-n}. We denote by $\mathcal{H}$ the subclass of $\mathcal{OMBA}_{\mathbb{Q}_2}$ consisting of the homogeneous structures.

Proposition 5.1. The class $\mathcal{H}$ is cofinal under embeddability in $\mathcal{OMBA}_{\mathbb{Q}_2}$.

Proof. Let $\boldsymbol{B} \in \mathcal{OMBA}_{\mathbb{Q}_2}$, with atoms $b_1 < \cdots < b_n$ and say $\mu(B_i) = \frac{m_i}{2^N}$ (so that $\sum_{i=1}^n m_i = 2^N$). Consider the Boolean algebra $\boldsymbol{A}_0$ with 2^N atoms $b_{i,1}, \dots, b_{i,m_1}; b_{2,1}, \dots, b_{2,m_2}; \dots; b_{n,1}, \dots, b_{n,m_n}$. Assign to each measure equal to 2^{-N}. Also order them as follows:

$$\begin{aligned} b_{i,1} <' \cdots <' b_{1,m_1} <' b_{2,1} \quad &<' \cdots <' b_{2,m_2} <' \dots \\ &<' b_{n,1} <' \cdots <' b_{n,m_n}. \end{aligned}$$

Call $\boldsymbol{A} \in \mathcal{OMBA}_{\mathbb{Q}_2}$ the resulting structure.

Clearly $\boldsymbol{A} \in \mathcal{H}$. The map

$$b_i \mapsto \bigvee_{j=1}^{m_i} b_{i,j}$$

is an embedding of $\boldsymbol{B}$ into $\boldsymbol{A}$. □

Call also $\boldsymbol{A} \in \mathcal{MBA}_{\mathbb{Q}_2}$ *homogeneous* if every atom has the same measure and denote by $\mathcal{H}_0$ the class of such measure algebras. If $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{H}$ and $\boldsymbol{A}_0, \boldsymbol{B}_0 \in \mathcal{H}_0$ are their reducts (in which we drop the ordering), then

$$\boldsymbol{A} \cong \boldsymbol{B} \Leftrightarrow \boldsymbol{A}_0 \cong \boldsymbol{B}_0.$$

(Note that this fails for the class $\mathcal{OMBA}_{\mathbb{Q}_2}$ itself.) Using this it is easy to check that the following holds:

$$\mathcal{H} \text{ has the Ramsey Property} \Leftrightarrow \mathcal{H}_0 \text{ has the Ramsey Property.}$$

We now propose the following problem:

Problem 5.2. Is it true that the class $\mathcal{H}_0$ of finite homogeneous measure algebras with measure taking values in the dyadic rationals has the Ramsey Property?

Next we use a positive answer to Problem 5.2 to derive the Ramsey Property for $\mathcal{OMBA}_{\mathbb{Q}_2}$. For that we need the following general result.

Proposition 5.3. Let $\mathcal{K}$ be an order Fraïssé class and $\mathcal{K}^* \subseteq \mathcal{K}$ a cofinal (under embeddability) subclass closed under isomorphism. If $\mathcal{K}^*$ has the Ramsey Property, so does $\mathcal{K}$.

Proof. It should be kept in mind below that, since $\mathcal{K}$ is an order Fraïssé class, if $\boldsymbol{C} \cong \boldsymbol{D}$ are in $\mathcal{K}$, then there is a unique isomorphism between $\boldsymbol{C}, \boldsymbol{D}$.

Fix $\boldsymbol{A} \leq \boldsymbol{B}$ in $\mathcal{K}$. Let $\boldsymbol{A}^* \in \mathcal{K}^*$ be such that $\boldsymbol{A} \leq \boldsymbol{A}^*$. For each $\boldsymbol{A}_0^* \cong \boldsymbol{A}^*$, let $\ell(\boldsymbol{A}_0^*)$ be the isomorphic copy of $\boldsymbol{A}$ in $\boldsymbol{A}_0^*$, whose universe is lexicographically least in the ordering of $\boldsymbol{A}_0^*$.

Lemma 5.4. *We can find $\boldsymbol{B}^* \in \mathcal{K}^*$ with $\boldsymbol{A}^* \leq \boldsymbol{B}^*$ such that there is a copy $\boldsymbol{B}_0 \cong \boldsymbol{B}$ with $\boldsymbol{B}_0 \subseteq \boldsymbol{B}^*$ with the property that for any copy $\boldsymbol{A}_0 \cong \boldsymbol{A}, \boldsymbol{A}_0 \subseteq \boldsymbol{B}_0$, there is a copy $\boldsymbol{A}_0^* \cong \boldsymbol{A}^*, \boldsymbol{A}_0^* \subseteq \boldsymbol{B}^*$ with $\ell(\boldsymbol{A}_0^*) = \boldsymbol{A}_0$.*

Proof. Let $e_1 : \boldsymbol{A} \to \boldsymbol{B}, \dots, e_n : \boldsymbol{A} \to \boldsymbol{B}$ be all the embeddings of $\boldsymbol{A}$ into $\boldsymbol{B}$. Using amalgamation, we will construct recursively $\boldsymbol{B}_1, \dots, \boldsymbol{B}_n \in \mathcal{K}$ containing, resp., copies $\boldsymbol{B}_1^0, \dots, \boldsymbol{B}_n^0$ of $\boldsymbol{B}$ such that for $1 \leq i \leq n$, if $\boldsymbol{A}_1^0, \dots, \boldsymbol{A}_i^0$ are the copies of $\boldsymbol{A}$ in $\boldsymbol{B}_i^0$ corresponding to the embeddings $e_1, \dots, e_i$, then there are $\boldsymbol{A}_1^*, \dots, \boldsymbol{A}_i^*$, copies of $\boldsymbol{A}^*$ in $\boldsymbol{B}_i$, for which $\ell(\boldsymbol{A}_j^*) = \boldsymbol{A}_j^0$, $\forall j \leq i$. Then $\boldsymbol{B}_n$ has the property that it contains a copy $\boldsymbol{B}_n^0 = \boldsymbol{B}_0$ of $\boldsymbol{B}$ such that for any copy $\boldsymbol{A}_0$ of $\boldsymbol{A}$ in $\boldsymbol{B}_0$, there is a copy $\boldsymbol{A}_0^*$ of $\boldsymbol{A}^*$ in $\boldsymbol{B}_n$ with $\ell(\boldsymbol{A}_0^*) = \boldsymbol{A}_0$. Extend then $\boldsymbol{B}_n$ to $\boldsymbol{B}^* \in \mathcal{K}^*$ to complete the proof.

We first construct $\boldsymbol{B}_1$ by amalgamating $e_1 : \boldsymbol{A} \to \boldsymbol{B}$ and $f_1 : \boldsymbol{A} \to \boldsymbol{A}^*$, whose range $f_1(\boldsymbol{A})$ is such that $\ell(\boldsymbol{A}^*) = f_1(\boldsymbol{A})$. Let $\boldsymbol{B}_1^0$ be the image of $\boldsymbol{B}$ in this amalgamation. Assume now that $\boldsymbol{B}_i$ has been constructed. Let $\bar{e}_{i+1} : \boldsymbol{A} \to \boldsymbol{B}_i^0$ be the embedding of $\boldsymbol{A}$ into $\boldsymbol{B}_i^0$ that corresponds to e_{i+1}. Since $\boldsymbol{B}_i^0 \subseteq \boldsymbol{B}_i$, view $\bar{e}_{i+1} : \boldsymbol{A} \to \boldsymbol{B}_i$ as an embedding of $\boldsymbol{A}$ into $\boldsymbol{B}_i$. Let $\boldsymbol{B}_{i+1}$ be constructed by amalgamating $\bar{e}_{i+1}$ and $f_1 : \boldsymbol{A} \to \boldsymbol{A}^*$ and let $\boldsymbol{B}_{i+1}^0$ be the image of $\boldsymbol{B}_i^0$ under this amalgamation. □

Using this lemma and the Ramsey Property for $\mathcal{K}^*$, given any k, we can find $\boldsymbol{C}^* \in \mathcal{K}^*$ such that $\boldsymbol{A}^* \leq \boldsymbol{B}^* \leq \boldsymbol{C}^*$ and $\boldsymbol{C}^* \to (\boldsymbol{B}^*)^{\boldsymbol{A}^*}_k$. We will show that $\boldsymbol{C}^* \to (\boldsymbol{B})^{\boldsymbol{A}}_k$. Indeed, fix a coloring

$$c : \binom{\boldsymbol{C}^*}{\boldsymbol{A}} \to \{1, \dots, k\}.$$

Define then the coloring

$$c^* : \binom{\boldsymbol{C}^*}{\boldsymbol{A}^*} \to \{1, \dots, k\}$$

by

$$c^*(\boldsymbol{A}^*_0) = c(\ell(\boldsymbol{A}^*_0)).$$

Let $\boldsymbol{B}^*_0 \cong \boldsymbol{B}^*, \boldsymbol{B}^*_0 \subseteq \boldsymbol{C}^*$ be homogeneous for c^*. By the lemma there is a copy $\boldsymbol{B}_0$ of $\boldsymbol{B}, \boldsymbol{B}_0 \subseteq \boldsymbol{B}^*_0$ such that for any copy $\boldsymbol{A}_0$ of $\boldsymbol{A}$ in $\boldsymbol{B}_0$ there is a copy $\boldsymbol{A}^*_0$ of $\boldsymbol{A}^*$ in $\boldsymbol{B}^*_0$ with $\ell(\boldsymbol{A}^*_0) = \boldsymbol{A}_0$. Then $c(\boldsymbol{A}_0) = c^*(\boldsymbol{A}^*_0)$, so that $\boldsymbol{B}_0$ is homogeneous for c. $\square$

THEOREM 5.5. *If Problem 5.2 has a positive answer, then the class* $\mathcal{K} = \mathcal{OMBA}_{\mathbb{Q}_2}$ *has the Ramsey Property.*

PROOF. By the positive answer to Problem 5.2 and the remarks preceding it, $\mathcal{H}$ has the Ramsey Property. The result then follows from Propositions 5.1 and 5.3. $\square$

We note here that $\mathcal{MBA}_{\mathbb{Q}_2}$ itself *does not* have the Ramsey Property. (Since $\mathcal{H}_0$ is cofinal in $\mathcal{MBA}_{\mathbb{Q}_2}$ and has the Ramsey Property, it follows that the use of orderings is crucial in Proposition 5.3.) To see this, let $\boldsymbol{A}_0$ be the measure Boolean algebra with two atoms a, b such that $\mu(a) = \frac{1}{4}, \mu(b) = \frac{3}{4}$. Let $\boldsymbol{B}_0$ be the measure Boolean algebra with 4 atoms of measure $\frac{1}{4}$. Suppose, towards a contradiction, that $\boldsymbol{C}_0$ is a measure algebra with

$$\boldsymbol{C}_0 \to (\boldsymbol{B}_0)^{\boldsymbol{A}_0}_2.$$

Take a natural order $\prec$ on $\boldsymbol{C}_0$ and color all copies $\boldsymbol{A}'_0$ of $\boldsymbol{A}_0$ in $\boldsymbol{C}_0$ as follows:

If a', b' are the two atoms of $\boldsymbol{A}'_0$ (corresponding to a, b), put

$$c(\boldsymbol{A}'_0) = \begin{cases} 0, & \text{if } a' \prec b', \\ 1, & \text{if } a' \succ b'. \end{cases}$$

We claim that there is no copy of $\boldsymbol{B}_0$ that verifies $\boldsymbol{C}_0 \to (\boldsymbol{B}_0)^{\boldsymbol{A}_0}_2$. Indeed assume $\boldsymbol{B}'_0$ was one, say with atoms $a_1 \prec a_2 \prec a_3 \prec a_4$. Let $\boldsymbol{A}'_0$ be the copy of $\boldsymbol{A}_0$ in $\boldsymbol{B}'_0$ with atoms $a' = a_1, b' = a_2 \vee a_3 \vee a_4$. Then, as $a' \prec b'$, we have $c(\boldsymbol{A}'_0) = 0$. Next let $\boldsymbol{A}''_0$ be the copy of $\boldsymbol{A}_0$ in $\boldsymbol{B}'_0$ with atoms $a'' = a_4, b'' = a_1 \vee a_2 \vee a_3$. Then $a'' \succ b''$, so $c(\boldsymbol{A}''_0) = 1$.

Finally, using the general result in [**KPT**, 10.3], a positive answer to Problem 5.2 and Proposition 6.3 below, one can compute the Ramsey degree (see [**KPT**, 10] for the definition) of each $\boldsymbol{A}_0 \in \mathcal{MBA}_{\mathbb{Q}_2}$ in the class $\mathcal{MBA}_{\mathbb{Q}_2}$: It is equal to $\frac{n!}{|\mathrm{Aut}(\boldsymbol{A}_0)|}$, where n is the number of atoms in $\boldsymbol{A}_0$. If $0 < \lambda_1 < \cdots < \lambda_m < 1$, where $1 \leq m \leq n$, are the values of the measure of $\boldsymbol{A}_0$ at its atoms, and the value λ_i is obtained by n_i atoms, so that $\sum_{i=1}^m n_i = n$, clearly $|\mathrm{Aut}(\boldsymbol{A}_0)| = n_1!n_2!\dots n_m!$, so the Ramsey degree is $\frac{(n_1+\cdots+n_m)!}{n_1!n_2!\dots n_m!}$.

6. Applications to extreme amenability and calculation of the universal minimal flow

If Problem 5.2 has a positive answer, then $\mathcal{K} = \mathcal{OMBA}_{\mathbb{Q}_2}$ has the Ramsey Property and its Fraïssé limit is $\langle \boldsymbol{B}_\infty, \lambda, \prec\rangle$, so we immediately obtain, using [**KPT**, 6.1], the following result.

THEOREM 6.1. *If Problem 5.2 has a positive answer, the group* $\mathrm{Aut}(\boldsymbol{B}_\infty, \lambda, \prec)$ *is extremely amenable.*

Using Theorem 6.1 and Theorem 4.1, this would then give a new proof of the following result of Giordano-Pestov [**GP**], a proof that has the feature that it uses Ramsey theory as opposed to the original proof that used concentration of measure techniques.

THEOREM 6.2 (Giordano-Pestov [**GP**]). *The automorphism group* $\mathrm{Aut}(X, \mu)$ *of a standard probability measure space is extremely amenable.*

Our next application conditionally computes the universal minimal flow of the group of measure preserving homeomorphisms of $2^{\mathbb{N}}$. First we note the following fact.

PROPOSITION 6.3. The class $\mathcal{K} = \mathcal{OMBA}_{\mathbb{Q}_2}$ has the ordering property.

PROOF. Let $\boldsymbol{A}_0 \in \mathcal{MBA}_{\mathbb{Q}_2}$ with corresponding measure μ. Let $a_1, \ldots, a_n$ be the atoms of $\boldsymbol{A}_0$ with $\mu(a_i) = \frac{m_i}{2^N}, 1 \leq m_i \leq 2^N$, so that $\sum_{i=1}^n m_i = 2^N$. Let $\boldsymbol{B}_0 \in \mathcal{MBA}_{\mathbb{Q}_2}$ be the homogeneous algebra with 2^N atoms. We claim that for any natural orderings $\prec, \prec'$ of $\boldsymbol{A}_0, \boldsymbol{B}_0$, resp., there is an embedding of $\langle \boldsymbol{A}_0, \prec\rangle$ into $\langle \boldsymbol{B}_0, \prec'\rangle$. Say $a_{j_1} \prec \cdots \prec a_{j_n}$. Since $n \leq 2^N$, let $b_1 \prec' \cdots \prec' b_n$ be the last n atoms of $\boldsymbol{B}_0$ in the ordering $\prec'$. Let $X_1, \ldots, X_n$ be a partition of the atoms of $\boldsymbol{B}_0$ such that $b_i \in X_i$ and $|X_i| = m_{j_i}$. Then $a_{j_i} \mapsto \bigvee X_i$ is an embedding of $\langle \boldsymbol{A}_0, \prec\rangle$ into $\langle \boldsymbol{B}_0, \prec'\rangle$. □

We view the group of measure preserving homeomorphisms of $2^{\mathbb{N}}$ as a closed subgroup of the Polish group of homeomorphisms of $2^{\mathbb{N}}$. It can therefore be identified with the automorphism group $\mathrm{Aut}(\boldsymbol{B}_\infty, \lambda)$. Recall that $\langle \boldsymbol{B}_\infty, \lambda\rangle$ is the Fraïssé limit of $\mathcal{K} = \mathcal{MBA}_{\mathbb{Q}_2}$. It follows then from [**KPT**, 7.5] and the preceding results that we can compute the universal minimal flow of $\mathrm{Aut}(\boldsymbol{B}_\infty, \lambda)$ as follows:

THEOREM 6.4. *If Problem 5.2 has a positive answer, then the universal minimal flow of the group of measure preserving homeomorphisms of* $2^{\mathbb{N}}, \mathrm{Aut}(\boldsymbol{B}_\infty, \lambda)$*, is (its canonical action on) the space of orderings on* $\boldsymbol{B}_\infty$*, whose restrictions to finite subalgebras of* $\boldsymbol{B}_\infty$ *are natural.*

From this and [**KPT**, 8.2], we see that if Problem 5.2 has a positive answer, then the groups of homeomorphisms of $2^{\mathbb{N}}$ and measure preserving homeomorphisms of $2^{\mathbb{N}}$ have the same underlying compact metric space as their universal minimal flow (and the action of the second group is the restriction of the action of the first group).

7. Reformulation of the Ramsey Property for homogeneous measure algebras

We start with a few remarks. Recall first the Dual Ramsey Theorem.

Theorem 7.1 (The Dual Ramsey Theorem, Graham-Rothschild [**GR**]). *For any positive integers $N, k < \ell$, there is a positive integer $m > \ell$ with the following property: For every coloring $c : EQ_k \to \{1, \dots, N\}$ of the set of all equivalence relations on $\{1, \dots, m\}$ which have exactly k classes, there is an equivalence relation E on $\{1, \dots, m\}$ with exactly ℓ classes such that c is constant on the set of all equivalence relations $F \supseteq E$ with exactly k classes.*

Identifying a finite set of size m with the Boolean algebra with m atoms, $\boldsymbol{B}_m$, we note that an equivalence relation E of $\{1, \dots, m\}$ with k classes corresponds exactly to an isomorphic copy of $\boldsymbol{B}_k$ in $\boldsymbol{B}_m$ (the copy having as atoms the (joins of the) equivalence classes of E). Thus the Dual Ramsey Theorem is exactly the Ramsey Property for the class $\mathcal{BA}$ of finite Boolean algebras.

Theorem 7.2 (The Dual Ramsey Theorem). *The class $\mathcal{BA}$ of finite Boolean algebras has the Ramsey Property.*

Keeping this interpretation in mind, one can reformulate (a somewhat stronger version) of Problem 5.2 as a Homogeneous Dual Ramsey Theorem. Recall that an equivalence relation is *homogeneous* if all its classes have the same size. Below we let

$$k << \ell \Leftrightarrow k < \ell \text{ and } k \text{ divides } \ell$$

Problem 7.3 (Homogeneous Dual Ramsey). Is it true that for any positive integers $N, k << \ell$, there is a positive integer $m >> \ell$ such that for every coloring $c : EQ_k^{\text{hom}} \to \{1, \dots, N\}$ of the set of all homogeneous equivalence relations on $\{1, \dots m\}$ which have exactly k classes, there is a homogeneous equivalence relation E on $\{1, \dots, m\}$ with exactly ℓ classes so that c is constant on the set of all homogeneous equivalence relations $F \supseteq E$ with exactly k classes?

Indeed, translated to Boolean algebras, this is simply the Ramsey Property for the class of all homogeneous measure Boolean algebras (with arbitrary measure - not necessarily in $\mathbb{Q}_2$).

We finally state another reformulation of Problem 7.3. For convenience we will use below the following notation and terminology:

We use $k, \ell, m, \dots$ for positive integers. A *k-algebra* is a finite Boolean algebra with exactly k atoms. A *homogeneous k-subalgebra* of a finite Boolean algebra $\boldsymbol{B}$ is a Boolean subalgebra $\boldsymbol{A}$ of $\boldsymbol{B}$ with k atoms, so that each atom of $\boldsymbol{A}$ contains exactly the same number of atoms of $\boldsymbol{B}$. Given a finite Boolean algebra $\boldsymbol{B}$, we denote by

$$\binom{\boldsymbol{B}}{k}_{\text{hom}}$$

the set of all homogeneous k-subalgebras of $\boldsymbol{B}$. We then have the following reformulation of Problem 7.3.

Problem 7.4 (Homogeneous Dual Ramsey). Is it true that for every positive integers $N, k << \ell$, there is $m >> \ell$ such that for every m-algebra $\boldsymbol{C}$ and coloring $c : \binom{\boldsymbol{C}}{k}_{\text{hom}} \to \{1, \dots, N\}$, there is $\boldsymbol{B} \in \binom{\boldsymbol{C}}{\ell}_{\text{hom}}$ so that c is constant on $\binom{\boldsymbol{B}}{k}_{\text{hom}}$ (and moreover, if k, ℓ are powers of 2, m can be chosen to be a power of 2)?

Remark 7.5. Tony Wang has verified, using Mathematica, that Problem 7.3 has a positive answer for $N = 2, k = 2, l = 4$ if we take $m = 8$.

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