A. S. Kechris: Classical Descriptive Set Theory; Corrections and Updates (October 18, 2017)

Page 3, line 9-: add after "spaces" ",with $d_i < 1$,"

Page 8, line 11: $D_\varphi \to D(\varphi)$

Page 22, line 6: $x_0 \in e \to x \in e$

Page 22: In the proof of 2.12 add that $s_n \neq \emptyset$

Page 24, line 21: add "less than" after "within"

Page 27, 4.31: drop separability

Page 35, line 10: Add after "clopen sets.": "Also the space of p-adic numbers $\mathbb{Q}_p$ and the space of p-adic integers $\mathbb{Z}_p$ are zero-dimensional (with their usual topologies). Moreover, $\mathbb{Q}_p$ is locally compact and $\mathbb{Z}_p$ is compact."

Page 42, 8.7: add "nonempty" before "perfect"

Page 53, line 4: add "Hausdorff" before "space"

Page 60, line 1: $(f_i)_{j \in J} \to (f_j)_{j \in J}$

Page 61, 9.11, line 3: subspace $\to$ linear subspace

Page 66, line 12-: replace first $\mathcal{S}$ by X

Page 70, line 7-: add "$0 <$" before "$|p - x|$"

Page 76, line 2-: add "nonempty" before "Polish"

Page 77, line 1: Delete "Assume that $X \neq \emptyset$ and"; capitalize "fix"

Page 78, 12.17: After "Theorem." add "(Dixmier)"

Page 84, line 2: $x_n \in A_n \to x_n \in A_{x|n}$

Page 86, line 7: delete "additionally"

Page 91, line 12: add "contains $\emptyset$ and" before "is"

Page 92, 15.10, line 2: add "a bijection" after "Then"

Page 111, line 4-: replace second $\overline{\lim}$ by $\overline{\lim}_n$

Page 117, 17.44, line 3: replace "$\vee a_n$" by "$\bigvee a_n$"

Page 117, 17.44, line 4: replace "A." by "A, where $\bigvee a_n$ is the least upper bound of $\{a_n : n \in \mathbb{N}\}$, also called the **join** of (a_n)"

Page 120, line 9: delete "here" after "interest"

Page 122, 18.6: In the second paragraph of the proof, add after "We will show that it is Borel." the sentence: "Clearly the domain of f, $\mathrm{proj}_X(P)$, is Borel, since $x \in \mathrm{proj}_X(P) \Leftrightarrow P_x \notin I_x$."

One can also formulate a stronger version of this theorem as follows:

Theorem 18.6*. *Let X, Y be standard Borel spaces and $P \subseteq X \times Y$ be Borel with $A = \mathrm{proj}_X(P) \subseteq X$. Let $x \in A \mapsto I_x$ be a map assigning to each $x \in A$ a σ-ideal of subsets of P_x such that:*

(i) For each Borel $R \subseteq P$, there is a $\mathbf{\Sigma}^1_1$ set $S \subseteq X$ and a $\mathbf{\Pi}^1_1$ set $T \subseteq X$ such that

$$x \in A \Rightarrow [R_x \in I_x \Leftrightarrow x \in S \Leftrightarrow x \in T],$$

(ii) $x \in A \Rightarrow P_x \notin I_x$.

Then there is a Borel uniformization of P, and in particular A is Borel.

Proof. Using the same notation as in the proof of 18.6, define $f : A \to Y$ exactly as before. We will check that the graph of f is Borel. Indeed we have:

$$f(x) = y \Leftrightarrow (x, y) \in P \;\&\; \forall n \exists s \in \mathbb{N}^n [P^s_x \notin I_x \;\&\; \forall t <_{\mathrm{lex}} s (P^t_x \in I_x) \;\&\; y \in P^s_x],$$

which by (i) above is both $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$, thus Borel. $\square$

Page 129, 19.1, line 1: replace "Mycielski, Kuratowski" by "Kuratowski, Mycielski"

Page 132, line 5: consider now $\to$ now consider

Page 154, line 17-: delete "unique"

Page 156, line 10: erase comma after "similarly"

Page 164, line 6: in $\to$ is

Page 165, line 17-: insert comma after "$\psi \circ \varphi$"; insert comma after "$y \in C_{\mathrm{x}}$"

Page 181, line 17 (Ex. 23.8): $n \to n > 1$

Page 191, line 1: $\bigcup_{j<i} A_n^{(i)} \to \bigcup_{j<i} A_n^{(j)}$; $\bar{A}_1^{(1)}, \ldots, \bar{A}_k^{(i)} \to \bar{A}_n^{(1)}, \ldots, \bar{A}_n^{(k)}$

Page 191, line 14: $\epsilon \to \in$ (in "$A_i^k \epsilon \mathbf{\Delta}^0_{\xi+1}$")

Page 191, line 16: $f^{(k)}(x) = y_i \to f^{(k)}(x) = y_i^{(k)}$

Page 194, line 17-: on $\mathcal{B}_1 \to$ in $\mathcal{B}_1$

Page 199, line 1: $P \to Q$

Page 201, line 8: second $R_s \to (R_s)$ (in 25.15 i))

Page 201, line 13: first $Q^s \to (Q^s)$ (in 25.15 ii))

Page 201, line 16: first $Q_s \to (Q_s)$ (in 25.15 ii))

Page 201, 7-: ")." $\to$ ".)"

Page 203, line 14: Fix now $\to$ Now fix

Page 203, line 15: after "$\subseteq \mathcal{S}$" add comma; after "$s \in \mathbb{N}^{<\mathbb{N}}$" add comma

Page 205, line 15-: discuss next $\to$ next discuss

Page 207, line 1-: of blocks $\to$ of nonempty blocks

Page 207, line 1-, and 208, 1: eliminate parenthetical remark

Page 213: Replace the first paragraph of 27.D by the following:

"Consider now the relation of isomorphism $\cong$ between elements of X_L, $L = \{R\}$, R binary, i.e.,

$$x \cong y \Leftrightarrow \mathcal{A}_x \cong \mathcal{A}_y.$$

It is clearly $\mathbf{\Sigma}^1_1$ (in $X_L \times X_L$). It was shown by H. Friedman and L. Stanley [1989] that it is $\mathbf{\Sigma}^1_1$-complete, even restricted to LO. Their proof used methods of effective descriptive set theory, which we do not develop here. An alternative proof based only on 27.12 was later found by R. Dougherty."

Also replace "To see that $\cong$ is not Borel," in the second paragraph of 27.D by "One can also give a different (and simpler) proof, using a result from Section 31, that $\cong$ on LO is not Borel (which also implies, using $\mathbf{\Sigma}^1_1$-Detereminacy, that it is $\mathbf{\Sigma}^1_1$-complete). To see this"

Page 216, line 3: Peter Komjath pointed out that this result has been earlier proved in B.S. Sodnomov, On arithmetical sums of sets, *Doklady Akad. Nauk SSSR (N.S.)*, **80** (1951), 173–175. See also B.S. Sodnomov, Example of two sets of type G_δ whose arithmetic sum is non-B-measurable, *Dokl. Akad. Nauk SSSR (N.S.)*, **99** (1954), 507–510.

Page 220, line 11-: $\mathrm{proj}_X A \to \mathrm{proj}_X(A)$

Page 227, line 3-: add space after comma

Page 229, line 8-: $\epsilon = 2 \to \epsilon/2$; $\epsilon = 2^{n+1} \to \epsilon/2^{n+1}$

Page 229, 29.16: R. Kaufman points out the following: "It is possible to derive a proof of this directly from 29.12 and 29.13. The difficulty is that the class of μ^*-measurable sets does not admit covers. But we can assume that $\mu^*(A)$ is finite and use the outer measure $\mu^{**}(B) = \mu^*(A \cap B)$. This admits covers since the total measure is finite."

Page 229: add after (29.14):

Note that 29.14 and 19.14 give another proof of 21.9.

Page 247, Fig. 33.1: insert ... between the last two segments

Page 256, line 8: replace "open" by "an intersection of an open and a closed set"

Page 256, line 11-: show now → now show

Page 256, line 19: $U_n \neq \emptyset\} \to U_n \neq \emptyset)$

Page 263, line 9-: add "(by 12.14)" after "it is enough"

Page 263, line 10: $l^1 \to \ell^1$

Page 267, line 3-: after Δ add ", provided Γ is closed under continuous preimages"

Page 268, 34.1, 3rd line: add "if" before φ; add comma after "rank"; delete "iff"

Page 269, line 16-: add "regular" before "Γ-rank"

Page 283, 35.5, 3rd line: $\mathcal{C} \to D$

Page 297, 35.47, 3rd line: add "compact," before K_σ

Page 304, line 5: add "a" between "is" and "function"

Page 322, line 3: delete "does so"

Page 324, line 3: properties $\to$ property

Page 342, line 17: after "$A \subseteq X \times \mathcal{N}$" add ", $A \in \Gamma$,"

Page 347, line 7: after "group representation theory," add "ergodic theory,"

Page 347, line 7-: before "all" add ", under definable determinacy,"

Page 350, line 4: add "a" before "unique"

Page 353, line 6-: express $\to$ expressing

Page 354, line 2-: "resp." $\to$ "resp.,"

Page 358, line 1: $G \to X$

Page 359, 17.43, second line: add space around first "="

Page 360, line 2: $f(\mathcal{N}_s) \to f(N_s)$

Page 360, line 8: $f \to g$

Page 360, line 3-: replace "J. Mycielski [1973] and K. Kuratowski [1973]" by "K. Kuratowski [1973] and J. Mycielski [1973]"

Page 362: in the hint for 23.4, line 4, after "$x \in A$" add "and the finite sets"

Page 363: after item **25.19** add item: **26.8.** See J.R. Steel [1980]

Page 364: after **27.10** add:

27.D. Dougherty (private communication) has found another proof, avoiding methods of effective descriptive set theory, that $\cong$ on LO is $\mathbf{\Sigma}^1_1$-complete. It goes as follows:

The set of linear orderings which are not well-orderings is $\mathbf{\Sigma}^1_1$-complete (see 27.12), so it suffices to produce two Borel maps A, B from LO to LO such that x is not a wellordering iff $A(x)$ is isomorphic to $B(x)$.

For a linear ordering $\mathcal{L}$, let $\mathcal{L}^*$ be the set of finite strictly decreasing sequences from $\mathcal{L}$, ordered lexicographically. Note that there is always a least member of $\mathcal{L}^*$, namely the empty sequence (when we refer to an element of a structure we of course mean an element of its universe).

Claim 1. $\mathcal{L}^* \times \mathcal{M}^*$ is isomorphic to $(\mathcal{L}+\mathcal{M})^*$ (where $\times$ refers to multiplication and $+$ to addition of linear orderings).

Proof. If $s \in \mathcal{L}^*$ and $t \in \mathcal{M}^*$, then the concatenation of t, s is in $(\mathcal{L} + \mathcal{M})^*$; it is easy to check that this gives an order-preserving bijection from $\mathcal{L}^* \times \mathcal{M}^*$ to $(\mathcal{L} + \mathcal{M})^*$.

Claim 2. If $\mathcal{L}$ is a wellordering, then $\mathcal{L}^*$ is a well-ordering.

Proof. Suppose not; say we have an infinite decreasing sequence of members of $\mathcal{L}^*$. Since $\mathcal{L}$ is a wellordering, the first coordinates of these sequences must eventually stabilize; then the second coordinates must reach a limiting value, and so on. Thus this sequence of limiting values will be an infinite decreasing sequence in $\mathcal{L}$, a contradiction.

Claim 3. If $\mathcal{L}$ is a countable linear ordering with no least element, then $\mathcal{L}^*$ is isomorphic to $1 + \mathbb{Q}$.

Proof. It suffices to prove that $\mathcal{L}^*$ is dense and has no greatest element (we already know it has a least element). Given $s \in \mathcal{L}^*$, choose $i \in \mathcal{L}$ less than all members of s; then the concatenation of s with i is greater than s in $\mathcal{L}^*$. Now suppose that $s < t$ in $\mathcal{L}^*$. If t is an extension of s, let i be the first coordinate of t beyond s, and find $i' < i$ in $\mathcal{L}$; then the concatenation of s, i' is between s and t in $\mathcal{L}^*$. If s and t disagree at some coordinate, find i in $\mathcal{L}$ less than the last coordinate of s; then the concatenation of s, i is between s and t in $\mathcal{L}^*$.

Now, if $\mathcal{L}$ is a wellordering, then $\mathcal{L}^*$ is a wellordering, so $\mathcal{L}^* + \mathcal{L}^*$ is not isomorphic to $\mathcal{L}^*$. If $\mathcal{L}$ is countable but not a wellordering, let $\mathcal{L}_1$ be the maximal well-founded initial segment of $\mathcal{L}$; then we have $\mathcal{L} \cong \mathcal{L}_1 + \mathcal{L}_2$, where $\mathcal{L}_2$ has no least element. We now have that $\mathcal{L}^*$ is isomorphic to $\mathcal{L}_1^* \times (1 + \mathbb{Q})$; since $1 + \mathbb{Q} + 1 + \mathbb{Q}$ is isomorphic to $1 + \mathbb{Q}$, $\mathcal{L}^* + \mathcal{L}^*$ is isomorphic to $\mathcal{L}^*$. (It could happen that $\mathcal{L}_1$ is empty, in which case $\mathcal{L}^* \cong 1 + \mathbb{Q}$.) Therefore, $\mathcal{L}$ is not a wellordering iff $\mathcal{L}^* \cong (\mathcal{L}^* + \mathcal{L}^*)$. It is easy now to find Borel maps A, B from LO to LO such that for $x \in$ LO, $\mathcal{A}_{A(x)} \cong (\mathcal{A}_x)^*$ and $\mathcal{A}_{B(x)} \cong ((\mathcal{A}_x)^* + (\mathcal{A}_x)^*)$, and the proof is complete.

Page 367, 37.4: (R. Kaufman) replace "$\exists w \in \mathrm{WO} \ldots w(f(s), f(t)) = 1)$" by "$\exists T \in \mathrm{WF} \forall s (x \in P_s \Rightarrow s \in T)$"

Page 370, line 1: add comma after "KAHANE"

Page 371, line 15: add comma after "GLADDINES"

Page 378: add comma after "MOSCHOVAKIS"

Page 379: after entry "S. M. SRIVASTAVA" add entry:

J. R. STEEL
[1980] Analytic sets and Borel isomorphisms, *Fund. Math.*, **108**, 83-88

Page 382, line11-: add $\bigvee$ after $\vee$

Page 383, line 5: $\mathbf{\Sigma^1_1} \to \mathbf{\Sigma^1_1}(X)$

Page 398: add after the entry "reduction property 170" the entry "reflection theorems 285,286"

Page 400: In entry "Steel, J. R." add after "Martin-Steel [1989] 206" the item
Steel [1980] 363