Realizations of countable Borel equivalence relations

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1 Introduction

(A) This paper is a contribution to the theory of countable Borel equivalence relations (CBER), a recent survey of which can be found in [K2]. One of our main concerns is the subject of well-behaved, in some sense, realizations of CBER. Given CBER E, F on standard Borel spaces X, Y, resp., a Borel isomorphism of E with F is a Borel bijection $f\colon X \to Y$ which takes E to F. If such f exists, we say that E, F are **Borel isomorphic**, in symbols $E \cong_B F$. Generally speaking a realization of a CBER E is a CBER $F \cong_B E$ with desirable properties.

To start with, a **topological realization** of E is an equivalence relation F on a *Polish space* Y such that $E \cong_B F$, in which case we say that F is a topological realization of E in the space Y. It is clear that every E admits a topological realization in some Polish space but we will look at topological realizations that have additional properties.

Recall here the Feldman-Moore Theorem that asserts that every CBER is induced by a Borel action of a countable group (see, e.g., [K2, 2.3]). By [K95, 13.11] there is a Polish topology with the same Borel structure in which this action is continuous. Thus every CBER admits a topological realization in some Polish space, which is induced by a continuous action of some countable (discrete) group. We will look again at such **continuous action realizations** for which the space and the action have additional properties.

To avoid uninteresting situations, unless it is otherwise explicitly stated or clear from the context, all the standard Borel or Polish spaces below will *be uncountable and all* CBER *will be aperiodic, i.e., have infinite classes. We will denote by* $\boldsymbol{\mathcal{AE}}$ *the class of all aperiodic* CBER *on uncountable standard Borel spaces.*

Concerning topological realizations, we first show the following (in Theorem 3.1):

Theorem 1.1. *For every equivalence relation $E \in \mathcal{AE}$ and every perfect Polish space Y, there is a topological realization of E in Y in which every equivalence class is dense.*

This has in particular as a consequence a stronger new version of a *marker lemma* (for the original form of the Marker Lemma see, e.g., [K2, 2.15]). Let E be a CBER on a standard Borel space X. A **Lusin marker scheme** for E is a family $\{A_s\}_{s\in\mathbb{N}^{<\mathbb{N}}}$ of Borel sets such that

(i) $A_\emptyset = X$;

(ii) $\{A_{sn}\}_n$ are pairwise disjoint and $\bigsqcup_n A_{sn} \subseteq A_s$;

(iii) Each A_s is a complete section for E (i.e., it meets every E-class).

We have two types of Lusin marker schemes:

(1) The Lusin marker scheme $\{A_s\}_{s\in\mathbb{N}^{<\mathbb{N}}}$ for E is of **type I** if in (ii) above we actually have that $\bigsqcup_n A_{sn} = A_s$ and moreover the following holds:

(iv) For each $x \in \mathcal{N} = \mathbb{N}^{\mathbb{N}}$, $\bigcap_n A_{x|n}$ is a singleton.

(Then in this case, for each $x \in \mathcal{N}$, $A^x_n = A_{x|n} \setminus \bigcap_n A_{x|n}$ is a vanishing sequence of markers (i.e., $\bigcap_n A^x_n = \emptyset$).)

(2) The Lusin marker scheme $\{A_s\}_{s\in\mathbb{N}^{<\mathbb{N}}}$ for E is of **type II** if it satisfies the following:

(v) If for each n, $B_n = \bigsqcup\{A_s \colon s \in \mathbb{N}^n\}$, then $\{B_n\}$ is a vanishing sequence of markers.

We now have (see Theorem 3.3):

Theorem 1.2. *Every $E \in \mathcal{AE}$ admits a Lusin marker scheme of type* I *and a Lusin marker scheme of type* II.

We next look at continuous action realizations. One such realization of $E \in \mathcal{AE}$ would be a realization F on a compact Polish space, where F is generated by a continuous action of a countable (discrete) group. We call these **compact action realizations**. Excluding the case of smooth relations (i.e., those that admit a Borel transversal), for which such a realization is impossible, we show the following (in Theorem 3.10). We use the following terminology: A CBER E on X is **compressible** if there is a Borel injection $f\colon X \to X$ with $f(C) \subsetneqq C$, for every E-class C. A CBER E is **hyperfinite** if $E = \bigcup_n E_n$, where each E_n is a **finite** CBER (i.e., all its classes are finite) and $E_n \subseteq E_{n+1}$. Finally, a **minimal, compact action realization** is a compact action realization in which the group acts minimally, i.e., all the orbits are dense.

Theorem 1.3. *Every non-smooth hyperfinite equivalence relation in $\mathcal{AE}$ has a minimal, compact action realization. Moreover if the equivalence relation is not compressible, the acting group can be taken to be $\mathbb{Z}$.*

We discuss other cases of CBER which admit such realizations in Section 3.C. For each infinite countable group Γ, let $F(\Gamma, 2^{\mathbb{N}})$ be the equivalence relation induced by the shift action of Γ on $(2^{\mathbb{N}})^{\Gamma}$ restricted to its free part. Every equivalence relation induced by a free Borel action of Γ is Borel isomorphic to the restriction of $F(\Gamma, 2^{\mathbb{N}})$ on an invariant Borel set. Also a CBER is **universal** if every CBER can be Borel reduced to it. As opposed to Theorem 1.3, the next results (see Theorem 3.16 and Corollary 3.25) show that some very complex CBER have compact action realizations.

Theorem 1.4. *(i) For every infinite countable group Γ, $F(\Gamma, 2^{\mathbb{N}})$ admits a compact action realization.*

(ii) Every compressible, universal CBER *admits a compact action realization.*

In particular, it follows that arithmetical equivalence on $2^{\mathbb{N}}$ has a compact action realization but it is not unknown if Turing equivalence has such a realization. More generally, we do not know whether *every* non-smooth CBER has a compact action realization. We also do not know if *every* non-smooth CBER even admits some other kinds of realizations, for example transitive (i.e., having at least one dense orbit) continuous action realizations on arbitrary or special types of Polish spaces. These problems as well as the situation with smooth CBER in such realizations are discussed in Section 3.B. In Section 3.E we discuss some special properties of continuous actions of countable groups on compact Polish spaces, related to compressibility and paradoxical decompositions, that may be relevant to these questions.

Clinton Conley also raised the question of whether every $E \in \mathcal{AE}$ admits a K_σ realization in a Polish space. We show in Proposition 3.59 that this is equivalent to asking whether such a realization can be found as a $K_\sigma = F_\sigma$

relation in a compact Polish space and this raises the related question of whether every $E \in \mathcal{AE}$ admits an F_σ realization in a Polish space with some additional properties, like having one or all classes dense. We show in Proposition 3.60 that any $E \in \mathcal{AE}$ has an F_σ realization in some Polish space with a dense class and moreover in a compact Polish space if it is smooth. In view of Theorem 1.3, every non-smooth hyperfinite equivalence relation in $\mathcal{AE}$ has an F_σ realization on a compact Polish space with all classes dense and Solecki in [S02] has shown that this fails for smooth relations, but this is basically the extent of our knowledge in this matter. All this is discussed in Section 3.F.

(B) In connection with these realization problems, we were also led to consider the following quasi-order on CBER, which we call the **Borel inclusion order**. Given CBER E, F on standard Borel spaces, we put $E \subseteq_B F$ if there is $E' \cong_B E$ with $E' \subseteq F$.

Below, unless otherwise explicitly stated or understood from the context, by a **measure** *on a standard Borel space we will always mean a Borel probability measure.*

For each CBER E, we denote by EINV_E the set of ergodic, invariant measures for E and by $|\mathrm{EINV}_E| \in \{0, 1, 2, \dots \aleph_0, 2^{\aleph_0}\}$ its cardinality.

Recall here Nadkarni's Theorem (see, e.g., [K2, 4.C]) which asserts that for a CBER E the following are equivalent:

(i) E has no invariant measure;

(ii) $|\mathrm{EINV}_E| = 0$;

(iii) E is compressible.

We now have the following result (see Proposition 2.3, Theorem 2.8 and Corollary 2.11), where $\mathcal{AH}$ is the class of hyperfinite relations in $\mathcal{AE}$.

Theorem 1.5. *(i) If $E \subseteq_B F$ are in $\mathcal{AE}$, then $|\mathrm{EINV_E}| \geq |\mathrm{EINV_F}|$ and if $E, F \in \mathcal{AH}$, then $E \subseteq_B F \iff |\mathrm{EINV_E}| \geq |\mathrm{EINV_F}|$.*

(ii) For any $E \in \mathcal{AE}$, there is $F \in \mathcal{AH}$ with $F \subseteq E$ such that moreover $\mathrm{EINV}_E = \mathrm{EINV}_F$.

Using this and the classification theorem for hyperfinite CBER from [DJK94, 9.1], one can then prove the next result (see Theorem 2.10 and Proposition 2.13), where we use the following terminology and notation:

For each CBER E and standard Borel space S, SE is the direct sum of "S" copies of E (see Section 2.A). We let E_0 be the equivalence relation on $2^{\mathbb{N}}$ given by $xE_0y \iff \exists m \forall n \geq m(x_n = y_n)$; E_t is the equivalence relation on

$2^{\mathbb{N}}$ given by $xE_ty \iff \exists m \exists n \forall k (x_{m+k} = y_{n+k})$; $I_{\mathbb{N}} = \mathbb{N}^2$; E_∞ is a universal under Borel embeddability CBER; and $E \times F$ is the product of E and F. Finally $\subset_B$ is the strict part of $\sqsubseteq_B$ and for any quasi-order $\preceq$ with strict part $\prec$ on a set Q and $q, r \in Q$, we say that r is a successor to q if $q \prec r$ and $(s \prec r \implies s \preceq q)$. Finally, for each cardinal $\kappa \in \{0, 1, 2, 3, \ldots, \aleph_0, 2^{\aleph_0}\}$, let $\mathcal{AE}_\kappa$ be the class of all $E \in \mathcal{AE}$ such that $|\mathrm{EINV}_E| = \kappa$. Thus by Nadkarni's Theorem $\mathcal{AE}_0$ is the class of compressible relations. We also let for $\kappa > 0$, $\kappa E = SE$, where S is a standard Borel space of cardinality κ.

Theorem 1.6. *(i)* $\mathbb{R}E_0 \subset_B \mathbb{N}E_0 \subset_B \cdots \subset_B 3E_0 \subset_B 2E_0 \subset_B E_0 \subset_B E_t$, *each equivalence relation in this list is a successor in* $\sqsubseteq_B$ *of the one preceding it and* $\mathbb{N}E_0$ *is the infimum in* $\sqsubseteq_B$ *of the* $nE_0, n \in \mathbb{N} \setminus \{0\}$.

(ii) $\mathbb{R}I_{\mathbb{N}} \subset_B E_t$ *and* E_t *is a successor of* $\mathbb{R}I_{\mathbb{N}}$ *in* $\sqsubseteq_B$.

(iii) $\mathbb{R}I_{\mathbb{N}}$ *is* $\sqsubseteq_B$*-minimum in* $\mathcal{AE}_0$ *and* E_t *is* $\sqsubseteq_B$*-minimum among the non-smooth elements of* $\mathcal{AE}_0$*. (B. Miller) Also* $E_\infty \times I_{\mathbb{N}}$ *is* $\sqsubseteq_B$*-maximum in* $\mathcal{AE}_0$*.*

(iv) For each $\kappa > 0$, κE_0 *is a* $\sqsubseteq_B$*-minimum element of* $\mathcal{AE}_\kappa$ *but* $\mathcal{AE}_\kappa$ *has no* $\sqsubseteq_B$*-maximum element.*

(v) Let $\kappa \leq \lambda$*. Then for every* $E \in \mathcal{AE}_\lambda$*, there is* $F \in \mathcal{AE}_\kappa$ *such that* $E \sqsubseteq_B F$*.*

In particular $\mathbb{R}E_0$ is $\sqsubseteq_B$-minimum non-smooth in $\mathcal{AE}$ and $E_\infty \times I_{\mathbb{N}}$ is $\sqsubseteq_B$-maximum in $\mathcal{AE}$. Thus one has the following version of the Glimm-Effros Dichotomy for $\sqsubseteq_B$ (see Corollary 2.12):

Theorem 1.7. *Let* $E \in \mathcal{AE}$*. Then exactly one of the following holds:*

(i) E *is smooth,*

(ii) $\mathbb{R}E_0 \sqsubseteq_B E$*.*

(C) We next look at a different kind of realization of CBER. For each infinite countable group Γ and standard Borel space X consider the shift action of Γ on X^Γ and let $E(\Gamma, X)$ be the associated equivalence relation and $E^{ap}(\Gamma, X)$ be its aperiodic part, i.e., the restriction of $E(\Gamma, X)$ to the set of points with infinite orbits. Consider now a Borel action of Γ on an uncountable standard Borel space, which we can assume is equal to $\mathbb{R}$. Then the map $f \colon X \to \mathbb{R}^\Gamma$ given by $x \mapsto p_x$, where $p_x(\gamma) = \gamma^{-1} \cdot x$, is an equivariant Borel embedding of this action to the shift action on $\mathbb{R}^\Gamma$. Thus every aperiodic CBER E induced by a Borel action of Γ can be realized as (i.e., is Borel isomorphic to) the restriction of $E^{ap}(\Gamma, \mathbb{R})$ to an invariant Borel set. By a result in [JKL02, 5.5] we also have $E^{ap}(\Gamma, \mathbb{R}) \cong_B E^{ap}(\Gamma, \mathbb{N})$, so such realizations exist for $E^{ap}(\Gamma, \mathbb{N})$ as well. We consider here the question of whether these realizations can be achieved in the optimal form, i.e., replacing $E^{ap}(\Gamma, \mathbb{N})$ by $E^{ap}(\Gamma, 2)$. This is equivalent to the statement that $E^{ap}(\Gamma, \mathbb{R}) \cong_B E^{ap}(\Gamma, 2)$. If this happens then we call the group Γ **2-adequate**.

Using a recent result of Hochman-Seward, we show the following (see Theorem 4.4):

Theorem 1.8. *Every infinite countable amenable group is 2-adequate.*

This in particular answers in the negative a question of Thomas [T12, Page 391], who asked whether there are infinite countable amenable groups Γ for which $E(\Gamma, \mathbb{R})$ is not Borel reducible to $E(\Gamma, 2)$.

We also show the following (see Corollary 4.9 and Proposition 4.11):

Theorem 1.9. *(i) The free product of any countable group with a group that has an infinite amenable factor and thus, in particular, the free groups* $\mathbb{F}_n, 1 \leq n \leq \infty$, *are* 2*-adequate.*

(ii) Let Γ *be* n*-generated,* $1 \leq n \leq \infty$*. Then* $\Gamma \times \mathbb{F}_n$ *is* 2*-adequate. In particular, all products* $\mathbb{F}_m \times \mathbb{F}_n$, $1 \leq m, n \leq \infty$, *are* 2*-adequate.*

On the other hand there are groups which are not 2-adequate (see Theorem 4.12).

Theorem 1.10. *The group* $\mathrm{SL}_3(\mathbb{Z})$ *is not 2-adequate.*

We do not know if there is a characterization of 2-adequate groups.

(D) In the course of the previous investigations two other classes of groups have been considered. A countable group Γ is called **hyperfinite generating** if for every $E \in \mathcal{AH}$ there is a Borel action of Γ that generates E. We provide equivalent formulations of this property in Proposition 5.1 and show in Corollary 5.2 that all countable groups with an infinite amenable factor are hyperfinite generating, while no infinite countable group with property (T) has this property (see Proposition 5.3).

Finally we say that an infinite countable group Γ is **dynamically compressible** if every $E \in \mathcal{AE}$ generated by a Borel action of Γ can be Borel reduced to a compressible $F \in \mathcal{AE}$ induced by a Borel action of Γ. We show in Proposition 5.7 that every infinite countable amenable group is dynamically compressible and the same is true for any countable group that contains a non-abelian free group (see Proposition 5.8). However there are infinite countable groups that fail to satisfy these two conditions but they are still dynamically compressible (see Proposition 5.9). We do not know if *every* infinite countable group is dynamically compressible.

(E) The paper is organized as follows. In Section 2, we study the structure of the Borel inclusion order on countable Borel equivalence relations. In Section 3, we consider topological realizations of countable Borel equivalence relations. In Section 4, we introduce and study the concept of 2-adequate groups, and in Section 5 we discuss results concerning the concepts of hyperfinite generating groups and dynamically compressible groups.

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2 The Borel inclusion order of countable Borel equivalence relations

2.A General properties

Definition 2.1. Let E, F be CBER on standard Borel spaces X, Y, resp. We put $E \subseteq_B F$ if there is a Borel isomorphism $f\colon X \to Y$ with $f(E) \subseteq F$.

It is clear that $\subseteq_B$ is a quasi-order on CBER, which we call the **Borel inclusion order**. We also let $E \subset_B F \iff E \subseteq_B F \,\&\, F \nsubseteq_B E$ be the strict part of this order.

Recall that a **homomorphism** of an equivalence relation E on X to an equivalence relation F on Y is a map $f\colon X \to Y$ such that $xEy \implies f(x)Ff(y)$. Thus $E \subseteq_B F$ iff there is a bijective Borel homomorphism of E to F.

We will study in this section the structure of this inclusion order on aperiodic CBER in uncountable standard Borel spaces.

We first prove some basic facts concerning the Borel inclusion order that will be repeatedly used in the sequel. Recall that a CBER E on X is **smooth** if it admits a Borel selector and **compressible** if there is Borel injection $f\colon X \to X$ such that for each E-class C, $f(C) \subsetneqq C$. We also let $I_{\mathbb{N}}$ be the equivalence relation $\mathbb{N}^2$ on $\mathbb{N}$ and for each equivalence relation E on X and standard Borel space S, we let SE be the direct sum of "S" copies of E, i.e., the equivalence relation on $S \times X$ defined by $(s,x)SE(t,y) \iff s = t \;\&\; xEy$. It is clear that there is a unique up to Borel isomorphism (which we denote by $\cong_B$), smooth aperiodic CBER, namely $\mathbb{R}I_{\mathbb{N}}$.

Proposition 2.2. *(i) If $E \subseteq_B F$ and F is smooth, then E is smooth.*

(ii) E is compressible iff $\mathbb{R}I_{\mathbb{N}} \subseteq_B E$. Therefore if $E \subseteq_B F$ and E is compressible, then F is compressible.

Proof. (i) By the Feldman-Moore Theorem (see, e.g., [K2, 2.B]), there is a Borel action of a countable group $\Gamma = \{\gamma_n\}$ on X (the space of F) which induces F, i.e., $xFy \iff \exists \gamma \in \Gamma(\gamma \cdot x = y)$. Let f be a Borel selector for F and define for each $x \in X$, $n(x) =$ the least n with $\gamma_n \cdot f(x) Ex$. Then $g(x) = \gamma_n \cdot f(x)$ is a Borel selector for E.

(ii) This follows from [K2, 2.23]. □

The number of ergodic, invariant probability Borel measures for a CBER E will play an important role in the sequel. We denote by $\mathrm{EINV_E}$ the set of ergodic, invariant probability measures and by $|\mathrm{EINV_E}|$ its cardinality. Since $\mathrm{EINV_E}$ can be viewed in a canonical way as a standard Borel space (see, e.g., [K2, 4.10]) we have that $|\mathrm{EINV_E}| \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$. Moreover by Nadkarni's Theorem, see [K2, 4.C], we have that $|\mathrm{EINV_E}| = 0$ iff E is compressible.

We note here the following basic fact:

Proposition 2.3. *If $E \subseteq_B F$, then $|\mathrm{EINV_E}| \geq |\mathrm{EINV_F}|$.*

Proof. This is clear when $|\mathrm{EINV_F}| = 0$. Otherwise assume that $E \subseteq F$ live on a space X and F admits at least one invariant measure. Consider then the ergodic decomposition $\{X_e\}_{e \in \mathrm{EINV_F}}$ of F, see [K2, 4.11]. Then for each $e \in \mathrm{EINV_F}$, X_e is E-invariant and e is an invariant measure for $E|X_e$, thus X_e supports at least one ergodic, invariant measure for E, say e'. Since the map $e \mapsto e'$ is injective the proof is complete. □

We will next show that many subclasses of $\mathcal{AE}$, including $\mathcal{AE}$ itself, admit maximum under $\subseteq_B$ elements. This was proved for $\mathcal{AE}$ by Ben Miller, see [K2, 11.E], and the proof below is an adaptation of his argument to a more general context. Later we will show the existence of a minimum under $\subseteq_B$ non-smooth element of $\mathcal{AE}$ (see the paragraph following Corollary 2.12).

Below for equivalence relations E, F on spaces X, Y, resp., we let $E \sqsubseteq_B F$ iff there is a Borel injection $f \colon X \to Y$ such that $xEy \iff f(x)Ff(y)$.

Again $\sqsubseteq_B$ is a quasi-order on CBER. Also we let $E \times F$ be the equivalence relation on $X \times Y$ given by $(x,y) E \times F (x',y') \iff (xEx' \ \&\ yFy')$. We now have:

Theorem 2.4. *Let $\mathcal{E} \subseteq \mathcal{AE}$ be a class of* CBER *such that $\mathcal{E}$ contains a maximum under $\sqsubseteq_B$ element E such that $E \times I_{\mathbb{N}} \in \mathcal{E}$. Then $E \times I_{\mathbb{N}} \in \mathcal{E}$ is $\subseteq_B$-maximum for $\mathcal{E}$.*

Proof. We start with the following fact, where for two equivalence relations F, G, $F \oplus G$ is their direct sum.

Lemma 2.5. *Let R be compressible. Then for any $S \in \mathcal{AE}$, $S \subseteq_B R \oplus S$.*

Proof. Suppose S lives on the space X. Then there is an S-invariant Borel set $X_0 \subseteq X$ such that $S|X_0 \cong_B \mathbb{R}I_{\mathbb{N}}$. Since $\mathbb{R}I_{\mathbb{N}} \oplus \mathbb{R}I_{\mathbb{N}} \cong_B \mathbb{R}I_{\mathbb{N}}$, we have, by Proposition 2.2, that $S \cong_B \mathbb{R}I_{\mathbb{N}} \oplus S \subseteq_B R \oplus S$. □

Let now $F \in \mathcal{E}$ in order to show that $F \subseteq_B E \times I_{\mathbb{N}}$. Since $F \sqsubseteq_B E$, there is G such that $F \oplus G \subseteq_B E$. Recalling (see, e.g., [K2, 2.23]) that for any CBER R, $R \times I_{\mathbb{N}}$ is compressible, we have, by Lemma 2.5, that $F \subseteq_B F \oplus (F \times I_{\mathbb{N}}) \oplus (G \times I_{\mathbb{N}})$. Note now that $F \oplus (F \times I_{\mathbb{N}}) \subseteq_B F \times I_{\mathbb{N}}$, therefore $F \subseteq_B F \oplus (F \times I_{\mathbb{N}}) \oplus (G \times I_{\mathbb{N}}) \subseteq_B (F \times I_{\mathbb{N}}) \oplus (G \times I_{\mathbb{N}}) \cong_B (F \oplus G) \times I_{\mathbb{N}} \subseteq_B E \times I_{\mathbb{N}}$. □

In particular this applies to the following classes $\mathcal{E}$: hyperfinite, α-amenable (see [K2, 8.B]), treeable, $\mathcal{AE}$.

2.B Hyperfiniteness

We will discuss here the inclusion order on the hyperfinite equivalence relations. Recall first the following well-known fact (see, e.g., [K2, 7.13]):

Proposition 2.6. *If E is hyperfinite and $F \subseteq_B E$, then F is hyperfinite.*

Thus the class $\mathcal{AH}$ of hyperfinite aperiodic CBER forms an initial segment in $\subseteq_B$. It is also downwards cofinal in $\subseteq_B$ in view of the following standard result (see, e.g., [K2, 2.10]):

Theorem 2.7. *For any $E \in \mathcal{AE}$, there is $F \in \mathcal{AH}$ with $F \subseteq E$.*

We will actually need a more precise version of this result, see [K2, 7.12]. Since a proof of this result has not appeared in print before, we will include it below.

Theorem 2.8. *For any $E \in \mathcal{AE}$, there is $F \in \mathcal{AH}$ with $F \subseteq E$ such that moreover* $\mathrm{EINV}_E = \mathrm{EINV}_F$.

Proof. We will need the following lemma. Below E_0 is the equivalence relation on $2^{\mathbb{N}}$ defined by $xE_0y \iff \exists m \forall n \geq m(x_n = y_n)$ and μ_0 is the product measure on $2^{\mathbb{N}}$, where $2 = \{0, 1\}$ is given the uniform $(\frac{1}{2}, \frac{1}{2})$ measure. Then μ_0 is the unique element of EINV_{E_0}.

Lemma 2.9. *Let E be a* CBER *on a standard Borel space X and let $\mu \in$* EINV_E. *Then there is an E-invariant Borel set $X_0 \subseteq X$ with $\mu(X_0) = 1$, an E_0-invariant Borel set $C_0 \subseteq 2^{\mathbb{N}}$ with $\mu_0(C_0) = 1$ and a Borel isomorphism $f\colon C_0 \to X_0$ such that $f_*\mu_0 = \mu$ and $f(E_0|C_0) \subseteq E$.*

Proof. This follows from the proof of Dye's Theorem, see, e.g., [KM04, Section 7] and [K94, 5.26]. □

If E is compressible, then the result follows from Proposition 2.2, (ii). Otherwise by Nadkarni's Theorem (see, e.g., [K2, 4.C]) EINV_E is nonempty. Consider then the ergodic decomposition $\{X_e\}_{e \in \mathrm{EINV_E}}$ of E (see, e.g., [K2, 4.11]). For each $e \in \mathrm{EINV_E}$, by Lemma 2.9, there is an E-invariant Borel set $X_{0,e} \subseteq X_e$ with $e(X_{0,e}) = 1$, an E_0-invariant Borel set $C_{0,e} \subseteq 2^{\mathbb{N}}$ with $\mu_0(C_{0,e}) = 1$ and a Borel isomorphism $f_e\colon C_{0,e} \to X_{0,e}$ such that $(f_e)_*\mu_0 = e$ and $F_e = f_e(E_0|C_{0,e}) \subseteq E$. Note that F_e admits a unique ergodic, invariant measure, namely e.

The proof of Lemma 2.9 is effective enough (see, e.g., the proof of [DJK94, 9.6]), so that $X_0 = \bigcup_e X_{0,e}$ is Borel and $F_0 = \bigcup_e F_e$, which lives on X_0, is also Borel and hyperfinite. Let $X' = X \setminus X_0$. Then by the properties of the ergodic decomposition $F|X'$ is compressible, so by the compressible case above there is a hyperfinite compressible equivalence relation $F' \subseteq E|X'$. Finally put $F = F_0 \cup F'$. This clearly works. □

Recall that the classification theorem for hyperfinite CBER, see [DJK94, 9.1], shows that, up to Borel isomorphism, $\mathcal{AH}$ consists exactly of the following equivalence relations, where E_t is the equivalence relation on $2^{\mathbb{N}}$ given by $xE_ty \iff \exists m \exists n \forall k(x_{m+k} = y_{n+k})$:

$$\mathbb{R}I_{\mathbb{N}}, E_t, E_0, 2E_0, 3E_0, \ldots, \mathbb{N}E_0, \mathbb{R}E_0.$$

Moreover $|\mathrm{EINV}_E|$, for E in this list, is respectively $0, 0, 1, 2, 3, \ldots, \aleph_0, 2^{\aleph_0}$.

Below for a quasi-order $\preceq$ with strict part $\prec$ on a set Q and $q, r \in Q$, we say that r is a **successor** to q if $q \prec r$ and $(s \prec r \implies s \preceq q)$.

We now have:

Theorem 2.10. *(i)* $\mathbb{R}E_0 \subset_B \mathbb{N}E_0 \subset_B \cdots \subset_B 3E_0 \subset_B 2E_0 \subset_B E_0 \subset_B E_t$, *each equivalence relation in this list is a successor in* $\subseteq_B$ *of the one preceding it and* $\mathbb{N}E_0$ *is the infimum in* $\subseteq_B$ *of the* $nE_0, n \in \mathbb{N} \setminus \{0\}$.

(ii) $\mathbb{R}I_{\mathbb{N}} \subset_B E_t$ *and* E_t *is a successor of* $\mathbb{R}I_{\mathbb{N}}$ *in* $\subseteq_B$.

Proof. (i) Clearly $E_0 \subseteq E_t$ and thus $E_0 \subset_B E_t$ as E_0 is not compressible. To see that $2E_0 \subseteq_B E_0$, note that $2^{\mathbb{N}} = X_0 \sqcup X_1$, where $X_i = \{x \in 2^{\mathbb{N}} \colon x_0 = i\}$, and $E_0|X_i \cong_B E_0$. From this it follows immediately that $(n+1)E_0 \subseteq_B nE_0$, for each $n \in \mathbb{N}, n \geq 1$.

To show that $\mathbb{N}E_0 \subseteq_B nE_0$, for each $n \in \mathbb{N} \setminus \{0\}$, it is enough to show that $\mathbb{N}E_0 \subseteq_B E_0$. Let $s_n = 1^n 0$ be the finite sequence staring with n 1's followed by one 0, for $n \in \mathbb{N}$. Let X_n be the subset of $2^{\mathbb{N}}$ consisting of all sequences starting with s_n, let $\bar{1}$ be the constant 1 sequence and put $X = 2^{\mathbb{N}} \setminus \{\bar{1}\}$. Then $X = \bigsqcup_n X_n$ and $E_0 \cong_B E|X \cong_B E|X_n$, for each $n \in \mathbb{N}$, which completes the proof that $\mathbb{N}E_0 \subseteq_B E_0$.

Finally to show that $\mathbb{R}E_0 \subseteq_B \mathbb{N}E_0$, it is enough to show that $\mathbb{R}E_0 \subseteq_B E_0$. To prove this, let for each $y \in 2^{\mathbb{N}}$, $X_y = \{x \in 2^{\mathbb{N}} \colon \forall n \in \mathbb{N}(x_{2n} = y_n)\}$. Then $2^{\mathbb{N}} = \bigsqcup_y X_y$ and $E_0|X_y \cong_B E_0, \forall y \in 2^{\mathbb{N}}$, which immediately implies that $\mathbb{R}E_0 \subseteq_B E_0$.

This establishes the non-strict orders in the list of (i). The strict orders and the last two statements of (i) now follow from Proposition 2.3.

(ii) Since E_t is compressible and not smooth, by Proposition 2.2, $\mathbb{R}I_{\mathbb{N}} \subset_B E_t$. It is also clear that E_t is a successor of $\mathbb{R}I_{\mathbb{N}}$. □

The following is an immediate corollary of Theorem 2.10:

Corollary 2.11. *Let* $E, F \in \mathcal{AH}$. *Then*

$$E \subseteq_B F \iff |\mathrm{EINV}_E| \geq |\mathrm{EINV}_F|.$$

The next result is a version of the Glimm-Effros Dichotomy, see [K2, 5.5], for the inclusion order $\subseteq_B$ instead of $\sqsubseteq_B$. It is an immediate corollary of Theorem 2.10 and Theorem 2.8.

Corollary 2.12. *Let* $E \in \mathcal{AE}$. *Then exactly one of the following holds:*

(i) E *is smooth,*

(ii) $\mathbb{R}E_0 \subseteq_B E$.

Denote by E_∞ a universal CBER, in the sense that every CBER F satisfies $F \sqsubseteq_B E_\infty$, see, e.g., [K2, 5.C]. Then, by Corollary 2.12, $\mathbb{R}E_0$ is a $\subseteq_B$-minimum among all the non-smooth relations in $\mathcal{AE}$ and, by Theorem 2.4, $E_\infty \times I_\mathbb{N}$ is a $\subseteq_B$-maximum relation in $\mathcal{AE}$.

2.C A global decomposition

For each cardinal $\kappa \in \{0, 1, 2, 3, \dots, \aleph_0, 2^{\aleph_0}\}$, let $\mathcal{AE}_\kappa$ be the class of all $E \in \mathcal{AE}$ such that $|\mathrm{EINV}_E| = \kappa$. Clearly $\mathcal{AE} = \bigsqcup_\kappa \mathcal{AE}_\kappa$ and each $\mathcal{AE}_\kappa$ is invariant under the equivalence relation associated with the quasi-order $\subseteq_B$, by Proposition 2.3. We also let for $\kappa > 0$, $\kappa E = SE$, where S is a standard Borel space of cardinality κ.

Proposition 2.13. *(i) $\mathbb{R}I_\mathbb{N}$ is $\subseteq_B$-minimum in $\mathcal{AE}_0$ and E_t is $\subseteq_B$-minimum among the non-smooth elements of $\mathcal{AE}_0$. (B. Miller) Also $E_\infty \times I_\mathbb{N}$ is $\subseteq_B$-maximum in $\mathcal{AE}_0$.*

(ii) For each $\kappa > 0$, κE_0 is a $\subseteq_B$-minimum element of $\mathcal{AE}_\kappa$ but $\mathcal{AE}_\kappa$ has no $\subseteq_B$-maximum element.

(iii) Let $\kappa \leq \lambda$. Then for every $E \in \mathcal{AE}_\lambda$, there is $F \in \mathcal{AE}_\kappa$ such that $E \subseteq_B F$.

(iv) (with R. Chen) The map $E \mapsto E \oplus E_0$ is an order embedding of the non-smooth elements of $\mathcal{AE}$ into $\mathcal{AE}$, i.e., for non-smooth $E, F \in \mathcal{AE}$, $E \subseteq_B F \iff E \oplus E_0 \subseteq_B F \oplus E_0$. It maps $\mathcal{AE}_\kappa$ into $\mathcal{AE}_{\kappa+1}$, if κ is finite, and $\mathcal{AE}_\kappa$ into itself, if κ is infinite.

Proof. (i) That $\mathbb{R}I_\mathbb{N}$ is $\subseteq_B$-minimum in $\mathcal{AE}_0$ follows from Proposition 2.2 and that $E_\infty \times I_\mathbb{N}$ is $\subseteq_B$-maximum in $\mathcal{AE}_0$ follows from Theorem 2.4. Finally we have to show that if $E \in \mathcal{AE}_0$ is not smooth, then $E_t \subseteq_B E$.

Since E is not smooth, we have that $E_t \sqsubseteq_B E$ (see [K2, 5.5 and 7.3]), so, as E_t is compressible, $E_t \sqsubseteq_B^i E$ (see [K2, 2.27]), i.e., E_t is Borel isomorphic to the restriction of E to an E-invariant Borel set. So if E lives on X, we have a Borel partition $X = Y \sqcup Z$ into E-invariant Borel sets such that $E|Y \cong_B E_t$. Since $E|Z$ is compressible, we see, using Lemma 2.5, that $E_t \subseteq_B E_t \oplus E|Z \cong_B E|Y \oplus E|Z \cong_B E$.

(ii) The fact that κE_0 is a $\subseteq_B$-minimum element of $\mathcal{AE}_\kappa$ is clear from Theorem 2.8. That $\mathcal{AE}_\kappa$ has no $\subseteq_B$-maximum element can be seen as follows.

Assume that E is such a $\subseteq_B$-maximum, towards a contradiction. Say E lives on the space X. Fix an invariant measure μ for E. We will show that every infinite countable group Γ embeds algebraically into $[E]$, the measure theoretic full group of E with respect to μ, contradicting a result of Ozawa, see [K10, page 29].

The group Γ admits a free Borel action on a standard Borel space Y, with associated equivalence relation G that has exactly κ ergodic, invariant measures. To see this, consider the free part of the shift action of Γ on 2^Γ, which has $2^{\aleph_0}$ ergodic components, and restrict the action to κ many ergodic components. Since E is $\subseteq_B$-maximum in $\mathcal{AE}_\kappa$, let $f\colon Y \to X$ be a Borel isomorphism such that $f(G) = F \subseteq E$. Then Γ acts freely in a Borel way on X inducing F, so that Γ can be algebraically embedded in $[F]$, the measure theoretic full group of F with respect to μ (which is clearly invariant for F). But $[F] \leq [E]$, so Γ embeds algebraically into $[E]$.

(iii) We can of course assume that $\kappa > 0$. Let $E \in \mathcal{AE}_\lambda$. Let $\{X_e\}_{e\in \mathrm{EINV}_E}$ be the ergodic decomposition of E, which has λ many components. If E lives on X, let Y be a Borel E-invariant subset of X consisting of exactly κ many ergodic components. Put $Z = X \setminus Y$. Then let $E' = E|Y$ and let G be a compressible equivalence relation on Z with $G \supseteq E|Z$. Let $F = E' \cup G$. Then $E \subseteq F$ and $F \in \mathcal{AE}_\kappa$.

(iv) We show that $E \mapsto E \oplus E_0$ is an order embedding on non-smooth aperiodic CBERs (on uncountable standard Borel spaces). (Note that the only failure is that $E_t \oplus E_0 \cong_B \mathbb{R}I_{\mathbb{N}} \oplus E_0$.)

Clearly, if $E \subseteq_B F$, then $E \oplus E_0 \subseteq_B F \oplus E_0$. Conversely, suppose that $E \oplus E_0 \subseteq_B F \oplus E_0$. We want to show that $E \subseteq_B F$.

We can write $E \cong_B R \oplus R'$ and $E_0 \cong_B S \oplus S'$ with $R \oplus S \subseteq_B F$ and $R' \oplus S' \subseteq_B E_0$. Note that R', S, S' are all aperiodic hyperfinite (maybe on a countable space), and since $E_0 \cong_B S \oplus S'$, exactly one of S or S' must be E_0, and the other is compressible hyperfinite. Also since E is non-smooth, we have $E \cong_B E \oplus E_t$, and similarly for F.

We have two cases:

1. If $S = E_0$ and S' is compressible, then since $R' \oplus S' \subseteq_B E_0$, we must have $R' \subseteq_B E_0 = S$.

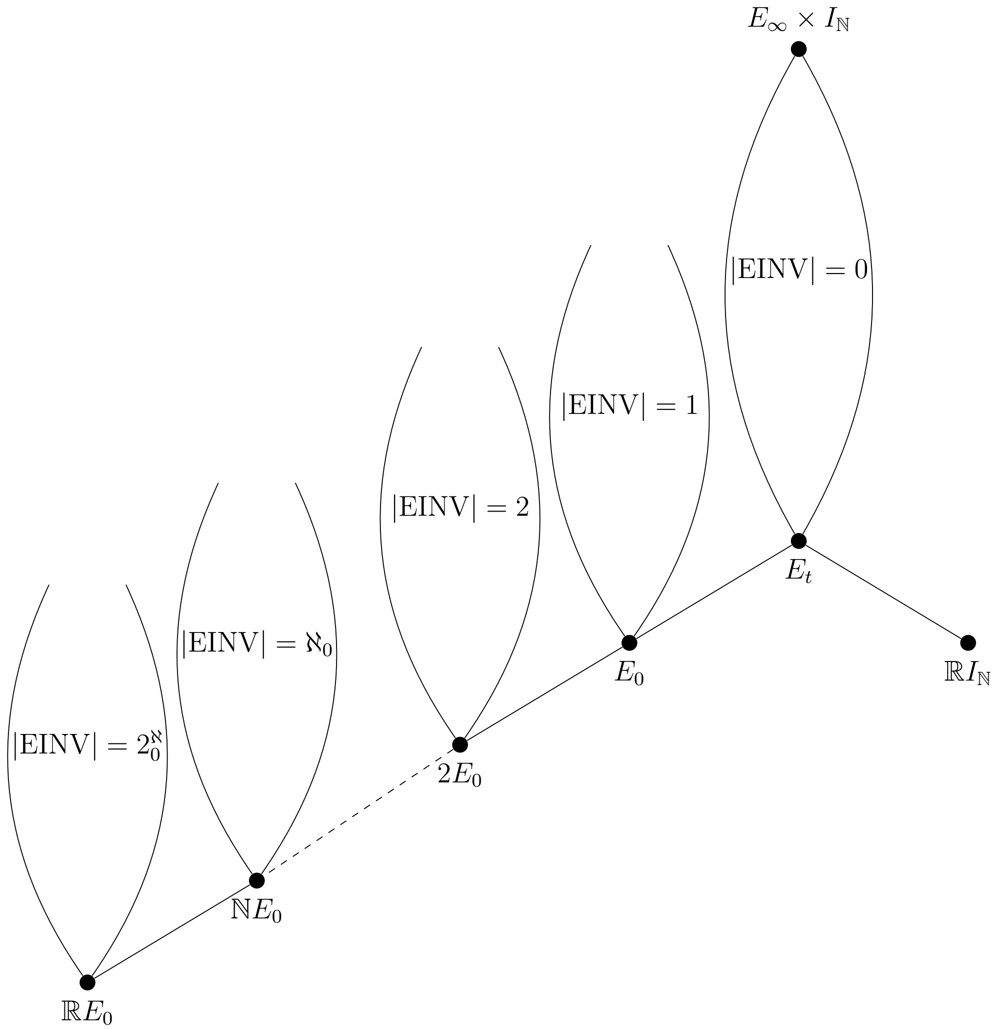
2. If S is compressible, then we have $R' \oplus E_t \subseteq_B S \oplus E_t$, since $R' \oplus E_t$ is hyperfinite and $S \oplus E_t \cong_B E_t$.

In both cases, we get:

$$E \cong_B E \oplus E_t \cong_B R \oplus R' \oplus E_t \subseteq_B R \oplus S \oplus E_t \subseteq_B F \oplus E_t \cong_B F$$

□

The following picture illustrates parts (i) and (ii) of Proposition 2.13.



It is interesting to consider the problem of existence of $\subseteq_B$-maximum elements in $\mathcal{E}_\kappa = \mathcal{AE}_\kappa \cap \mathcal{E}$ for other classes $\mathcal{E} \subseteq \mathcal{AE}$. This is clearly the case if $\kappa = 0$ and $\mathcal{E}$ satisfies the conditions of Theorem 2.4, so we will consider $\kappa \geq 1$.

Clearly κE_0 is $\subseteq_B$-maximum in $\mathcal{AH}_\kappa$. Denote by $\mathcal{AT}$ the subclass of $\mathcal{AE}$ consisting of the treeable equivalence relations.

Problem 2.14. *Let $\kappa \geq 1$. Does $\mathcal{AT}_\kappa$ have a $\subseteq_B$-maximum element?*

If E is $\subseteq_B$-maximum in $\mathcal{AT}_1$, then κE is $\subseteq_B$-maximum in $\mathcal{AT}_\kappa$, for every $1 \leq \kappa \leq \aleph_0$, so we will concentrate in the case $\kappa = 1$, i.e., the class of **uniquely ergodic** elements of $\mathcal{AT}$. We do not know the answer to this problem but we would like to point out that a positive answer has an implication in the context of the theory of measure preserving CBER, see [K1].

Fix a standard Borel space X and a measure μ on X. We will consider as in [K1] **pmp** CBER on X, i.e., μ-measure preserving CBER on X, where we identify two such relations if they agree μ-a.e. Inclusion of pmp relations is also understood in the μ-a.e. sense. Such a relation is treeable if it has this property μ-a.e. We also denote by $\mathrm{Aut}(X, \mu)$ the group of measure preserving automorphisms of (X, μ).

Proposition 2.15. *If E on a standard Borel space X is a $\subseteq_B$-maximum uniquely ergodic, equivalence relation in $\mathcal{AT}$, with (unique) invariant measure μ, then for every treeable pmp relation F on (X, μ), there is an automorphism $T \in \mathrm{Aut}(X, \mu)$ such that $T(F) \subseteq E$.*

Proof. We will use the following lemma.

Lemma 2.16. *Let G be a treeable pmp* CBER *on (X, μ). Then there is an ergodic, treeable pmp* CBER *H on (X, μ) with $G \subseteq H$.*

Proof. For each $T \in \mathrm{Aut}(X, \mu)$ denote by E_T the equivalence relation induced by T. By [CM14, Theorem 8] the set of $T \in \mathrm{Aut}(X, \mu)$ such that E_T is independent of G (see [KM04, Section 27] for the notion of independence) is comeager in $\mathrm{Aut}(X, \mu)$, equipped with the usual weak topology. So is the set of all ergodic $T \in \mathrm{Aut}(X, \mu)$, see [K10, Theorem 2.6]. Thus there is an ergodic $T \in \mathrm{Aut}(X, \mu)$ such that E_T is independent of G. Then put $H = E_T \vee G$, the smallest equivalence relation containing E_T and G. □

By Lemma 2.16, we can assume that F is ergodic. We can also assume that there is $F' \in \mathcal{AT}$ which agrees with F μ-a.e. By considering the ergodic decomposition of F', we can also assume that μ is the unique invariant measure for F'. Fix then a Borel automorphism $T \colon X \to X$ such that $T(F') \subseteq E$. Then both $T_*\mu$ and μ are $T(F')$-invariant. Since $T(F')$ is uniquely ergodic, it follows that $T_*\mu = \mu$, i.e., $T \in \mathrm{Aut}(X, \mu)$ and the proof is complete. □

Remark 2.17. We note here that an analog of the conclusion of Proposition 2.15 is valid for the class $\mathcal{AH}$. More precisely, let $X = 2^{\mathbb{N}}$ and let μ be the usual product measure on X. Then for every hyperfinite pmp relation F on (X, μ), there is an automorphism $T \in \mathrm{Aut}(X, \mu)$ such that $T(F) \subseteq E_0$. This can be seen as follows: By [K10, 5.4] (in which the aperiodicity of E is not needed), we can find a hyperfinite pmp relation F' such that $F \subseteq F'$. By Dye's Theorem (see, e.g., [K10, 3.13]) there is an automorphism $T \in \mathrm{Aut}(X, \mu)$ such that $T(F') = E_0$ and thus $T(F) \subseteq E_0$.

3 Topological realizations

3.A Dense realizations and Lusin marker schemes

We will first use the results in Section 2 to prove the following:

Theorem 3.1. *For every equivalence relation $E \in \mathcal{AE}$ and every perfect Polish space Y, there is a topological realization of E in Y in which every equivalence class is dense.*

Proof. First, since for every perfect Polish space Y there is a continuous bijection from the Baire space $\mathcal{N}$ onto Y (see [K95, 7.15]), we can assume that $Y = \mathcal{N}$. Moreover by Corollary 2.12, it is enough to prove this result for $E = \mathbb{R}E_0$ and $E = \mathbb{R}I_{\mathbb{N}}$.

Case 1: $\mathbb{R}E_0$.

Consider the shift map of $\mathbb{Z}$ on $2^{\mathbb{Z}}$ with associated equivalence relation F'. Let $Y = \{x \in 2^{\mathbb{Z}} \colon [x]_{F'} \text{ is dense in } 2^{\mathbb{Z}}\}$, Clearly Y is a dense, co-dense G_δ set in $2^{\mathbb{Z}}$, so, in particular, it is a zero-dimensional Polish space (with the relative topology from $2^{\mathbb{Z}}$). We next check that every compact set in Y has empty interior. Indeed let $K \subseteq Y$ be compact in Y. Then K is compact in $2^{\mathbb{Z}}$. If now V is open in $2^{\mathbb{Z}}$ and $\emptyset \neq V \cap Y \subseteq K$, then since Y is dense in $2^{\mathbb{Z}}$, by looking at $V \setminus K$ we see that $V \subseteq K$, contradicting that Y is also co-dense in $2^{\mathbb{Z}}$.

By [K95, 7.7] Y is homeomorphic to $\mathcal{N}$. Moreover if $F = F'|Y$, F has dense classes and $|\mathrm{EINV}_F| = 2^{\aleph_0}$, so $F \cong_B \mathbb{R}E_0$.

Case 2: $\mathbb{R}I_{\mathbb{N}}$.

Consider the equivalence relation R on $\mathcal{N}$ given by

$$xRy \iff \exists m \forall n \geq m (x_n = y_n).$$

Let $A \subseteq \mathcal{N}$ be an uncountable Borel partial transversal for R (i.e., no two distinct elements of A are in R). Then, as R is not smooth, denoting by $B = [A]_R$ the R-saturation of A, we also have that $Y = \mathcal{N} \setminus B$ is uncountable. Fix then a Borel bijection $f\colon A \to Y$ and let F be the equivalence relation obtained by adding to each $[a]_R$, $a \in A$, the point $f(a)$. Then F is a smooth CBER, so $F \cong_B \mathbb{R}I_{\mathbb{N}}$, and every F-class is dense in $\mathcal{N}$. □

A **complete section** of an equivalence relation E on X is a subset $Y \subseteq X$ which meets every E-class. Recall that a **vanishing sequence of markers** for a CBER E is a decreasing sequence of complete Borel sections $\{A_n\}$ for E such that $\bigcap_n A_n = \emptyset$. A very useful result in the theory of CBER is the Marker Lemma, which asserts that every $E \in \mathcal{AE}$ admits a vanishing sequence of markers, see, e.g., [K2, 2.15]. We will see next that Theorem 3.1 implies a strong new version of a marker lemma.

Definition 3.2. Let E be a CBER on a standard Borel space X. A **Lusin marker scheme** for E is a family $\{A_s\}_{s \in \mathbb{N}^{<\mathbb{N}}}$ of Borel sets such that

(i) $A_\emptyset = X$;

(ii) $\{A_{sn}\}_n$ are pairwise disjoint and $\bigsqcup_n A_{sn} \subseteq A_s$;

(iii) Each A_s is a complete section for E.

We have two types of Lusin marker schemes:

(1) The Lusin marker scheme $\{A_s\}_{s \in \mathbb{N}^{<\mathbb{N}}}$ for E is of **type I** if in (ii) above we actually have that $\bigsqcup_n A_{sn} = A_s$ and moreover the following holds:

(iv) For each $x \in \mathcal{N} = \mathbb{N}^{\mathbb{N}}$, $\bigcap_n A_{x|n}$ is a singleton.

(Then in this case, for each $x \in \mathcal{N}$, $A^x_n = A_{x|n} \setminus \bigcap_n A_{x|n}$ is a vanishing sequence of markers.)

(2) The Lusin marker scheme $\{A_s\}_{s \in \mathbb{N}^{<\mathbb{N}}}$ for E is of **type II** if it satisfies the following:

(v) If for each n, $B_n = \bigsqcup \{A_s \colon s \in \mathbb{N}^n\}$, then $\{B_n\}$ is a vanishing sequence of markers.

Theorem 3.3. *Every $E \in \mathcal{AE}$ admits a Lusin marker scheme of type* I *and a Lusin marker scheme of type* II.

Proof. Type I: By Theorem 3.1, we can assume that E lives on $\mathcal{N}$ and that every equivalence class is dense. Let then for each $s \in \mathbb{N}^n$, $A_s = \{x \colon x|n = s\}$.

Type II: By Theorem 3.1, we can assume that E lives on $\mathbb{R}$ and that every equivalence class is dense. By induction on n, we can easily construct open sets $A_s, s \in \mathbb{N}^n$, such that $\{A_s\}_{s \in \mathbb{N}^{<\mathbb{N}}}$ is a Lusin marker scheme for E and moreover it has the following properties:

(a) Each $A_s, s \in \mathbb{N}^n, n \geq 1$, is contained in (n, ∞);

(b) Each $A_s, s \in \mathbb{N}^n, n \geq 1$, has non-empty intersection with the interval $(k, k+1)$ for every $k \geq n$.

Then clearly $\{A_s\}_{s\in\mathbb{N}^{<\mathbb{N}}}$ is of type II. □

Remark 3.4. (a) We can also easily see that every $E \in \mathcal{AE}$ admits a **Cantor marker scheme** $\{A_s\}_{s\in 2^{<\mathbb{N}}}$ of each type, which is defined in an analogous way.

(b) By applying Theorem 3.3 to $\mathbb{R}E$, and using the ccc property for category, we can see that every $E \in \mathcal{AE}$ admits a variant of a Lusin marker scheme of type I, where condition (iv) in Definition 3.2 is replaced by the following condition:

(iv)' For each $x \in \mathcal{N}$, $\bigcap_n A_{x|n}$ has at most one element and for a comeager set of x it is empty.

3.B Continuous action realizations

Any CBER has a continuous action realization, i.e., a topological realization induced by a continuous action of a countable group on a Polish space. We will consider what additional properties of the action and the Polish space of the realization are possible. For example, we have the following:

Proposition 3.5. *Every $E \in \mathcal{AE}$ has a continuous action realization on the Baire space $\mathcal{N}$.*

Proof. By the usual change of topology arguments, we can assume that E is induced by a continuous action of a countable group on a 0-dimensional space X. Let $P \subseteq X$ be the perfect kernel of X, which is clearly invariant under the action. Since $X \setminus P$ is countable, it is easy to see that $E|P \cong_B E$, so we can assume that X is perfect. Let then D be a countable dense subset of X which is also invariant under the action and put $Y = X \setminus D$. Then again $E \cong_B E|Y$. The space Y is a nonempty, 0-dimensional Polish space in which every compact set has empty interior and thus is homeomorphic to the Baire space (see [K95, Theorem 7.7]). □

Definition 3.6. (i) A **transitive action realization**, resp., **minimal action realization** of a CBER is a topological realization induced by a continuous, topologically transitive action of a countable group (i.e., one which has a dense orbit), resp., induced by a continuous, topologically minimal action of a countable group (i.e., one for which all orbits are dense).

(ii) A **σ-compact action realization**, resp., **locally compact action realization**, resp., **compact action realization** of a CBER is a topological realization induced by a continuous action of a countable group on a σ-compact, resp., locally compact, resp., compact Polish space.

(iii) A **transitive, σ-compact action realization** is a topological realization induced by a continuous, topologically transitive action of a countable group on a σ-compact Polish space. Similarly we define the concepts of

transitive, locally compact action realization,
transitive, compact action realization,
minimal, σ-compact action realization,
minimal, locally compact action realization,
minimal, compact action realization.

We first note the following fact:

Proposition 3.7. *If $E \in \mathcal{AE}$ has a compact action realization or a transitive action realization on a perfect Polish space or a minimal action realization, then E is not smooth.*

Proof. Suppose a smooth E has a compact action realization F, towards a contradiction. Then there is a compact invariant subset K in which the action is minimal. Since $F|K$ is also smooth, by [K95, 8.46] some orbit in K is non-meager in K, thus consists of isolated points in K. Minimality then implies that K consists of a single infinite orbit, contradicting compactness.

The proof of the case of a transitive action realization on a perfect Polish space or a minimal action realization follows also from [K95, 8.46]. □

We first note here that the hypothesis of perfectness in Proposition 3.7 is necessary.

Proposition 3.8. *Every smooth equivalence relation in $\mathcal{AE}$ has a transitive locally compact action realization (in some non-perfect space).*

Proof. Let $\mathbb{N} = \bigsqcup_{q\in\mathbb{Q}} N_q$ be a decomposition of $\mathbb{N}$ into infinite sets indexed by the rationals. Define then recursively $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$, with Im $z_n > 0$, Im $z_{n+1} <$ Im z_n, Im $z_n \to 0$, and pairwise disjoint closed squares S_n with center z_n with Im $S_n > 0$ as follows:

If $0 \in N_q$, choose $z_0 \in \{q\} \times \mathbb{R}$ and let S_0 be a very small square around z_0. At stage $n+1$, if $n+1 \in N_q$, choose $z_{n+1} \in \{q\} \times \mathbb{R}$ so that $0 <$ Im $z_n <$

$\frac{1}{n+1}$, Im $z_{n+1} <$ Im z_n, $z_{n+1} \notin \bigcup_{m \leq n} S_m$, and then choose S_{n+1} to be a small square around z_{n+1} so that it has empty intersection with all $S_m, m \leq n$.

Put $X = \mathbb{R} \cup \{z_n\}_{n \in \mathbb{N}}$. Then X is closed in $\mathbb{C}$, so it is locally compact. Next define $T \colon X \to X$ as follows:

If $x \in \mathbb{R}$, then $T(x) = x + 1$. If $x = z_n$ with $n \in N_q$, so that $x \in \{q\} \times \mathbb{R}$, and if in the increasing enumeration of N_q, n is the ith element, then put $T(x) = z_m$, where m is the ith element in the increasing enumeration of N_{q+1}. It is not hard to check that T is a homeomorphism of X. For example, to check that T is continuous (a similar argument works for T^{-1}) let $w_n, w \in X$, with $w_n \to w$, in order to show that $T(w_n) \to T(w)$. We can assume of course that $w_n \notin \mathbb{R}, w \in \mathbb{R}, \text{Im } w_n \to 0$. Now Re $T(w_n) =$ Re $w_n + 1$ and Im $T(w_n) \to 0$, thus $T(w_n) = \text{Re } w_n + 1 + i \text{ Im } T(w_n) \to w + 1 = T(w)$.

Next for each pair $(m, n) \in \mathbb{N}^2$, let $T_{m,n}$ be the homeomorphism of X that switches z_m with z_n and keeps every other point of X fixed. Then the group generated by all $T_{m,n}$ and T acts continuously on X. One of its orbits is $\{z_n\}$ which is dense in X, thus the action is topologically transitive. The equivalence relation F it generates has as classes the set $\{z_n\}$ and the sets of the form $x + \mathbb{Z}$, for $x \in \mathbb{R}$, so it is aperiodic and smooth, with transversal $\{z_0\} \cup [0, 1)$. □

Also the hypothesis of compactness in Proposition 3.7 is necessary.

Proposition 3.9. *Every smooth equivalence relation in $\mathcal{AE}$ has a locally compact action realization on a perfect space, in fact one in the space $2^{\mathbb{N}} \setminus \{\bar{1}\}$, where $\bar{1}$ is the constant 1 sequence.*

Proof. We use an example in [DJK94, page 200, (b)]. Consider the space $X = 2^{\mathbb{N}} \setminus \{\bar{1}\}$. For each $m \neq n$, let $h_{m,n}$ be the homeomorphism of X defined by: $h_{m,n}(1^m 0 y) = 1^n 0 y, h_{m,n}(1^n 0 y) = 1^m 0 y, h_{m,n}(x) = x$, otherwise. Then the group generated by these homeomorphisms acts continuously on X and generates the equivalence relation F given by: $xFy \iff \exists z(x = 1^m 0 z \ \& \ y = 1^n 0 z)$, which is smooth aperiodic. □

We next show that non-smooth hyperfinite equivalence relations in $\mathcal{AE}$ have the strongest kind of topological realization.

Theorem 3.10. *Every non-smooth hyperfinite equivalence relation in $\mathcal{AE}$ has a minimal, compact action realization on the Cantor space $2^{\mathbb{N}}$. Moreover if the equivalence relation is not compressible, the acting group can be taken to be $\mathbb{Z}$.*

Proof. We first consider the compressible case, namely E_t. Then E_t is generated by a continuous action of $\mathbb{F}_2$, see [K2, 2.B].

Next assume that $E \in \mathcal{AH}$ is non-compressible and let $\kappa = |\mathrm{EINV}_E| > 0$. By a theorem of Downarowicz [D91] (see also [M18]) for every metrizable Choquet simplex K there is a minimal continuous action of $\mathbb{Z}$ on $2^{\mathbb{N}}$ such that K is affinely homeomorphic to the simplex of invariant measures for this action. In particular the cardinality of the set of ergodic, invariant measures for the action is the same as the cardinality of the set of extreme points of K. Fix now a compact Polish space X of cardinality κ and let K be the Choquet simplex of measures on X. The extreme points are the Dirac measures, so there are exactly κ many of them. Thus we can find a minimal continuous action of $\mathbb{Z}$ on $2^{\mathbb{N}}$ with exactly κ many ergodic, invariant measures and therefore if F is the equivalence relation induced by this action, we have that $E \cong_B F$. □

Although E_t does not have a minimal, compact action realization where the acting group is amenable (otherwise it would have an invariant measure), we have the following:

Proposition 3.11. *A compressible, non-smooth, hyperfinite* CBER *has a minimal, locally compact action realization where the acting group is $\mathbb{Z}$.*

Proof. It is known that there are minimal homeomorphisms on uncountable locally compact spaces with no invariant measure, which thus generate a compressible non-smooth hyperfinite CBER; see, e.g., [D01, Section 2]. Below we give a simple example:

Let $X \subseteq 4^{\mathbb{N}}$ be the set of sequences which eventually consist only of 1 and 2. Let $X_n = 4^n \times \{1,2\}^{\mathbb{N}}$, so that $X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$ and $X = \bigcup_n X_n$. We give X_n the usual product topology, so that X_n is clopen in X_{n+1}, and X the inductive limit topology, so that $U \subseteq X$ is open iff $\forall n(U \cap X_n$ is open in $X_n)$. This is Hausdorff, locally compact and second countable, with basis $\bigcup_n \mathcal{B}_n$, where $\mathcal{B}_n$ is a countable basis for X_n. Thus X is a locally compact Polish space, see, e.g., [K95, 5.3].

Let now $\varphi : 4^{\mathbb{N}} \to 4^{\mathbb{N}}$ be the odometer map, i.e., addition by 1 with carry, which is a homeomorphism of $4^{\mathbb{N}}$. Note that $\varphi(X) \subseteq X$ and $\varphi^{-1}(X) \subseteq X$. We next check that $\varphi|X$ is a homeomorphism of X. It enough to check that $\varphi|X_n \colon X_n \to X$ and $\varphi^{-1}|X_n \colon X_n \to X$ are continuous. This follows noticing that $\varphi(X_n) \subseteq X_{n+1}$ and $\varphi^{-1}(X_n) \subseteq X_{n+1}$.

Let E be the equivalence relation on X induced by $\varphi|X$. Denote by E'_0 the equivalence relation on $4^{\mathbb{N}}$ defined by $xE'_0y \iff \exists m \forall n \geq m(x_n = y_n)$. Then $E = E'_0|X$ and $E|X_n = E'_0|X_n$, so $\varphi|X$ is minimal.

Finally E is compressible as witnessed by the following Borel map $f\colon X \to X$: Given $x \in X$, let n be least such that $x \in X_n$. If $x(n) = 1$, let $f(x) = y$ be equal to x, except that $y(n) = 0$. If $x(n) = 2$, let $f(x) = y$ be equal to x, except that $y(n) = 3$. □

Remark 3.12. Here are also some other minimal, locally compact action realizations of a compressible, non-smooth, hyperfinite CBER (but where the acting group is not $\mathbb{Z}$)..

(i) Let X be the locally compact space constructed in the proof of Proposition 3.8, whose notation we use below. For each $q \in \mathbb{Q}$, let $T_q\colon X \to X$ be the homeomorphism which is translation by q on $\mathbb{R}$ and defined on $\{z_n\}$ in a way similar to translation by 1 in the proof of Proposition 3.8. Also define a homeomorphism $T\colon X \to X$ as follows: T is the identity on $\mathbb{R}$. Next let for each $q \in \mathbb{Q}$, $N_q = \{n_0^q < n_1^q < n_2^q < \dots\}$ be the increasing enumeration of N_q and define $T(z_{n^q_{2n+3}}) = z_{n^q_{2n+1}}, T(z_{n^q_1}) = z_{n^q_0}, T(z_{n^q_{2n}}) = z_{n^q_{2n+2}}, n \in \mathbb{N}$.

The group generated by $T, T_q, q \in \mathbb{Q}$ is abelian and acts continuously on X. The orbits consist of $\{z_n\}$ and the sets of the form $x + \mathbb{Q}$ for $x \in \mathbb{R}$, so the action is minimal. Finally there is clearly no invariant measure for this action.

(ii) Another construction, where the acting group is actually $\mathbb{Z}^2$ is the following: Let S be a minimal homeomorphism on an uncountable compact metric space K, inducing the equivalence relation F, and let $X = K \times \mathbb{Z}$. Then let $\mathbb{Z}^2$ act by homeomorphisms on X, where one of the generators acts like S on K and the other as translation by 1 on $\mathbb{Z}$. The associated equivalence relation of this action is Borel isomorphic to $F \times I_N$ so it is compressible, non-smooth and hyperfinite by [K2, 7.27].

Below for a Borel action of a countable group Γ on a standard Borel space X and a probability measure ζ on Γ, we say that a measure μ on X is **ζ-stationary** if $\mu = \int \gamma_* \mu \, \mathrm{d}\zeta(\gamma)$.

It is easy to see that μ is quasi-invariant under the action, i.e., the action sends μ-null sets to μ-null sets. Next we check that if the action has infinite orbits, then μ is non-atomic. Let $x \in X$ be such that $\mu(\{x\}) > 0$, towards a contradiction. Since $\mu(\{x\}) = \int \mu(\gamma^{-1} \cdot \{x\}) \, \mathrm{d}\zeta(\gamma)$, if $\mu(\gamma^{-1} \cdot \{x\}) \leq \mu(\{x\}), \forall \gamma$, then as $\mu(\gamma^{-1} \cdot \{x\}) > 0$, we must have that $\mu(\gamma^{-1} \cdot \{x\}) = \mu(\{x\}), \forall \gamma$, a contradiction. Thus we see that for every $x \in X$ with $\mu(\{x\}) >$

0, there is $x' \in \Gamma \cdot x$, with $\mu(\{x'\}) > \mu(\{x\})$. So we can find $x_0, x_1, x_2, \ldots$ with $\mu(\{x_0\}) < \mu(\{x_1\}) < \mu(\{x_2\}) < \ldots$, a contradiction.

We use these facts and Theorem 3.10 to prove the following:

Proposition 3.13. *Let $E \in \mathcal{AE}$ be an equivalence relation on a standard Borel space X. Then the following are equivalent:*

(i) E is not smooth;

(ii) There is a Borel action of a countable group Γ on X generating E, such that for every *measure ζ on Γ there is a ζ-stationary, ergodic for this action measure on X.*

(iii) There is a Borel action of a countable group Γ on X generating E, such that for some *measure ζ on Γ there is a ζ-stationary, ergodic for this action measure on X.*

Proof. If (iii) holds, then E admits a non-atomic, ergodic, quasi-invariant measure, so it is not smooth. We next prove that (i) implies (ii).

Since E is not smooth, by the Glimm-Effros dichotomy, there is an E-invariant Borel set $Y \subseteq X$ such that $E|Y$ is non-smooth, hyperfinite. Then, by Theorem 3.10, there is a continuous action of $\Gamma = \mathbb{F}_\infty$ on a compact space Z inducing an equivalence relation $F \cong_B E|Y$. Let ζ by any measure on Γ. Then there is a ζ-stationary for this action measure on Z, see, e.g., [CKM13]. The set of ζ-stationary for this action measures is thus a non-empty compact, convex set of measures, so it has an extreme point which is therefore ergodic. Transferring this back to Y and extending the Γ action to X so that it generates $E|(X \setminus Y)$ on $X \setminus Y$, we see that (ii) holds. □

The following question is open:

Problem 3.14. *Does* every *non-smooth $E \in \mathcal{AE}$ have any of the topological realizations stated in Definition 3.6?*

In particular it is unknown whether every non-smooth $E \in \mathcal{AE}$ admits a compact action realization. We will consider this case in the next two sections.

Recall that a **reduction** of an equivalence relation E on X to an equivalence relation F on Y is a map $f \colon X \to Y$ such that $xEy \iff f(x)Ff(y)$. If such a Borel reduction exists, we say that E is **Borel reducible** to F and write $E \leq_B F$. If $E \leq_B F$ and $F \leq_B E$, then E, F are **Borel bireducible**, in symbols $E \sim_B F$. We note here that the following weaker version of Problem 3.14 is also open:

Problem 3.15. *Is* every *non-smooth $E \in \mathcal{AE}$ Borel bireducible to some $F \in \mathcal{AE}$ which has any of the topological realizations stated in Definition 3.6?*

3.C Compact action realizations

We have seen in Theorem 3.10 that the answer to Problem 3.14 is affirmative in the strongest sense for hyperfinite E but the situation for general E is unclear. The following results provide some cases of non-hyperfinite equivalence relations that admit compact action realizations.

For each infinite countable group Γ, let $F(\Gamma, 2^{\mathbb{N}})$ be the equivalence relation induced by the shift action of Γ on $(2^{\mathbb{N}})^{\Gamma}$ restricted to its free part X. Every equivalence relation induced by a free Borel action of Γ is Borel isomorphic to the restriction of $F(\Gamma, 2^{\mathbb{N}})$ on an invariant Borel set. We now have:

Theorem 3.16. *For every infinite countable group Γ, $F(\Gamma, 2^{\mathbb{N}})$ has a compact action realization on the Cantor space $2^{\mathbb{N}}$.*

Proof. By a result of Elek [E18], $F(\Gamma, 2^{\mathbb{N}})$ is Borel isomorphic to $F(\Gamma, 2^{\mathbb{N}})|Y$, where $Y \subseteq X$ is invariant under the shift and moreover, if $\overline{Y}$ is the closure of Y in $(2^{\mathbb{N}})^{\Gamma}$, then $\overline{Y} \subseteq X$. It follows that $F(\Gamma, 2^{\mathbb{N}}) \cong_B F(\Gamma, 2^{\mathbb{N}})|\overline{Y}$ and the latter is induced by a continuous action of Γ on the compact space $K = \overline{Y}$. As in the proof of Proposition 3.5 we can replace $\overline{Y}$ by its perfect kernel, which is homeomorphic to the Cantor space. □

We next note the following fact, which can be used to provide more examples of CBER that admit compact action realizations.

Proposition 3.17. *Let F be an aperiodic* CBER *on a standard Borel space X. Let $Z \subseteq X$ be a Borel invariant set and put $Y = X \setminus Z$ and $E = F|Y$. If E is not smooth and $F|Z$ is hyperfinite, compressible, then $E \cong_B F$. So if F has a compact action realization, so does E.*

Proof. If $F|Z$ is smooth, then $F|Z \cong_B \mathbb{R}I_{\mathbb{N}}$ is Borel isomorphic to a direct sum of copies of $I_{\mathbb{N}}$, while if it is not smooth $F|Z \cong_B E_t$. Thus, by the Glimm-Effros Dichotomy and [K2, 7.3 and 2.27] in the second case, we can find a decomposition $Y = Y_0 \sqcup Y_1 \sqcup Y_2 \sqcup \cdots \sqcup Y_\infty$ into invariant Borel sets such that $F|Z \cong_B F|Y_n, \forall n \in \mathbb{N}$. Let π_0 be a Borel isomorphism of $F|Z$ with $F|Y_0$ and for $n > 0$, let π_n be a Borel isomorphism of $F|Y_{n-1}$ with $F|Y_n$. Finally let π_∞ be the identity on Y_∞. Then $\bigcup_{n\in\mathbb{N}} \pi_n \cup \pi_\infty$ is a Borel isomorphism of F and E. □

Corollary 3.18. *Let F be an aperiodic* CBER *on a Polish space X. Then there is meager, invariant Borel set $M \subseteq X$ such that for any invariant Borel set $Y \supseteq M$, if $E = F|Y$ is not smooth, then $E \cong_B F$.*

Proof. By [KM04, 12.1 and 13,3], there is an invariant, comeager Borel set $C \subseteq X$ such that $F|C$ is compressible, hyperfinite. Put $M = X \setminus C$. If $Y \supseteq M$ is invariant Borel such that $E = F|Y$ is not smooth and $Z = X \setminus Y$, then we can apply Proposition 3.17. □

For example, let Γ be a countable group and consider a continuous, topologically transitive action of Γ on a compact Polish space X with infinite orbits. Then there is an invariant dense G_δ set $C \subseteq X$ consisting of points with dense orbits in X and such that if F is the equivalence relation induced by the action, then $F|C$ is compressible, hyperfinite and non-smooth (as the action of Γ on C is topologically transitive). So $F|C \cong_B E_t \cong_B \mathbb{R}E_t$. Then by the countable chain condition for category, some copy of E_t in $E|C$ is meager, so can subtract it from C and assume that if $M = X \setminus C$, then $F|M$ is not smooth. It follows that for any invariant Borel set $Y \supseteq M$, if $E = F|Y$, then $E \cong_B F$, so that E has a compact action realization

Since for every $E \in \mathcal{AE}$ on a Polish space X there is an invariant comeager Borel set $Y \subseteq X$ such that $E|Y$ is hyperfinite, it follows that if $E \in \mathcal{AE}$ is not smooth when restricted to any invariant comeager Borel set, then there is an invariant comeager Borel set $Y \subseteq X$ such that $E|Y$ admits a minimal, compact action realization. Whether this holds for measure instead of category is an open problem.

Problem 3.19. *Let $E \in \mathcal{AE}$ be on a standard Borel space X and let μ be a measure on X such that the restriction of E to any invariant Borel set of measure 1 is not smooth. Is there is an invariant Borel set $Y \subseteq X$ with $\mu(Y) = 1$ such that $E|Y$ admits a compact action realization?*

We next describe a "gluing" construction of two continuous actions of groups on compact Polish spaces at an orbit of one of the actions. We thank Aristotelis Panagiotopoulos for a useful discussion on this construction.

Let the countable group Γ act continuously on the compact Polish space X and let $X_0 \subseteq X$ be an infinite orbit of this action. Let also the countable group Δ act continuously on the compact Polish space Y with a fixed point $y_0 \in Y$. Fix compatible metrics $d_X \leq 1$ and $d_Y \leq 1$ for X and Y, respectively. Fix also a map $x \mapsto |x|$ from X_0 to $\mathbb{R}^+$ such that $\lim_{x\to\infty} |x| = +\infty$, i.e., for every $M \in \mathbb{R}^+$, there if finite $F \subseteq X_0$ such that $x \notin F \implies |x| > M$. For each $x \in X_0$, let Y_x be a set and let π_x be a bijection $\pi_x : Y \to Y_x$ such that $\pi_x(y_0) = x$ and $x_1 \neq x_2 \implies Y_{x_1} \cap Y_{x_2} = \emptyset$. Put $Y'_x = Y_x \setminus \{x\}$ and let $Z = X \sqcup \bigsqcup_{x \in X_0} Y'_x$. Define a metric d_x on Y_x as follows:

$$d_x(y_1, y_2) = \frac{d_Y(\pi_x^{-1}(y_1), \pi_x^{-1}(y_2))}{|x|}.$$

Then define a metric d_Z on Z as follows:

$$d_Z(x_1, x_2) = d_X(x_1, x_2), \text{if } x_1, x_2 \in X,$$

$$d_Z(y_1, y_2) = d_x(y_1, y_2), \text{if } y_1, y_2 \in Y_x, x \in X_0,$$

$$d_Z(y, x') = d_x(y, x) + d_X(x, x'), \text{if } y \in Y_x, x \in X_0, x' \in X,$$

$$d_Z(y_1, y_2) = d_{x_1}(y_1, x_1) + d_X(x_1, x_2) + d_{x_2}(x_2, y_2), \text{if } y_1 \in Y_{x_1}, y_2 \in Y_{x_2}, x_1 \neq x_2.$$

Remark 3.20. We note here that in the preceding "gluing" construction, if the spaces X, Y are 0-dimensional, so is the space Z. To see this we start with metrics d_X, d_Y as above which are actually ultrametrics (these exist since X, Y are 0-dimensional). Then it is enough to show that for every $z \in Z$, there is an $\varepsilon_z > 0$ such that every open ball (in the metric d_Z) $B_z(\varepsilon)$, for $\varepsilon < \varepsilon_z$, is closed. Below recall that open balls in ultrametrics are closed.

Consider first the case where $z \in X$ and fix $z_1, z_2, \dots \in B_z(\varepsilon)$ with $z_n \to z_\infty$. If infinitely many z_n are in X, then clearly $z_\infty \in B_z(\varepsilon)$ as d_X is an ultrametric. Otherwise, we can assume that all z_n are in $Z \setminus X$. If now there is some $x \in X_0$ such that infinitely many $z_n \in Y'_x$, so that $z_\infty \in Y_x$, we have $d_Z(z_n, z) = d_x(z_n, x) + d_X(x, z)$, so $d_x(z_n, x) < \varepsilon - d_X(x, z)$, thus, since d_x is an ultrametric, $d_x(z_\infty, x) < \varepsilon - d_X(x, z)$ and thus $d_Z(z_\infty, z) < \varepsilon$. Otherwise there is a subsequence (z_{n_i}) and $x_i \in X_0$ with $z_{n_i} \in Y'_{x_i}$ and x_i converges to $x \in X$ and thus $z_{n_i} \to z_\infty = x$ (since $d_Z(z_{n_i}, x_i) < \frac{1}{|x_i|}$). Now $d_Z(z, z_{n_i}) = d_X(z, x_i) + d_{x_i}(x_i, z_{n_i}) < \varepsilon$, so $d_X(z, x_i) < \varepsilon$ and, since d_X is an ultrametric, $d_Z(z, z_\infty) = d_X(z, z_\infty) < \varepsilon$.

The other case is when $z \in Y'_x$, for some $x \in X_0$. Take $\varepsilon_z = d_x(z, x)$. Then for $\varepsilon < \varepsilon_z$ the open ball $B_z(\varepsilon)$ is the same as the open ball of radius ε in the metric d_x, so the proof is complete.

Proposition 3.21. *(Z, d_Z) is a compact metric space.*

Proof. It is routine to check that d_Z is a metric on Z. We next verify compactness. Let (z_n) be a sequence in Z in order to find a converging subsequence. The other cases been obvious, we can assume that $z_n \in Y_{x_n}$ with $x_n \in X_0$ distinct and therefore $|x_n| \to \infty$, in which case, by going to a subsequence, we can also assume that $x_n \to x \in X$. Since $d_Z(z_n, x_n) \leq \frac{1}{|x_n|}$, it follows that $z_n \to x$. □

We next define an action of Δ on Z. Given $\delta \in \Delta$ and $z \in Z$ we define $\delta \cdot z$ as follows:

$$\delta \cdot z = \pi_x(\delta \cdot \pi_x^{-1}(z)), \text{if } z \in Y_x, x \in X_0,$$
$$\delta \cdot z = z, \text{if } z \in X.$$

If we identify each Y_x with Y, then this action "extends" the action of Δ on Y.

We finally extend the action of Γ from X to all of Z. Given $\gamma \in \Gamma$ and $z \in Z$ define $\gamma \cdot z$ as follows:

$$\gamma \cdot z = z, \text{if } z \in X,$$

$$\gamma \cdot z = \pi_{\gamma \cdot x}(\pi_x^{-1}(z)), \text{if } z \in Y_x, x \in X_0.$$

It is easy to see that these two actions commute, so they give an action of $\Gamma \times \Delta$ on Z.

Proposition 3.22. *The action of* $\Gamma \times \Delta$ *on* Z *is continuous.*

Proof. It is enough to check that the action of Γ on Z is continuous and so is the action of Δ.

Let first $\gamma \in \Gamma$ and $z_n \in Z$ be such that $z_n \to z$, in order to show that $\gamma \cdot z_n \to \gamma \cdot z$. It is enough to find a subsequence (n_i) such that $\gamma \cdot z_{n_i} \to \gamma \cdot z$. Again, the other cases being trivial, we can assume that $z_n \in Y_{x_n}$ with $x_n \in X_0$ distinct, so that also $|x_n| \to \infty$, in which case, by going to a subsequence, we can also assume that $x_n \to x \in X$. Then $\gamma \cdot x_n \to \gamma \cdot x$ and $d_Z(\gamma \cdot z_n, \gamma \cdot x_n) \leq \frac{1}{|\gamma \cdot x_n|} \to 0$, as the $\gamma \cdot x_n$ are also distinct and thus $|\gamma \cdot x_n| \to \infty$. Since $d_Z(z_n, x_n) \leq \frac{1}{|x_n|}$, clearly $x = z$, and thus $\gamma \cdot z_n \to \gamma \cdot z$.

Let now $\delta \in \Delta$ and $z_n \in Z$ be such that $z_n \to z$, in order to show that $\delta \cdot z_n \to \delta \cdot z$. It is enough again to find a subsequence (n_i) such that $\delta \cdot z_{n_i} \to \delta \cdot z$ and as before we can assume that $z_n \in Y_{x_n}$ with $x_n \in X_0$ distinct, so that also $|x_n| \to \infty$, in which case, by going to a subsequence, we can also assume that $x_n \to x \in X$. Then $\delta \cdot x_n = x_n \to \delta \cdot x = x$. Now $\delta \cdot z_n \in Y_{x_n}$, so that $\delta_Z(\delta \cdot z_n, x_n) \to 0$ and $d_Z(z_n, x_n) \to 0$. Thus $z = x$ and $\delta \cdot z_n \to \delta \cdot z = z$. $\square$

Let now E be the equivalence relation induced by the action of Γ on X, let F be the equivalence relation induced by the action of Δ on $Y \setminus \{y_0\}$ and finally let G be the equivalence relation induced by the action of $\Gamma \times \Delta$ on Z. Then it is easy to check the following;

Proposition 3.23. $G \cong_B E \oplus (F \times I_{\mathbb{N}})$

We present now an application of this construction to the problem of realization by compact group actions.

Theorem 3.24. *Let the* CBER F *be induced by a continuous action of a countable group on a locally compact Polish space. Then* $F \times I_{\mathbb{N}}$ *admits a compact action realization. In particular, if* F *is compressible,* F *admits a compact action realization.*

Moreover, if the locally compact space is 0-dimensional, $F \times I_{\mathbb{N}}$ *admits a compact action realization on the Cantor space* $2^{\mathbb{N}}$.

Proof. In the preceding "gluing" construction, take $X = 2^{\mathbb{N}}$ and a continuous action of $\Gamma = \mathbb{F}_2$ such that $E = E_t$. Fix also a countable group Δ and a continuous action of Δ on a locally compact space Y' which induces F. Let $Y = Y' \sqcup \{y_0\}$ be the one-point compactification of Y' (if Y' is already compact, we obtain Y by adding an isolated point to Y'). Then the action of Δ can be continuously extended to Y by fixing y_0. Thus we have by Proposition 3.23 that $E \oplus (F \times I_{\mathbb{N}})$ admits a compact action realization. Since F is not smooth, we have, as in the proof of Proposition 3.17, that $E \oplus (F \times I_{\mathbb{N}}) \cong_B (F \times I_{\mathbb{N}})$ and the proof is complete.

In the case that Y' is 0-dimensional, by Remark 3.20 $F \times I_{\mathbb{N}}$ admits a compact action realization on a 0-dimensional space Z. By going to the perfect kernel of Z, we can assume that Z is perfect (see the proof of Proposition 3.5), thus homeomorphic to the Cantor space. $\square$

Recall that a CBER E is **universal** if for every CBER F we have $F \leq_B E$. The following is a very interesting consequence of Theorem 3.24:

Corollary 3.25. *Let* E *be a compressible, universal* CBER. *Then* E *admits a transitive, compact action realization on the Cantor space* $2^{\mathbb{N}}$.

Proof. Let us first note that there exists a compressible, universal CBER F that is generated by a continuous action of a countable group on $2^{\mathbb{N}}$. Indeed, let $E(\mathbb{F}_2, 2)$ be the equivalence relation generated by the canonical action of $\mathbb{F}_2$ on $2^{\mathbb{F}_2}$. Consider the equivalence relation $F = E(\mathbb{F}_2, 2) \times I_{\mathbb{N}}$. This equivalence relation is compressible, universal. By Theorem 3.24, F has a continuous action realization on $2^{\mathbb{N}}$. An inspection of the "gluing" construction involved in the proof of Theorem 3.24 shows that this action is topologically transitive.

Let now E be a compressible, universal CBER and F be as above. Then, $E \sim_B F$. By [CK18, Proposition 3.27 (ii)] it follows that $E \cong_B F$. □

Corollary 3.26. *Let E be a compressible, universal* CBER*. Then E admits a minimal action realization on the Baire space $\mathcal{N}$.*

Proof. By Corollary 3.25 consider a continuous action a of a countable group G on $2^{\mathbb{N}}$, which induces an equivalence relation F which is Borel isomorphic to E. Then consider the Borel map f that sends $x \in 2^{\mathbb{N}}$ to the closure of its orbit (which is a member of the space of all compact subsets of $2^{\mathbb{N}}$), By [MSS16, Theorem 3.1], there is some K such that $F|f^{-1}(K)$ is universal. But clearly $Z = f^{-1}(K)$ is a G_δ set, so a Polish, 0-dimensional space, invariant under the action a. Moreover this action restricted to Z is minimal. As in the proof of Proposition 3.5, we can find a subspace Y of Z homeomorphic to $\mathcal{N}$ invariant under the action, such that $F|Z \cong_B F|Y$. Thus $F|Z$ is induced by a minimal action on the Baire space and is compressible, universal. As in the proof of Corollary 3.25, this shows that every compressible, universal CBER admits a minimal action realization on the Baire space. □

The following is an open problem:

Problem 3.27. *Does an arbitrary (not necessarily compressible) aperiodic, universal* CBER *admit a compact action realization; a transitive, compact action realization; a minimal, compact action realization on the Cantor space? We also do not know if every compressible, universal* CBER *admits a minimal, compact action realization on the Cantor space.*

We do not even know if *some* aperiodic, universal CBER admits a minimal, compact action realization on the Cantor space? It turns out that this is equivalent to the following. Consider the shift action of $\mathbb{F}_\infty$ on $(2^{\mathbb{N}})^{\mathbb{F}_\infty}$. A point in this space is **minimal** or **almost periodic** if the closure of its orbit is a minimal subshift. Let M be the set of minimal points. This is a Borel set (see, e.g., [GJS16, Lemma 2.4.5]). Denote by E the equivalence relation induced by the shift action on $(2^{\mathbb{N}})^{\mathbb{F}_\infty}$, which is a universal CBER.

Proposition 3.28. *The following are equivalent:*

i) Some aperiodic, universal CBER *admits a minimal, compact action realization on the Cantor space.*

ii) The equivalence relation $E|M$ is universal.

Proof. That i) implies ii) is a consequence of the standard fact that every continuous action of $\mathbb{F}_\infty$ on a 0-dimensional compact Polish space can be topologically embedded in the shift action of $\mathbb{F}_\infty$ on $(2^{\mathbb{N}})^{\mathbb{F}_\infty}$, see, e.g., [GJS16, Lemma 2.7.2].

To see that ii) implies i), assume that $E|M$ is universal and consider the Borel map f that sends $x \in M$ to the closure of its orbit (which a member of the space of all compact subsets of $(2^{\mathbb{N}})^{\mathbb{F}_\infty}$). By [MSS16, Theorem 3.1], there is some K such that $E|f^{-1}(K)$ is universal. But clearly $f^{-1}(K) = K$ and the shift action restricted to K is minimal. By going to the perfect kernel of K, we find an aperiodic, universal CBER induced by a minimal action of a countable group on the Cantor space. □

For a sequence (E_n) of CBER, we let $\bigoplus_n E_n$ be the direct sum of this sequence. If E_n is on the space X_n, then $E = \bigoplus_n E_n$ is the equivalence relation on the space $\bigsqcup_n X_n$, where $xEy \iff \exists n(x, y \in X_n \text{ and } xE_ny)$.

Corollary 3.29. *Let each $E_n \in \mathcal{AE}$ admit a compact action realization. Then $\bigoplus_n E_n \times I_{\mathbb{N}}$ also admits a compact action realization. In particular, if also every E_n is compressible, $\bigoplus_n E_n$ admits a compact action realization.*

3.D Turing and arithmetical equivalence

Below let $\equiv_T$ denote **Turing equivalence** and $\equiv_A$ **arithmetical equivalence**.

The following is an immediate consequence of Corollary 3.25, since $\equiv_A$ is compressible and universal by [MSS16]:

Corollary 3.30. *Arithmetical equivalence $\equiv_A$ on $2^{\mathbb{N}}$ admits a compact action realization on $2^{\mathbb{N}}$.*

On the other hand the following is open:

Problem 3.31. *Does Turing equivalence $\equiv_T$ on $2^{\mathbb{N}}$ admit a compact action realization?*

A negative answer to this question will on the one hand provide a new proof of the non-hyperfiniteness of $\equiv_T$ and, more importantly, give a negative answer to the long standing problem of the universality of $\equiv_T$, see [DK].

Concerning Turing equivalence, we know from the general Proposition 3.5 that it admits a continuous action realization on the Baire space $\mathcal{N}$, i.e.,that there is a Borel isomorphism of $2^{\mathbb{N}}$ with $\mathcal{N}$ which sends $\equiv_T$ to an equivalence relation induced by a continuous action of a countable group on $\mathcal{N}$. We calculate below an upper bound for the Baire class of such a Borel isomorphism. A version of the next theorem was first proved by Andrew Marks, in response to an inquiry of the authors, with "Baire class 3" instead of "Baire class 2". The proof of Theorem 3.32 below uses some of his ideas along with other additional arguments.

Theorem 3.32. *There exists a Baire class 2 map $\Phi\colon 2^{\mathbb{N}} \to \mathcal{N}$ that is an isomorphism between $\equiv_T$ and an equivalence relation given by a continuous group action on $\mathcal{N}$.*

The most natural construction of the isomorphism will yield Proposition 3.33 below. We will show later that it in fact implies Theorem 3.32.

Proposition 3.33. *There exists a Baire class 2 map Ψ that is an isomorphism between $\equiv_T$ on $2^{\mathbb{N}}$ and an equivalence relation given by a continuous group action on a 0-dimensional Polish space.*

Proof. Let φ^i denote the partial function computed by the ith Turing machine, in some recursive enumeration of all the Turing machines, such that φ^0 is the identity on $2^{\mathbb{N}}$. That is, we consider Turing machines with oracle and input tapes, and $\varphi^i(x) = y$ iff for each n the ith Turing machine with oracle x and input n halts with the output $y(n)$.

We start with an easy observation. Below, for $s \in 2^{<\mathbb{N}}$, put $[s] = \{x \in 2^{\mathbb{N}}\colon s \subseteq x\}$.

Lemma 3.34. *Assume that $x \equiv_T y$. There exists an i with $\varphi^i(x) = y$ and $\varphi^i(y) = x$.*

Proof. We can assume that $x \neq y$. Pick $j, k \in \mathbb{N}$ with $\varphi^j(x) = y$ and $\varphi^k(y) = x$, and n with $x \restriction n \neq y \restriction n$. Then, an i with $\varphi^i \restriction [x \restriction n] = \varphi^j \restriction [x \restriction n]$ and $\varphi^i \restriction [y \restriction n] = \varphi^k \restriction [y \restriction n]$ clearly works. □

The idea is to define a coding function $\Psi = (\alpha, \beta, \gamma)$ that will serve as an isomorphism. The crucial property of $\Psi(x)$ is that α encodes for each i whether φ^i is an involution on x (and does this for every $y \equiv_T x$), β will ensure that Ψ is continuous, while γ will be used to code x and its $\equiv_T$ equivalence class.

Let us now give the precise definitions. Fix a function $\iota : \mathbb{N}^2 \to \mathbb{N}$ such that for each $i, j \in \mathbb{N}$ we have $\varphi^{\iota(i,j)} = \varphi^i \circ \varphi^j$.

Let $\beta : 2^{\mathbb{N}} \to (\mathbb{N} \cup \{*\})^{\mathbb{N}^3}$ be defined by $\beta(x)(i, j, m) = n$, if n is least such that both the ith and the jth Turing machines with oracle x and input m halt with the same output in at most n steps, and let $\beta(x)(i, j, m) = *$, if such an n does not exist.

Define a map $\alpha : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}^2}$ by letting

$$\alpha(x)(i, j) = 0 \iff \beta(x)(i, j) \in \mathbb{N}^{\mathbb{N}},$$

and

$$\alpha(x)(i, j) = m + 1 \iff m \text{ is least with } \beta(x)(i, j, m) = *.$$

Let $\gamma : 2^{\mathbb{N}} \to (2 \cup \{*\})^{\mathbb{N}^2}$ be defined by

$$\gamma(x)(i, k) = *, \text{ if } \alpha(x)(\iota(i, i), 0) \neq 0,$$

and

$$\gamma(x)(i, k) = \varphi^i(x)(k),$$

otherwise.

Note that by the choice of φ^0 for each $x \in 2^{\mathbb{N}}$ we have that $\gamma(x)(0) = x$.

Finally, let $\Psi(x) = (\alpha(x), \beta(x), \gamma(x))$. Let us denote the space $\mathbb{N}^{\mathbb{N}^2} \times (\mathbb{N} \cup \{*\})^{\mathbb{N}^3} \times (2 \cup \{*\})^{\mathbb{N}^2}$ by X, where $\mathbb{N} \cup \{*\}$ and $2 \cup \{*\}$ are endowed with the discrete topology.

Lemma 3.35. $\Psi(2^{\mathbb{N}})$ *is closed in* X.

Proof. Assume that $(\alpha(x_k), \beta(x_k), \gamma(x_k))_k$ is a convergent sequence. It follows from the choice of φ^0, the definition of γ, and $\gamma(x_k)(0) \to \gamma(x)(0)$ that $x_k \to x$ holds. Take any $i, j, m \in \mathbb{N}$. It is clear from the definition of β that $\beta(x)(i, j, m) = n$ holds for some $n \in \mathbb{N}$ if and only if $\beta(x_k)(i, j, m) = n$ is true for every large enough k. This shows that $\beta(x_k) \to \beta(x)$.

Using this, it is easy to check that $\alpha(x_k) \to \alpha(x)$ holds as well.

Finally, by $\alpha(x_k) \to \alpha(x)$, we have that $\alpha(x_k)(i, j) = \alpha(x)(i, j)$ for each large enough k. This of course implies $\gamma(x_k) \to \gamma(x)$ by the continuity of the functions φ^i. $\square$

Since φ^0 is the identity and by the definition of γ, we have that $\gamma(x)(0) = x$ for every x. In particular, Ψ is injective.

For $i \in \mathbb{N}$ define a map δ_i from X to itself as follows:

$$\delta_i(\Psi(x)) = \Psi(x) \text{ if } \alpha(x)(\iota(i,i),0) \neq 0,$$

otherwise

$$\delta_i(\Psi(x)) = \Psi(\varphi^i(x)).$$

Lemma 3.36. *The maps $(\delta_i)_{i\in\mathbb{N}}$ are $\Psi(2^{\mathbb{N}}) \to \Psi(2^{\mathbb{N}})$ homeomorphisms.*

Proof. Fix $i \in \mathbb{N}$. It is easy to check that on the set $\{\Psi(x) : \alpha(x)(\iota(i,i),0) = 0\}$ for each $i', j', m \in \mathbb{N}$ we have that:

$$\delta_i(\alpha(x), \beta(x), \gamma(x))(0)(i', j') = (\alpha(x))(\iota(i', i), \iota(j', i)),$$

$$\delta_i(\alpha(x), \beta(x), \gamma(x))(1)(i', j', m) = (\beta(x))(\iota(i', i), \iota(j', i), m),$$

and

$$\delta_i(\alpha(x), \beta(x), \gamma(x))(2)(i', m) = (\gamma(x))(i, m), \text{ if } \alpha(\iota(\iota(i', i'), i), i) \neq 0,$$

while

$$\delta_i(\alpha(x), \beta(x), \gamma(x))(2)(i', m) = (\gamma(x))(\iota(i', i), m), \text{ otherwise.}$$

As δ_i is equal to identity on a relatively clopen set, while it selects and permutes some of the coordinates on its complement, it follows that δ_i is continuous.

Finally, we show that $\delta_i(\delta_i(\Psi(x))) = \Psi(x)$ holds for each x. Indeed, the set $\{x : \alpha(x)(\iota(i,i),0) = 0\}$ is the collection of binary sequences on which φ^i is an involution, so it follows from the definition of δ_i that on the Ψ image of this set our lemma holds. Moreover, on the complement of this set, δ_i is the identity, which finishes the proof of the lemma. □

Let E_Δ be the equivalence relation on $\Psi(2^\omega)$ generated by the maps $\{\delta_i : i \in \mathbb{N}\}$.

Lemma 3.37. *Ψ is an isomorphism between $\equiv_T$ and E_Δ.*

Proof. First, it is clear from the definition of δ_i that $\delta_i(\Psi(x)) = \Psi(y)$ implies that $x \equiv_T y$. So Ψ^{-1} is a homomorphism.

Second, assume that $x \equiv_T y$. Then by Lemma 3.34 there exists an i with $\varphi^i(x) = y$ and $\varphi^i(y) = x$. Then $\alpha(x)(\iota(i,i),0) = 0$, so $\delta_i(\Psi(x)) = \Psi(\varphi^i(x)) = \Psi(y)$, so $\Psi(x) E_\Delta \Psi(y)$. □

Now we turn to the calculation of the complexity of the map Ψ.

Lemma 3.38. *The map β is Baire class 1 and the maps α and γ are Baire class 2. Consequently, the map Ψ is Baire class 2.*

Proof. For β, take any $i, j, m \in \mathbb{N}$. Then for each natural number n, the set $\{x : \beta(x)(i, j, m) = n\}$ is open. Thus, the set $\{x : \beta(x)(i, j, m) = *\}$ is closed. This shows that β preimages of basic clopen sets are $\mathbf{\Delta}^0_2$.

For α, for a given i, the set $\{x : \alpha(x)(i, j) = 0\} = \{x : \forall m(\beta(x)(i, j, m) \in \omega)\}$ is $\mathbf{\Pi}^0_2$, and also, for $m \neq 0$ we have that $\{x : \alpha(x)(i, j) = m\} = \{x : \forall m' < m \ (\beta(x)(i, j, m') \in \mathbb{N} \wedge \beta(x)(i, j, m) = *)\}$, which shows that these sets are $\mathbf{\Pi}^0_2$ as well, and thus α is indeed Baire class 2. Finally, just note that γ depends continuously on α, so it must be Baire class 2. □

This completes the proof of Proposition 3.33 □

In order to finish the proof of Theorem 3.32 we need a last observation.

Lemma 3.39. *Assume that Γ acts continuously on the uncountable zero-dimensional Polish space X, so that the induced equivalence relation E_Γ is aperiodic. Then there exist invariant under the action $X' \subseteq X$ that is homeomorphic to $\mathcal{N}$, and an isomorphism φ between E_Γ and $E_\Gamma \restriction X'$ that moves only countably many points.*

Proof. As in the proof of Proposition 3.5. □

Proof of Theorem 3.32. By Proposition 3.33 there exists a Baire class 2 isomorphism between $\equiv_T$ and some equivalence relation of the form E_Γ, where Γ acts continuously on a zero-dimensional Polish space. Applying Lemma 3.39 we get an isomorphism with an equivalence relation on the Baire space. Moreover, as countable modifications of Baire class 2 functions do not change their class, we are done. □

We do not know if the complexity of the Borel isomorphism in Theorem 3.32 is optimal.

Problem 3.40. *Is there a Baire class 1 map that is an isomorphism between $\equiv_T$ and an equivalence relation given by a continuous group action on $\mathcal{N}$?*

Consider now *any* map $\Phi\colon 2^{\mathbb{N}} \to \mathcal{N}$ satisfying the conditions of Theorem 3.32. Then for some $p \in \mathcal{N}$ we have $x \equiv_T y \implies \langle \Phi(x), p\rangle \equiv_T \langle \Phi(y), p\rangle$. Thus if Martin's Conjecture is true, we have that on a cone $\langle \Phi(x), p\rangle$ is Turing equivalent to one of x, x', x'' and thus the same is true for $\Phi(x)$. For the *particular* Φ that was constructed in the proof of Theorem 3.32, it is easy to see that $\Phi(x) \equiv_T x''$, since x'' can be easily computed from the map α defined in the proof of Proposition 3.33. The next result shows that no such Φ can satisfy $\Phi(x) \equiv_T x$ on a cone but similarly to Problem 3.40, we do not know if Φ can be found such that $\Phi(x) \equiv_T x'$ on a cone. Below for $x, y \in \mathcal{N}$, $x \leq_T y$ iff x is recursive in y.

Proposition 3.41. *There is no Borel map $\Phi\colon 2^{\mathbb{N}} \to \mathcal{N}$ that is an isomorphism between $\equiv_T$ and an equivalence relation given by a continuous group action on $\mathcal{N}$ such that $\Phi(x) \leq_T x$ on a cone.*

Proof. Recall that a **pointed perfect tree** is a perfect binary tree $S \subseteq 2^{<\mathbb{N}}$ such that $x \in [S] \implies S \leq_T x$, where $[S] \subseteq 2^{\mathbb{N}}$ is the set of infinite branches of S. Below we will use certain properties of pointed perfect trees due to Martin, whose proofs can be found, for example, in [K21]. Assume that $\Phi(x) \leq_T x$ on a cone, towards a contradiction. Then by [K21, Theorem 1.3] there is a perfect pointed tree T such that $x \in [T] \implies \Phi(x) \leq_T x$. Then by [K21, Lemma 1.4], there is a perfect pointed subtree $S \subseteq T$ and $i \in \mathbb{N}$ such that if $x \in [S]$, then $\varphi^i(x)$ is defined and $\varphi^i(x) = \Phi(x)$. Thus Φ is continuous on $[S]$. It follows that $\equiv_T$ restricted to $[S]$ is $\boldsymbol{\Sigma^0_2}$. Let now Ψ be the canonical homeomorphism of $2^{\mathbb{N}}$ with $[S]$, so that $\Psi(x) \equiv_T x$, if $S \leq_T x$. It follows that for $S \leq_T x, y$, we have $x \equiv_T y \iff \Psi(x) \equiv_T \Psi(y)$, thus, in particular, for some $z \in 2^{\mathbb{N}}$, the Turing degree of z, i.e., the set $\{w \in 2^{\mathbb{N}} \colon w \equiv_T z\}$ is $\Sigma^0_2(z)$. This is false in view of the following well-known fact:

Lemma 3.42. *For any $z \in 2^{\mathbb{N}}$, the Turing degree of z is in $\Sigma^0_3(z)$ but not in $\Pi^0_3(z)$.*

Proof. It is easy to check that the Turing degree of z is $\Sigma^0_3(z)$. Assume now that it is in $\Pi^0_3(z)$, towards a contradiction. Then if $A = \{w \in 2^{\mathbb{N}} \colon w \leq_T z\}$, we have that A is also $\Pi^0_3(z)$, since $w \in A \iff \langle w, z\rangle \equiv_T z$. But then $2^{\mathbb{N}} \setminus A$ is a comeager $\Sigma^0_3(z)$ set, so by the relativized version of the basis theorem of Shoenfield [S58] it contains a recursive in z real, a contradiction. □

□

3.E Continuous actions on compact spaces, compressibility and paradoxicality

(1) In connection with Problem 3.14 we discuss some special properties of continuous actions of countable groups on compact Polish spaces that may have some relevance to this question.

Let Γ be a countable group and let a be a Borel action of Γ on a standard Borel space X. Put $\gamma \cdot x = a(\gamma, x)$. We denote by $\langle a \rangle$ the set of all Borel maps $T\colon X \to X$ such that $\forall x \exists \gamma \in \Gamma (T(x) = \gamma \cdot x)$. Equivalently this means that there is a Borel partition $X = \bigsqcup_{\gamma \in \Gamma} X_\gamma$ such that $T(x) = \gamma \cdot x$ for $x \in X_\gamma$. We also let $\langle a \rangle^f$ consist of all Borel maps $T\colon X \to X$ for which there is a *finite* subset $F \subseteq \Gamma$ such that $\forall x \exists \gamma \in F (T(x) = \gamma \cdot x)$. Equivalently this means that there is a Borel partition $X = \bigsqcup_{\gamma \in F} X_\gamma$ such that $T(x) = \gamma \cdot x$ for $x \in X_\gamma$.

We say that the action a is **compressible** (resp., **finitely compressible**) if there is an injective Borel map in $T \in \langle a \rangle$ (resp., $T \in \langle a \rangle^f$) such that for every orbit C of a, $T(C) \subsetneqq C$ or equivalently $\Gamma \cdot (X \setminus T(X)) = X$. Clearly the action a is compressible iff the associated equivalence relation is compressible. The action a is **paradoxical** (resp., **finitely paradoxical**) if there are two injective Borel maps T_1, T_2 in $\langle a \rangle$ (resp., in $\langle a \rangle^f$) such that $T_1(X) \cap T_2(X) = \emptyset, T_1(X) \cup T_2(X) = X$.

Clearly if a is paradoxical (resp., finitely paradoxical), then a is compressible (resp., finitely compressible). It is also known that if a is compressible, then a is paradoxical; see, e.g., [K2, 2.23].

Remark 3.43. It is easy to see that finite compressibility does not imply imply finite paradoxicality. Take for example $\mathbb{Z}$ acting on itself by translation. Since $\mathbb{Z}$ is amenable this action is not finitely paradoxical. On the other hand the map $T\colon \mathbb{Z} \to \mathbb{Z}$ such that $T(n) = n$, if $n < 0$, and $T(n) = n + 1$, if $n \geq 0$, shows that this action is finitely compressible.

Remark 3.44. One can easily see that finite paradoxicality is equivalent to the following strengthening of finite compressibility: There is an injective Borel map $T \in \langle a \rangle^f$ and a finite subset $F \subseteq \Gamma$ such that $F \cdot (X \setminus T(X)) = X$.

For $n \geq 1$, let $[n] = \{1, 2, \dots, n\}$. The ***n*-amplification** of a is the action a_n of the group $\Gamma \times S_n$ on $X \times [n]$ given by $(\gamma, \pi) \cdot (x, i) = (\gamma \cdot x, \pi(i))$, where S_n is the group of permutations of $[n]$. An **amplification** of a is an n-amplification of a, for some n.

Theorem 3.45. *Let a be a continuous action of a countable group Γ on a compact Polish space X. Then the following are equivalent:*

(i) a is compressible;

(ii) a is paradoxical;

(iii) an amplification of a is finitely compressible;

(iv) an amplification of a is finitely paradoxical.

Proof. (A) The proof will be based on Nadkarni's Theorem and the following two results. We use the following terminology:

Let X be a standard Borel space and let $B(X)$ be the σ-algebra of its Borel sets. A **finitely additive Borel probability measure** is a map $\mu\colon B(X) \to [0,1]$ such that $\mu(\emptyset) = 0, \mu(X) = 1$, and $\mu(A \cup B) = \mu(A) + \mu(B)$, if $A \cap B = \emptyset$. It is **countably additive** if moreover $\mu(\bigcup_n A_n) = \sum_n \mu(A_n)$, for any pairwise disjoint family (A_n). Recall that we call these simply *measures*. If a is a Borel action of a countable group Γ on X then μ is **invariant** if for any Borel set A and $\gamma \in \Gamma$, $\mu(\gamma \cdot A) = \mu(A)$.

Theorem 3.46 ([T15, 5.3])**.** *Let Γ be a countable group and let a be a continuous action of Γ on a compact Polish space X. If a admits an invariant finitely additive Borel probability measure, then it admits an invariant measure.*

Remark 3.47. The hypothesis that X is compact Polish is necessary here. From Remark 3.53 we see that there is a counterexample to this statement even with X Polish locally compact.

Theorem 3.48 ([TW16, 11.3])**.** *Let Γ be a countable group and let a be a Borel action of Γ on a standard Borel space X. Then the following are equivalent:*

(i) there is no invariant finitely additive Borel probability measure on X;

(ii) there is a finitely paradoxical amplification of a.

(B) We now prove Theorem 3.45. We have already mentioned the equivalence of (i) and (ii).

(i) $\implies$ (iv): If a is compressible, then by Nadkarni's Theorem it does not admit an invariant measure, so by Theorem 3.46 it does not admit an invariant finitely additive Borel probability measure. Then by Theorem 3.48 some amplification of a is finitely paradoxical.

(iv) $\implies$ (iii) is obvious.

(iii) $\implies$ (i): Assume that for some n the amplification a_n is finitely compressible but, towards a contradiction, a is not compressible. Then by Nadkarni's Theorem, a admits an invariant measure and thus so does a_n, contradicting the compressibility of a_n. □

Conjecture 3.49. *In Theorem 3.45, one cannot replace (iii) by "a is finitely compressible" and similarly for (iv).*

Remark 3.50. It follows from Theorem 3.45 that for a continuous action a of a countable group on a compact Polish space, the property "a has a finitely compressible (or finitely paradoxical) amplification" is a property of the induced equivalence relation E_a. More precisely, if a, b are two continuous actions of groups Γ, Δ on compact metrizable spaces X, Y, resp., and $E_a \cong_B E_b$, i.e., E_a, E_b are Borel isomorphic, then a admits a finitely compressible (or finitely paradoxical) amplification iff b admits a finitely compressible (or finitely paradoxical) amplification. In view of Conjecture 3.49, this may not be true for the property "a is finitely compressible" or "a is finitely paradoxical". In fact one way to try to prove Conjecture 3.49 is to search for two continuous actions a, b of countable groups Γ, Δ on a compact metrizable space X with $E_a = E_b$, for which a is finitely compressible (or finitely paradoxical) but b is not.

Remark 3.51. Theorem 3.45 fails if the space X is not compact. In fact there are even counterexamples with X Polish locally compact. Recall that an action of a group Γ on a set X is **amenable** if there is a finitely additive probability measure defined on all subsets of X and invariant under the action. Any action of a countable amenable group is amenable. Take now Γ to be a locally finite, infinite group and consider the left-translation action of Γ on itself. This action is not finitely compressible. Let then $X = 2^{\mathbb{N}} \times \Gamma$ (with Γ discrete). This is Polish locally compact and Γ acts on it continuously by the action a given by $\gamma \cdot (x, \delta) = (x, \gamma\delta)$. This action is clearly compressible via the map $T(x, \gamma) = (x, f(\gamma))$, where $f \colon \Gamma \to \Gamma$ is an injection with $f(\Gamma) \neq \Gamma$, so (i) in Theorem 3.45 holds. On the other hand, all amplifications a_n are amenable, so not finitely paradoxical and (iv) in Theorem 3.45 fails. Also all the actions a_n are not finitely compressible and (iii) in Theorem 3.45 also fails.

In this counterexample the action a is smooth. One can find another counterexample where the action a is not smooth as follows: Let Γ be as before and consider again the translation action of Γ on itself. Let also

Γ act on 2^Γ by shift and consider the action b of $\Delta = \Gamma^2$ on $X = 2^\Gamma \times \Gamma$ given by $(\gamma, \delta) \cdot (x, \varepsilon) = (\gamma \cdot x, \delta\varepsilon)$. This action is not smooth and is compressible but it is also amenable, since the action of each factor of Δ is amenable on the corresponding space and therefore the action of Δ is amenable by taking the product of finitely additive probability measures witnessing the amenability of these two actions. (By the product of a finitely additive probability measure μ defined on all subsets of a set A and a finitely additive probability measure ν defined on all subsets of a set B, we mean the finitely additive probability measure $\mu \times \nu$ on $A \times B$ defined by $\mu \times \nu(C) = \int_A \nu(C_x) \mathrm{d}\mu(x)$.) Also all the actions a_n are not finitely compressible.

Remark 3.52. Using Remark 3.43 it is easy to see that finite compressibility does not imply finite paradoxicality even for continuous actions of countable groups on compact Polish spaces. To see this, let G be a compact metrizable group containing a copy of $\mathbb{Z}$ (e.g., the unit circle under multiplication) and consider the left-translation action of $\mathbb{Z}$ on G.

Remark 3.53. Let E be a countable Borel equivalence relation on a standard Borel space X. We say that E is compressible (resp., finitely compressible, paradoxical, finitely paradoxical) if there is a Borel action a of a countable group Γ on X with $E = E_a$ and a is compressible (resp., finitely compressible, paradoxical, finitely paradoxical). Then it is easy to check that these conditions are equivalent. Indeed if E is compressible, then there is a smooth, aperiodic (i.e., having infinite classes) Borel equivalence relation F with $F \subseteq E$. Then $F \cong_B \mathbb{R} \times I_\mathbb{N}$, where $H = \mathbb{R} \times I_\mathbb{N}$ is the equivalence relation on $\mathbb{R} \times \mathbb{N}$ given by $(x, m) H (y, n) \iff x = y$. There is a transitive action of the free group with two generators $\mathbb{F}_2$ on $\mathbb{N}$ which is finitely paradoxical and thus a Borel action b of $\mathbb{F}_2$ on $\mathbb{R} \times \mathbb{N}$ with $F = E_b$ which is finitely paradoxical. Fix also a Borel action c of a countable group Δ with $E_c = E$. Then the action a of $\Gamma * \mathbb{F}_2$ that is equal to c on Γ and b on $\mathbb{F}_2$ is finitely paradoxical and $E_a = E$.

Remark 3.54. Ronnie Chen pointed out that (iv) $\implies$ (ii) in Theorem 3.45 can be also proved by using the cardinal algebra $K(E \times I_\mathbb{N})$ as in [C18] and the cancellation law for cardinal algebras.

Recall also that a CBER E admits an invariant measure iff *some* Borel action of a countable group that generates E has an invariant measure iff *every* Borel action of a countable group that generates E has an invariant measure (iff E is not compressible). On the other hand, there are aperiodic CBER E such that some Borel action of a countable group that generates E has an invariant finitely additive Borel probability measure but some other Borel action of a countable group that generates E has no invariant finitely additive Borel probability measure. For example, let $E = E_t$. There is a continuous action of $\mathbb{F}_2$ on $2^{\mathbb{N}}$ that generates E (see [K2, 2.B]) and this action has no invariant finitely additive Borel probability measure by Theorem 3.46. On the other hand, E_t is induced by a Borel action of $\mathbb{Z}$ and this action has in fact an invariant finitely additive probability measure defined on all subsets of $2^{\mathbb{N}}$.

However in view of Remark 3.53 we have the following equivalent formulation of existence of invariant measures for a CBER:

Proposition 3.55. *For every aperiodic* CBER *E, E admits an invariant measure iff every Borel action of a countable group that generates E admits an invariant finitely additive Borel probability measure.*

(2) The preceding results in part (1) of this subsection can be generalized as follows.

Let Γ be a countable group and let a be an action of Γ on a set X. Let also $\mathcal{A}$ be an algebra of subsets of X invariant under this action. For $A, B \in \mathcal{A}$, let $A \sim_{\mathcal{A}} B$ iff there are partitions $A = \bigsqcup_{i=1}^n A_i, B = \bigsqcup_{i=1}^n B_i$, where $A_i, B_i \in \mathcal{A}$, and $\gamma_i \in \Gamma$ such that $\gamma_i \cdot A_i = B_i$. We say that the action is **$\mathcal{A}$-finitely compressible** if $X \sim_{\mathcal{A}} Y$ with witnesses X_i, Y_i, γ_i as above, so that if $T\colon X \to X$ is such that $T(x) = \gamma_i \cdot x$, for $x \in X_i$, then for every orbit C of the action, $T(C) \subsetneqq C$. Also the action is **$\mathcal{A}$-finitely paradoxical** if there is a partition $X = Y \sqcup Z$, with $Y, Z \in \mathcal{A}$ and $X \sim_{\mathcal{A}} Y \sim_{\mathcal{A}} Z$.The concept of an invariant finitely additive probability measure on $\mathcal{A}$ is defined as before.

We extend the algebra $\mathcal{A}$ to an algebra $\mathcal{A}_n$ of subsets of $X \times [n]$ by letting $A \in \mathcal{A}_n \iff A = \bigcup_{i=1}^n A_i \times \{i\}$, where $A_i \in \mathcal{A}$. We say that a_n is $\mathcal{A}$-finitely compressible if it is $\mathcal{A}_n$-finitely compressible. Similarly we define what it means for a_n to be $\mathcal{A}$-finitely paradoxical.

We now have the following generalization of Theorem 3.45:

Theorem 3.56. *Let a be a continuous action of a countable group Γ on a compact Polish space X. Let $\mathcal{A}$ be an algebra of subsets of X which is invariant under the action and contains a basis for X. Then the following are equivalent:*

(i) there is no invariant finitely additive probability measure μ on $\mathcal{A}$;
(ii) there is no invariant measure ν;
(iii) an amplification of a is $\mathcal{A}$-finitely compressible;
(iv) an amplification of a is $\mathcal{A}$-finitely paradoxical.

The proof of Theorem 3.56 is similar to the proof of Theorem 3.45 using the following generalizations of Theorem 3.46 and Theorem 3.48.

Theorem 3.57 ([T15, 5.3])**.** *Let Γ be a countable group and let a be a continuous action of Γ on a second countable Hausdorff space X. Let $\mathcal{A}$ be an algebra of subsets of X which is invariant under the action and contains a basis for X and a compact set K. If there is an invariant finitely additive probability measure μ on $\mathcal{A}$ with $\mu(K) > 0$, then there is an invariant (Borel probability, countably additive) measure ν.*

Theorem 3.58 ([TW16, 11.3])**.** *Let Γ be a countable group and let a be an action of Γ on a set X. Let $\mathcal{A}$ be an algebra of subsets of X invariant under this action. Then the following are equivalent:*
(i) there is no invariant finitely additive probability measure on $\mathcal{A}$;
(ii) there is a $\mathcal{A}$-finitely paradoxical amplification of a.

As a particular case of Theorem 3.56 we have the following. Let a be a continuous action of a countable group Γ on a zero-dimensional compact Polish space X (e.g., the Cantor space). Let $\mathcal{C}$ be the algebra of clopen subsets of X. Then the following are equivalent:
(i) there is no invariant finitely additive probability measure μ on $\mathcal{C}$;
(ii) there is no invariant measure ν;
(iii) an amplification of a is $\mathcal{C}$-finitely compressible;
(iv) an amplification of a is $\mathcal{C}$-finitely paradoxical;
(v) a is compressible;
(vi) a is paradoxical;
(vii) an amplification of a is finitely compressible;
(viii) an amplification of a is finitely paradoxical.

Thus, rather surprisingly, for a continuous action of a countable group on a zero-dimensional compact Polish space, existence of a (countable Borel) paradoxical decomposition is equivalent to the existence of an amplification with a finite paradoxical decomposition using Borel pieces and also equivalent to the existence of an amplification with a finite paradoxical decomposition using *clopen* pieces.

3.F F_σ and K_σ realizations

Clinton Conley raised the following question: Does every aperiodic CBER have a realization as a K_σ equivalence relation in a Polish space?

We first note that this has the following equivalence formulation:

Proposition 3.59. *An aperiodic* CBER *has a K_σ realization in a Polish space iff it has an K_σ realization in a compact Polish space.*

Proof. Let $E \in \mathcal{AE}$ be a K_σ equivalence relation on a Polish space X, which is therefore itself K_σ. Let $K = \{0, \dots, \frac{1}{n}, \dots, \frac{1}{3}, \frac{1}{2}, 1\}$, a compact subset of $\mathbb{R}$. Finally let Y be a compactification of X (i.e., a compact Polish space which contains X as a dense subspace). Then X is $\mathbf{\Delta}^0_2$ in Y and thus so is $Y \setminus X$. Consider now the compact product space $Z = Y \times K$. Identify Y with $Y \times \{0\}$ and so X with $X \times \{0\}$. Consider then the equivalence relation F on Z whose classes are the E-classes in $X \times \{0\}$ and the sets $\{x\} \times (K \setminus \{0\})$, $x \in X$, and $\{y\} \times K$, $y \in (Y \setminus X)$. It is easy to check that F is F_σ, therefore K_σ, in Z. Now notice that there is a perfect set $C \subseteq X$ consisting of pairwise E-inequivalent elements. To see this, let P be the perfect kernel of X. Then $E \cap P^2$ is not comeager in P^2, so, by [K95, 8.26] and [K95, 19.11], there is a perfect set of pairwise E-inequivalent elements in P. Using C it is now easy to show that $F \cong_B E$. □

Of course on compact Polish spaces K_σ and F_σ relations are the same, so one can also ask more generally about F_σ realizations with additional properties. First it is clear that every aperiodic CBER has a realization in some Polish space which is induced by a continuous action of a countable group and thus it is F_σ. In fact by looking at the perfect kernel of such a Polish space we can assume that it is also perfect. We can actually find such an F_σ realization with at least one class dense.

Proposition 3.60. *Every aperiodic* CBER *admits an F_σ realization in some Polish space with a dense class. Every aperiodic smooth* CBER *admits such a realization in a compact Polish space.*

Proof. Let $E \in \mathcal{AE}$ and find a realization F of E on a Polish space X induced by a continuous action of a countable group. Then let $Q \subseteq X$ be a countable dense subset of X invariant under the action. The note that $F' = F \cup Q^2$ is an F_σ equivalence relation, $F' \cong_B F \cong_B E$ and Q is a dense class of F'.

For the second assertion, consider the equivalence relation R on $\mathbb{R}$ which is given by the translation action of $\mathbb{Z}$ on $\mathbb{R}$. Let $X = \mathbb{R} \cup \{\infty\}$ be the one-point compactification of $\mathbb{R}$ and let $F = R \cup (\mathbb{Z} \cup \{\infty\})^2$. Next fix a countable subset Q of X, let $[Q]_F$ be the F-saturation of Q and put $F' = F \cup ([Q]_F)^2$. Then F' is F_σ smooth aperiodic and has a dense class. □

The following is an open problem:

Problem 3.61. *Can every aperiodic* CBER *be realized as an F_σ equivalence relation on some Polish space with all classes dense?*

Again Theorem 3.10 shows that the answer is positive for all non-smooth relations in $\mathcal{AH}$, with the realization being in fact on a compact space. However we do not know if Problem 3.61 has a positive answer for smooth relations,

Problem 3.62. *Can every smooth aperiodic* CBER *be realized as an F_σ equivalence relation on some Polish space with all classes dense?*

Returning to K_σ realizations, we restate Conley's question and some stronger variants:

Problem 3.63. *(1) Can every aperiodic* CBER *be realized as a K_σ equivalence relation on some Polish space? In addition:*

(2) with a dense class

or even

(3) all classes dense?

Clearly the answer to Problem 3.63, (1) is positive for hyperfinite relations (by Theorem 3.10 and Proposition 3.60). Proposition 3.60 also gives a positive answer to Problem 3.63, (2) for smooth equivalence relations. Finally Problem 3.63, (3) has of course a positive answer for non-smooth hyperfinite relations and in compact spaces but it fails for smooth relations as proved by Solecki in [S02]:

Theorem 3.64 (Solecki, [S02, Corollary 3.2])**.** *Every K_σ equivalence relation on a Polish space with all classes dense and at least two classes is not smooth.*

In contrast, we note that by a variation of Case 2 in the proof of Theorem 3.1, one can show the following:

Proposition 3.65. *Every aperiodic smooth* CBER *can be realized as an equivalence relation which is a Boolean combination of K_σ relations in a compact Polish space and has all classes dense.*

Proof. Here are two such realizations:

1) Consider the equivalence relation E_0 in $2^{\mathbb{N}}$. Let A be a Cantor set in $2^{\mathbb{N}}$ which is a partial transversal for E_0. Let B be the E_0-saturation of A and put $Y = 2^{\mathbb{N}} \setminus B$. Then Y is G_δ, so a zero-dimensional Polish space (in the relative topology). Every compact subset of Y has empty interior in Y, so Y is homeomorphic to the Baire space $\mathcal{N}$ (see [K95, 7.7]). Therefore there is a continuous bijection $f\colon Y \to A$ (see [K95, 7.15]). Let F be the equivalence relation on $2^{\mathbb{N}}$ obtained by adding to each E_0 class $[a]_{E_0}$, with $a \in A$, the point $f^{-1}(a)$. Then F is smooth with all classes dense. Put

$$S(x,y) \iff x \in B \;\&\; y \in Y \;\&\; \exists z \in A(xE_0z \;\&f(y) = z)$$

and

$$T(x,y) \iff S(y,x).$$

Then each of S, T is the intersection of two K_σ relations with a G_δ relation and

$$xFy \iff (x, y \in B \;\&\; xE_0y) \vee S(x,y) \vee T(x,y),$$

so F is a Boolean combination of K_σ relations as well.

2) Let $X = \prod_{n\geq 1} 2^n$, where 2^n is the set of binary sequences of length n. Define $f\colon X \to 2^{\mathbb{N}}$ as follows: Let $Y = \{(x_n) \in X\colon \exists m \forall n \geq m(x_n \subseteq x_{n+1})\}$ and for $(x_n) \in Y$, let $f(x) = \bigcup_{n\geq m} x_n$, for all sufficiently large m. For $(x_n) \in Z = X \setminus Y$, let $f((x)_n)$ be the concatenation $x_1x_2x_3\cdots$. Let $xFy \iff f(x) = f(y)$. Then F is a smooth CBER with all classes dense and it is easy to check that $F = F_1 \cup F_2 \cup F_3 \cup F_4$, where F_1 is K_σ, F_2 and F_3 are intersections of a K_σ and a G_δ relation and F_4 is the equality relation on X. □

3.G A σ-ideal associated to a K_σ CBER

Suppose that X is an (uncountable) Polish space and E a CBER on X. Denote by $K(X)$ the space of compact subsets of X with the usual Vietoris topology (see [K95, 4.F]). Let

$$I_E = \{K \in K(X)\colon [K]_E \neq X\}.$$

Recall that a σ ideal of compact sets is a nonempty subset $I \subseteq K(X)$ such that $K \subseteq L \in I \implies K \in I$ (i.e., it is hereditary) and $K \in K(X), K = \bigcup_n K_n, K_n \in I, \forall n \implies K \in I$ (i.e., it is closed under countable unions which are compact).

Proposition 3.66. *Let X be a Polish space and E a K_σ CBER on X with all E-classes dense. Then I_E is a G_δ σ-ideal of compact sets.*

Proof. Here and in the sequel, notice that since E is K_σ, $X = \{x \in X : (x,x) \in E\}$ (and X^2) is also K_σ and $F_\sigma = K_\sigma$ on X (and X^2).

Clearly I_E is hereditary. To check closure under countable unions, we will actually show that if $K_n \in I_E, \forall n$, then $[\bigcup_n K_n]_E \neq X$. Notice that because E is K_σ, for each compact K the set $[K]_E$ is also K_σ and thus if $K \in I_E$, then $X \setminus [K]_E$ is dense G_δ. So if $K_n \in I_E, \forall n$, and $[\bigcup_n K_n]_E = \bigcup_n [K_n]_E = X$ this contradicts the Baire Category Theorem. Since

$$K \in I_E \iff \exists x \forall y (y \in K \implies \neg x E y),$$

clearly I_E is $\mathbf{\Sigma}^1_1$, thus by [KLW87, Theorem 11] (see also [MZ07, Theorem 1.4]) it is G_δ. □

Corollary 3.67. *If X, E are as in Proposition 3.66 and moreover E admits a meager complete section, then E admits a nowhere dense, compact complete section.*

Proof. We have a sequence K_n of nowhere dense compact sets with $[\bigcup_n K_n]_E = \bigcup_n [K_n]_E = X$. Thus for some n, $K_n \notin I_E$, so K_n is a nowhere dense, compact complete section. □

Below denote by $K_{\aleph_0}(X)$ the σ-ideal of countable compact subsets of X and by $\mathrm{MGR}(X)$ the σ-ideal of nowhere dense (i.e., meager) compact subsets of X.

Corollary 3.68. *If X, E are as in Corollary 3.67, then*

$$K_{\aleph_0}(X) \subsetneqq I_E \subsetneqq \mathrm{MGR}(X).$$

Corollary 3.69. *If X, E are as in Proposition 3.66, then E does not admit a K_σ transversal.*

Proof. If F is a K_σ transversal, we can write $F = F_1 \sqcup F_2$, where each F_i is also K_σ and nonempty. Then each F_i is the union of countably many compact sets in I_E, a contradiction. □

We say that a σ-ideal of compact sets I satisfies **Solecki's Property (*)** if for any sequence $K_n \in I, \forall n$, there is a G_δ set G such that $\bigcup_n K_n \subseteq G$ and $K(G) = \{K \in K(X) \colon K \subseteq G\} \subseteq I$; see [S11].

Proposition 3.70. *If X, E are as in Proposition 3.66, then I_E satisfies Solecki's Property (*).*

Proof. Let $K_n \in I_E, \forall n$. Then there is $x \in X$ such that $[x]_E \cap [\bigcup_n K_n]_E = \emptyset$ and thus if $G = X \setminus [x]_E$, G is G_δ and $K(G) \subseteq I_E$. □

In particular I_E admits a representation as in [S11, Theorem 3.1].

A σ-ideal I of compact sets is **ccc** if there is no uncountable collection of pairwise disjoint compact sets which are not in I. Since for any CBER E every $K \notin I_E$ is a complete section, it follows that I_E is ccc.

On the other hand, let I_E^* be the σ-ideal of subsets of X generated by I_E, i.e, for $A \subset X$, $A \in I_E^* \iff \exists (K_n)(K_n \in I_E, \forall n,$ and $A \subseteq \bigcup_n K_n)$. Then I_E^* is not ccc, in fact we have the following:

Proposition 3.71. *Let X, E be as in Proposition 3.66 and moreover for every nonempty open set $U \subseteq X$ there is a meager complete section contained in U. Then there is a homeomorphic embedding $f \colon 2^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to X$ such that for every $\alpha \in 2^{\mathbb{N}}$, we have $f(\{\alpha\} \times \mathbb{N}^{\mathbb{N}}) \notin I_E^*$.*

Proof. By [KS95, Section 3, Lemma 9], it is enough to show that for every nonempty open $U \subseteq X$, there is a nowhere dense compact set $K \subseteq U$ with $K \notin I_E$. This follows as in the proof of Corollary 3.67. □

A σ-ideal I of compact sets has the **covering property** if for every $\boldsymbol{\Sigma^1_1}$ set $A \subseteq X$, either $A \subseteq \bigcup_n K_n$, where $K_n \in I, \forall n$, or else $K(A) \subseteq I$. It is **calibrated** if whenever $K \in K(X)$ and $K_n \subseteq K$ are such that $K_n \in I, \forall n$, and $K(K \setminus \bigcup_n K_n) \subseteq I$, then $K \in I$.

Proposition 3.72. *Let X, E be as in Proposition 3.66. Then I_E does not have the covering property and is not calibrated.*

Proof. Fix $x \in X$ and let $G = X \setminus [x]_E$. This provides a counterexample to both properties. □

We next provide an example of a pair X, E satisfying all the properties of Proposition 3.71, and which therefore satisfies all the preceding propositions. We take X to be the collection of all subsets A of $\mathbb{N}$ such that $0 \in A, 1 \notin$

A, with the usual topology. We let then E be the restriction of many-one equivalence to X. It is easy to see that E is a K_σ CBER and every E-class is dense. Finally if U is an open subset of X, which we can assume that it has the form $U = \{A \in X \colon F_1 \subseteq A, F_2 \cap A = \emptyset\}$, for two disjoint finite subsets F_1, F_2 of $\mathbb{N}$, then for a large enough number n the set $K = \{A \in U \colon A \text{ contains only even numbers } > n\}$ is a meager complete section contained in U.

4 Generators and 2-adequate groups

For each infinite countable group Γ and standard Borel space X consider the shift action of Γ on X^Γ and let $E(\Gamma, X)$ be the associated equivalence relation and $E^{ap}(\Gamma, X)$ be its aperiodic part, i.e., the restriction of $E(\Gamma, X)$ to the set of points with infinite orbits. Consider now a Borel action of Γ on an uncountable standard Borel space, which we can assume is equal to $\mathbb{R}$. Then the map $f \colon X \to \mathbb{R}^\Gamma$ given by $x \mapsto p_x$, where $p_x(\gamma) = \gamma^{-1} \cdot x$, is an equivariant Borel embedding of this action to the shift action on $\mathbb{R}^\Gamma$. In particular for every aperiodic equivalence relation E induced by a Borel action of Γ we have that $E \sqsubseteq^i_B E(\Gamma, \mathbb{R})$, where for equivalence relations R, S on standard Borel spaces Y, Z, resp., we let $R \sqsubseteq^i_B S$ iff there is an injective Borel reduction $f \colon Y \to Z$ of R to S such that $f(Y)$ is S-invariant. Thus every aperiodic equivalence relation E induced by a Borel action of Γ can be realized as (i.e., is Borel isomorphic to) the restriction of $E^{ap}(\Gamma, \mathbb{R})$ to an invariant Borel set.

Now recall that for a Borel action of Γ on a standard Borel space X and $n \in \{2, 3, \dots, \dots, \mathbb{N}\}$ an ***n*-generator** is a Borel partition $X = \bigsqcup_{i<n} X_i$ such that $\{\gamma \cdot X_i \colon \gamma \in \Gamma, i < n\}$ generates the Borel sets in X. This is equivalent to having a Borel equivariant embedding of the action to the shift action on n^Γ.

It is shown in [JKL02] that for every such action with infinite orbits there exists an $\mathbb{N}$-generator. It follows that every aperiodic equivalence relation E induced by a Borel action of Γ can be realized as the restriction of $E^{ap}(\Gamma, \mathbb{N})$ to an invariant Borel set. In particular $E^{ap}(\Gamma, \mathbb{R}) \cong_B E^{ap}(\Gamma, \mathbb{N})$. However because of entropy considerations, even for the group $\Gamma = \mathbb{Z}$, it is not the case that every such action with invariant measure has a finite generator.

Weiss [W89] asked whether for $\Gamma = \mathbb{Z}$ any Borel action without invariant measure admits a finite generator. Tserunyan [T15] showed that answer is affirmative for *any* infinite countable group Γ if the action is Borel isomorphic to a continuous action on a σ-compact Polish space. Then Hochman [H15] provided a positive answer to Weiss' question (for $\mathbb{Z}$). Finally this work culminated in the following complete answer:

Theorem 4.1 (Hochman-Seward)**.** *Every Borel action of a countable group on a standard Borel space without invariant measure admits a* 2*-generator.*

This however leaves open the possibility that every aperiodic CBER E induced by a Borel action of Γ can be realized as the restriction of $E^{ap}(\Gamma, 2)$ to an invariant Borel set. This is clearly equivalent to the statement that $E^{ap}(\Gamma, \mathbb{R}) \cong_B E^{ap}(\Gamma, 2)$ and it also equivalent to the statement that there is a Borel action of Γ that generates E and has a 2-generator. This leads to the following concept.

Definition 4.2. An infinite countable group Γ is called **2-adequate** if

$$E^{ap}(\Gamma, \mathbb{R}) \cong_B E^{ap}(\Gamma, 2).$$

Remark 4.3. Thomas [T12] studies the question of when $E(\Gamma, \mathbb{R}) \sim_B E(\Gamma, 2)$.

The first result here is the following:

Theorem 4.4. *Every infinite countable amenable group is* 2*-adequate.*

Proof. Let $X = \mathbb{R}^\Gamma, Y = 2^\Gamma, E = E^{ap}(\Gamma, \mathbb{R}), F = E^{ap}(\Gamma, 2)$. Note that $|\mathrm{EINV}_F| = |\mathrm{EINV}_E| = 2^{\aleph_0}$, so fix a Borel bijection $\pi\colon \mathrm{EINV}_E \to \mathrm{EINV}_F$. Fix also the ergodic decompositions $\{X_e\}_{e \in \mathrm{EINV_E}}$ of E and $\{Y_f\}_{f \in \mathrm{EINV_F}}$ of F, resp. By the Ornstein-Weiss Theorem, see. e.g., [K2, 7.25], let Z_e be an E-invariant Borel subset of X_e such that $E|Z_e$ is hyperfinite with unique invariant measure e. Again the construction of Z_e is effective enough, so that $Z = \bigcup_e Z_e$ is Borel. Put $X' = X \setminus Z$, so that $E|X'$ is compressible.

Then, by Theorem 4.1, there is a Borel F-invariant subset Y' of Y such that $E|X' \cong_B F|Y'$, say by the Borel isomorphism $g\colon X' \to Y'$. Put $W' = Y \setminus Y'$. Then let W_f be an F-invariant Borel subset of Y_f such that $W_f \subseteq W'$ and $F|W_f$ is hyperfinite with unique invariant measure f. Again the construction of W_f is effective enough, so that $W = \bigcup_f W_f$ is Borel and there is a Borel isomorphism h_e of $E|Z_e$ with $F|W_{\pi(e)}$ such that moreover $h = \bigcup_e h_e$ is Borel and thus a Borel isomorphism of $E|Z$ with $F|W$. Then $g \cup h$ shows that $E \sqsubseteq_B^i F$ and the proof is complete. $\square$

Thomas [T12, Page 391] asked the question of whether there is an infinite amenable Γ such that $E(\Gamma, \mathbb{R}) \not\leq_B E(\Gamma, 2)$. Theorem 4.4 provides a negative answer in a strong form.

To discuss other examples of 2-adequate groups, we will need the following strengthening of Theorem 2.7.

Proposition 4.5. *Let $E \in \mathcal{AE}$ and let $R \subseteq E$ be hyperfinite. Then there is $R \subseteq F \subseteq E$ with $F \in \mathcal{AH}$.*

Proof. Suppose E lives on the standard Borel space X and let

$$Y = \{x \colon [x]_E \text{ contains a finite nonempty set of finite } R\text{-classes}\}.$$

Then Y is E-invariant and $E|Y$ is smooth, thus we can let $F = E$ on Y. Let $W = \{x \colon [x]_E \text{ contains no finite } R\text{-classes}\}$. Then we can take $F = R$ on W.

So we can assume that each E-class contains infinitely many finite R-classes. Let $Z = \{x \colon [x]_R \text{ is finite}\}$. Then $R|Z$ is R-invariant and smooth, so let S be a Borel selector and T the associated Borel transversal $T = \{x \colon S(x) = x\}$.. Then, by Theorem 2.7, let F' be a hyperfinite aperiodic Borel equivalence relation on T such that $F' \subseteq E|T$. Let then F'' be the equivalence relation on Z defined by $xF''y \iff S(x)F'S(y)$. It is clearly aperiodic, hyperfinite, and $R|Z \subseteq F'' \subseteq E|Z$. Finally put $F = F'' \cup R|(X \setminus Z)$. □

We also consider the following class of countable groups.

Definition 4.6. A countable group Γ is **hyperfinite generating** if for every $E \in \mathcal{AH}$ there is a Borel action of Γ that generates E.

We now have the next result that generalizes Proposition 4.5 from $\mathbb{Z}$ to any hyperfinite generating group. The proof is similar, noting that any smooth aperiodic CBER can be generated by a Borel action of any infinite countable group.

Proposition 4.7. *Let $E \in \mathcal{AE}$ and let $R \subseteq E$ be generated by a Borel action of Γ, where Γ is a hyperfinite generating group. Then there is $R \subseteq F \subseteq E$ with $F \in \mathcal{AE}$ generated by a Borel action of Γ.*

Proposition 4.8. *Let Γ be any countable group and Δ a hyperfinite generating, 2-adequate group. Then $\Gamma \star \Delta$ is 2-adequate.*

Proof. Fix a Borel action a of $\Gamma \star \Delta$ on an uncountable standard Borel space X generating an aperiodic equivalence relation that we denote by E_a. Let $b = a|\Delta, c = a|\Gamma$ and denote by E_b, E_c the associated equivalence relations, so that $E_a = E_b \vee E_c$. By Proposition 4.7 find a Borel action b' of Δ such that $E_{b'}$ is aperiodic and $E_b \subseteq E_{b'} \subseteq E_a$, so that $E_a = E_{b'} \vee E_c$. Let now a' be the action of $\Gamma \star \Delta$ such that $a'|\Delta = b', a'|\Gamma = c$, so that $E_{a'} = E_a$. Since b' has a 2-generator, so does a' and the proof is complete. □

It will be shown in Corollary 5.2 that all groups that have an infinite amenable factor are hyperfinite generating. Thus we have:

Corollary 4.9. *The free product of any countable group with a group that has an infinite amenable factor and thus, in particular, the free groups $\mathbb{F}_n, 1 \leq n \leq \infty$, are 2-adequate.*

The following is immediate:

Proposition 4.10. *If Γ, Δ are countable groups, every aperiodic equivalence relation induced by a Borel action of Γ can be also induced by a Borel action of Δ, Δ is a factor of Γ and Δ is 2-adequate, so is Γ. In particular, for any $1 \leq n \leq \infty$, every n-generated countable group that factors onto $\mathbb{F}_n$ is 2-adequate.*

The next two results owe a lot to some crucial observations by Brandon Seward.

Proposition 4.11. *Let Γ be n-generated, $1 \leq n \leq \infty$. Then $\Gamma \times \mathbb{F}_n$ is 2-adequate. In particular, all products $\mathbb{F}_m \times \mathbb{F}_n$, $1 \leq m, n \leq \infty$, are 2-adequate.*

Proof. Let $\{\gamma_i\}_{i<n}$ be generators for Γ and let $\{\alpha_i\}_{i<n}$ be free generators for $\mathbb{F}_n$. Consider a Borel action a of $\Gamma \times \mathbb{F}_n$ with E_a aperiodic. Then the equivalence relation E_i generated by $a|\langle \gamma_i, \alpha_i \rangle$ is generated by a Borel action of $\mathbb{Z}^2$ thus is hyperfinite, see, e.g., [K2, 7.F], and thus is given by a Borel action a_i of $\mathbb{Z}$. Let b the Borel action of $\mathbb{F}_n$ in which the generator α_i acts like a_i. Then $E_b = \bigvee_i E_i = E$ and the proof is complete by Proposition 4.10. □

Finally not every infinite countable group is 2-amenable. The argument below follows the pattern of the proofs in [T12, Section 6].

Theorem 4.12. *The group $\mathrm{SL}_3(\mathbb{Z})$ is not 2-adequate.*

Proof. Assume that $\Gamma = \mathrm{SL}_3(\mathbb{Z})$ is 2-adequate, towards a contradiction. Then in particular $E^{ap}(\Gamma, 3) \cong_B E^{ap}(\Gamma, 2)$, say via the Borel isomorphism f. Let μ be the usual product of the uniform measure on 3^Γ. Then $\nu = f_*\mu$ is an ergodic, invariant measure for the shift action of Γ on 2^Γ, thus by Stuck-Zimmer [SZ94] this shift action is free ν-a.e. This gives a contradiction by the arguments in [T12, Section 6]. □

We conclude this section with the following problem:

Problem 4.13. *Characterize the* 2*-adequate groups.*

5 Additional results

(A) Hyperfinite generating groups. We introduced in Section 4 the concept of hyperfinite generating groups. We will establish here some equivalent formulations of this concept and in particular prove the fact mentioned in the paragraph after Proposition 4.8. Below we let μ be the product of the uniform measure on $2^\mathbb{N}$ and by $[E_0] \leq \mathrm{Aut}(2^\mathbb{N}, \mu)$ the usual measure theoretic full of the pmp equivalence relation E_0. For a countable group $\Delta \leq [E_0]$, we denote by E_Δ the subequivalence relation of E_0 induced by the action of Δ on $2^\mathbb{N}$. This is again understood to be defined only μ-a.e.

Below an **IRS** on a countable group Γ is a measure on the space of subgroups of Γ invariant under conjugation. We say that an IRS μ has some property P if μ-almost all $\Delta \leq \Gamma$ have property P. Finally a subgroup $\Delta \leq \Gamma$ is **co-amenable** if the action of Γ on Γ/Δ is amenable, i.e., admits a finitely additive probability measure.

Proposition 5.1. *Let* Γ *be an infinite countable group. Then the following are equivalent:*

(i) Γ *is hyperfinite generating;*

(ii) There is a Borel action of Γ *that generates* E_0*;*

(iii) Γ *admits a Borel action which generates a non-compressible, aperiodic hyperfinite equivalence relation;*

(iv) Γ *admits a factor* $\Delta \leq [E_0]$ *such that* E_Δ *has a* μ*-positive set of infinite orbits.*

Moreover, if Γ *is hyperfinite generating,* Γ *admits a co-amenable* IRS *with infinite index.*

Proof. Clearly (i) $\implies$ (ii) $\implies$ (iii). We next prove that (iii) $\implies$ (iv). Indeed (iii) implies that there is a Borel action of Γ on a standard Borel space X generating an aperiodic equivalence relation E that has an ergodic, invariant measure μ. This action induces a homomorphism $\pi\colon \Gamma \to [E]$, the measure theoretic full group of E, with respect to μ. If $\Delta = \pi(\Gamma) \leq [E]$, then Δ generates E (again μ-a.e). But by Ornstein-Weiss and Dye, see, e.g., [K2, 7.8 and 7.25], E and E_0 are measure theoretically isomorphic, which proves (iv).

We now show that (iv) $\implies$ (i). Fix $E \in \mathcal{AH}$ which lives on a space X. If E is compressible, then it is generated by a Borel action of Γ, by [DJK94, 11.2]. Otherwise consider the ergodic decomposition $\{X_e\}_{e\in \mathrm{EINV}_E}$ of E. Now (iv) implies (iii) and it follows that Γ has a Borel action on a standard Borel space Z which generates an aperiodic hyperfinite equivalence relation F, which has an ergodic, invariant measure μ. Find then, using Dye's Theorem, see, e.g., [K2, 7.8], invariant Borel sets $Y_e \subseteq X_e$ with $e(Y_e) = 1$ and $Z_e \subseteq Z$ with $\mu(Z_e) = 1$ such that $E|Y_e$ and $F|Z_e$ are Borel isomorphic. Then $E|Y_e$ can be generated by a Borel action of Γ, and, by the effectivity of this construction, we also have that $Y = \bigcup_e Y_e$ is Borel and putting together the acton of Γ on each Y_e, we get a Borel action of Γ on Y which generates $E|Y$. Since $E|(X \setminus Y)$ is compressible, this shows that E is generated by a Borel action of Γ.

Finally the last statement follows as in the proof of (vii) $\implies$ (x) in the last paragraph of [BK17, Appendix D] (finite generation is not required there). □

Corollary 5.2. *Every countable group that has an infinite amenable factor is hyperfinite generating.*

Proof. If Γ is infinite amenable, consider its shift action on 2^Γ, equipped with the product of the uniform measure, with associated equivalence relation $E = E(\Gamma, 2)$. Then E and E_0 are measure theoretically isomorphic, so the measure theoretic full group of E is isomorphic to $[E_0]$. Since $\Gamma \leq [E]$ we have an embedding $\pi\colon \Gamma \to [E_0]$ such that if $\Delta = \pi(\Gamma)$, then $E_\Delta = E_0$, which completes the proof. □

It also immediately follows from [M06, Theorem 13] that every countable group that has a factor of the form $\Gamma \star \Delta$, where Γ, Δ are non-trivial subgroups of $[E_0]$, is hyperfinite generating.

On the other hand, not every infinite countable group is hyperfinite generating.

Proposition 5.3. *No infinite countable group with property* (T) *is hyperfinite generating.*

Proof. See, for example, the proof of [K10, Proposition 4.14]. □

It is also shown in [K10, page 29] that there are groups that do not have property (T) and are not hyperfinite generating.

The following is an open problem.

Problem 5.4. *Characterize the hyperfinite generating groups.*

(B) Dynamically compressible groups. In the course of the previous investigations the following property of countable groups came up. In the following it will be convenient to use the notation E_Γ^X for the equivalence relation induced by a Borel action of a countable group Γ on a standard Borel space X.

Definition 5.5. An infinite countable group Γ is called **dynamically compressible** if for every aperiodic E_Γ^X, there is a compressible E_Γ^Y with $E_\Gamma^X \leq_B E_\Gamma^Y$.

Here is an equivalent formulation of this notion.

Proposition 5.6. *A countable group Γ is dynamically compressible iff for every aperiodic E_Γ^X, $E_\Gamma^X \times I_\mathbb{N}$ is induced by a Borel action of Γ*

Proof. Since $E_\Gamma^X \times I_\mathbb{N} \leq_B E_\Gamma^X$, if $E_\Gamma^X \leq_B E_\Gamma^Y$, with E_Γ^Y compressible, then $E_\Gamma^X \times I_\mathbb{N} \leq_B E_\Gamma^Y$, therefore $E_\Gamma^X \times I_\mathbb{N} \sqsubseteq_B^i E_\Gamma^Y$ by [K2, 2.27]. □

We now have:

Proposition 5.7. *Every infinite countable amenable group is dynamically compressible.*

Proof. Consider any aperiodic $E = E_\Gamma^X$, which we can clearly assume is not compressible, so admits an invariant measure. Then let $\{X_e\}_{e \in \mathrm{EINV}_E}$ be its ergodic decomposition. Then there is a Borel set $Y_e \subseteq X_e$ with $e(Y_e) = 1$ such that $E|Y_e$ is hyperfinite, thus $E|Y_e \leq_B E_t$. As usual $Y = \bigcup_e Y_e$ is Borel and $E|Y \leq_B \mathbb{R}E_t \leq_B E_t$. Now $E|(X \setminus Y)$ is compressible and E_t is induced by a Borel action of Γ by [DJK94, 11.2], so the proof is complete. □

Proposition 5.8. *If $\mathbb{F}_2 \leq \Gamma$, then Γ is dynamically compressible.*

Proof. Let E_Γ^X be aperiodic. Then $E_\Gamma^X = E_{\mathbb{F}_\infty} \leq_B E_{\mathbb{F}_\infty} \times I_\mathbb{N} = E_{\mathbb{F}_\infty}^Y$, for $Y = X \times \mathbb{N}$. Now $\mathbb{F}_\infty \leq \Gamma$, so by using the inducing construction from the action of $\mathbb{F}_\infty$ on Y, see [BK96, 2.3.5], we have $E_{\mathbb{F}_\infty}^Y \leq_B E_Z^\Gamma$ for some compressible E_Z^Γ. □

Therefore only the groups that are not amenable but do not contain $\mathbb{F}_2$ can possibly fail to be dynamically compressible. But even among those there exist dynamically compressible groups.

Proposition 5.9. *Let Γ be a countable group for which there is an infinite group Δ such that $\Gamma \times \Delta \leq \Gamma$. Then Γ is dynamically compressible.*

Proof. Let E_Γ^X be aperiodic. Then for $Y = X \times \mathbb{N}$, $E_\Gamma^X \leq_B E_\Gamma^X \times I_\mathbb{N} = E_{\Gamma\times\Delta}^Y \leq_B E_\Gamma^Z$, where E_Γ^Z is obtained by inducing from the action of $\Gamma \times \Delta$ on Y. □

As a result any countable group of the form $\Gamma \times \Delta^{<\mathbb{N}}$, for an infinite Δ, is dynamically compressible. Take now Γ to be any group that is not amenable and does not contain $\mathbb{F}_2$ and consider $G = \Gamma \times \mathbb{Z}^{<\mathbb{N}}$. Then G is dynamically compressible and clearly is not amenable. Moreover it does not contain $\mathbb{F}_2$ because of the following standard fact.

Proposition 5.10. *Let G, H be two groups such that $\mathbb{F}_2 \leq G \times H$. Then $\mathbb{F}_2 \leq G$ or $\mathbb{F}_2 \leq H$.*

Proof. Let $\pi\colon \mathbb{F}_2 \to H$ be the second projection, If it has trivial kernel, then $\mathbb{F}_2 \leq H$. Else either $\mathbb{F}_2 \leq \ker(\pi) \leq G$ or $\ker(\pi) \cong \mathbb{Z}$. In the latter case, by [LS01, 3.110], $[\mathbb{F}_2 : \ker(\pi)]$ is finite, so by [LS01, 3.9],

$$[\mathbb{F}_2 : \ker(\pi)] = \frac{\operatorname{rank}(\ker(\pi)) - 1}{\operatorname{rank}(\mathbb{F}_2) - 1} = 0,$$

a contradiction. □

We now have the following open problem:

Problem 5.11. *Is every infinite countable group dynamically compressible?*

We note that Γ fails to be dynamically compressible iff there is some aperiodic E^X_Γ such that every $E^X_\Gamma \leq_B E^Y_\Gamma$ admits an invariant measure.

We conclude with the following interesting consequence of Proposition 5.8. Let $\Gamma = \mathrm{SL}_3(\mathbb{Z})$ and consider the shift action of Γ on $\mathbb{R}^\Gamma$ and denote by $E = F(\Gamma, \mathbb{R})$ the restriction of $E(\Gamma, \mathbb{R})$ to the free part of the action. Then, by Proposition 5.8, $E \times I_\mathbb{N}$ is induced by a Borel action of Γ. On the other hand, $E \times I_\mathbb{N}$ cannot be induced by a *free* Borel action of Γ, since if that was the case then $E \times I_\mathbb{N} \sqsubseteq^i_B E$, contradicting the Addendum following [CK18, 5.28].

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