THE FIRST NEGATIVE HECKE EIGENVALUE OF GENUS 2 SIEGEL CUSPFORMS WITH LEVEL $N \geq 1$

JIM BROWN

Abstract. In this short note we extend results of Kohnen and Sengupta on the sign of eigenvalues of Siegel cuspforms. We show that their bound for the first negative Hecke eigenvalue of a genus 2 Siegel cuspform of level 1 extends to the case of level $N > 1$. We also discuss the signs of Hecke eigenvalues of CAP forms.

1. Introduction

Let $\mathcal{S}_k(\mathrm{Sp}_4(\mathbb{Z}))$ be the space of Siegel cuspforms of weight k and level $\mathrm{Sp}_4(\mathbb{Z}) \subset \mathrm{GL}_4(\mathbb{Z})$. Denote the space of Maass spezialchars by $\mathcal{S}_k^{\mathrm{M}}(\mathrm{Sp}_4(\mathbb{Z})) \subset \mathcal{S}_k(\mathrm{Sp}_4(\mathbb{Z}))$. Let $F \in \mathcal{S}_k(\mathrm{Sp}_4(\mathbb{Z}))$ be a nonzero Hecke eigenform of all Hecke operators $T(n)$ with $n > 0$. Write $\lambda_F(n)$ for the eigenvalue of $T(n)$ acting on F. It was shown by Breulmann in [B99] that $F \in \mathcal{S}_k^{\mathrm{M}}(\mathrm{Sp}_4(\mathbb{Z}))$ if and only if $\lambda_F(n) > 0$ for all $n > 0$. Essentially this boiled down to an elementary calculation combined with the fact that $F \in \mathcal{S}_k^{\mathrm{M}}(\mathrm{Sp}_4(\mathbb{Z}))$ if and only if the Spinor L-function $L(s, F, \mathrm{Spin})$ has a pole at $s = k$ ([E81]). This result naturally leads one to ask the question of what can be said about the signs of $\lambda_F(n)$ for $F \notin \mathcal{S}_k^{\mathrm{M}}(\mathrm{Sp}_4(\mathbb{Z}))$. In [K07] it is shown that for such F the values $\lambda_F(n)$ change sign infinitely often. Furthermore, in [KS07] it is shown that if k is odd or $F \notin \mathcal{S}_k^{\mathrm{M}}(\mathrm{Sp}_4(\mathbb{Z}))$ then there exists $n \ll k^2 \log^{20} k$ such that $\lambda_F(n) < 0$ where the implied constant is absolute and effectively computable. It is then natural to ask what can be said in the case of level $\Gamma_0^2(N)$ for $N > 1$.

The natural generalization of the Maass spezialchars to the case of level $\Gamma_0^2(N)$ is the notion of CAP forms (see § 3.) We show (essentially a result of [PS08]) that Breulmann's result generalizes to the level $\Gamma_0^2(N)$ situation as well. See Theorem 3.1 for the precise result. Once we have dealt with CAP forms, we look at the case of non-CAP forms. We then generalize Kohnen and Sengupta's arguments from [KS07] to the case of level $\Gamma_0^2(N)$ to show that their bound of $n \ll k^2 \log^{20} k$ holds in this case as well. See Theorem 4.1 for the precise statement.

2000 *Mathematics Subject Classification.* Primary 11F46.

Key words and phrases. Siegel modular form, Hecke operator, CAP form, Maass subspace.

2. Notation and Set-up

Throughout the paper we write $A \ll B$ to mean there is an absolute constant c so that $A \leq cB$. If the constant is not absolute, say it depends on k, we write $A \ll_k B$.

Let $\mathrm{G} = \mathrm{GSp}_4$, i.e.,

$$\mathrm{G} = \{g \in \mathrm{GL}_4 : \, {}^t g J g = \lambda(g) J, \, \lambda(g) \in \mathrm{GL}_1\}$$

where $J = \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix}$. We have a natural map $\lambda : G \to \mathrm{GL}_1$. The kernel of this map is the familiar group Sp_4. For N a positive integer, set $\Gamma_0^2(N) \subset \mathrm{Sp}_4(\mathbb{Z})$ to be the subgroup defined by

$$\Gamma_0^2(N) = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Z}) : C \equiv 0 (\mathrm{mod}\, N) \right\}.$$

The group

$$\mathrm{G}^+(\mathbb{R}) = \{g \in \mathrm{G}(\mathbb{R}) : \lambda(g) > 0\}.$$

acts on the Siegel upper half-space

$$\mathfrak{h}^2 = \{Z \in \mathrm{M}_2(\mathbb{C}) : \, {}^t Z = Z, \, \mathrm{Im}(Z) > 0\}$$

by linear fractional transformation in the usual way.

Let $F : \mathfrak{h}^2 \to \mathbb{C}$ be a holomorphic function. The group $\mathrm{G}^+(\mathbb{R})$ acts on F via the slash operator

$$(F|_k g)(Z) = \lambda(g)^k j(g,Z)^{-k} F(gZ)$$

where $j(g,Z) = \det(CZ + D)$ is the usual automorphy factor. The space of Siegel modular forms of weight k and level $\Gamma_0^2(N)$ is the space of such F with the condition that $(F|_k g)(Z) = F(Z)$ for all $g \in \Gamma_0^2(N)$. This space is denoted $\mathcal{M}_k(\Gamma_0^2(N))$ and we denote the subspace of cusp forms by $\mathcal{S}_k(\Gamma_0^2(N))$. We have the usual Hecke operators $T(n)$ for $p \nmid N$ as defined in [A74]. We denote the Frobenius operators of Andrianov for $p \mid N$ by $T(p)$, see [A01] for example for the definition.

Let $F \in \mathcal{S}_k(\Gamma_0^2(N))$ be a Hecke eigenform. Associated to F is a cuspidal automorphic form Φ_F defined as follows. Write $N = \prod p^{r_p}$ (we set $r_p = 0$ for $p \nmid N$) and define $\mathrm{K}_0(N)$ by

$$\mathrm{K}_0(N) = \prod_{p \nmid \infty} \mathrm{K}_0(p^{r_p})$$

where

$$\mathrm{K}_0(p^{r_p}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{G}(\mathbb{Z}_p) : C \equiv 0 (\mathrm{mod}\, p^{r_p} \mathbb{Z}_p) \right\}.$$

Strong approximation for $\mathrm{G}(\mathbb{A})$ allows us to write

$$\mathrm{G}(\mathbb{A}) = \mathrm{G}(\mathbb{Q})\, \mathrm{G}^+(\mathbb{R})\, \mathrm{K}_0(N)$$

where

Thus, given $g \in \mathrm{G}(\mathbb{A})$ there exists $g_{\mathbb{Q}} \in \mathrm{G}(\mathbb{Q})$, $g_\infty \in \mathrm{G}^+(\mathbb{R})$, and $k_0 \in \mathrm{K}_0(N)$ such that $g = g_{\mathbb{Q}} g_\infty k_0$. Define $\Phi_F : \mathrm{G}(\mathbb{A}) \to \mathbb{C}$ by

$$\Phi_F(g) = (F \mid g_\infty)(i1_2).$$

Note that Φ_F is well defined since F has level $\Gamma_0^2(N)$ and

$$\Gamma_0^2(N) = \mathrm{G}(\mathbb{Q}) \cap \mathrm{G}^+(\mathbb{R})\, \mathrm{K}_0(N).$$

Let V_F denote the space of right translates of Φ_F. The group $\mathrm{G}(\mathbb{A})$ acts on V_F by right translation. We can decompose the space V_F into a finite direct sum of irreducible cuspidal automorphic representations of $\mathrm{G}(\mathbb{A})$. Let π_F be one of these irreducible components and write $\pi_F = \otimes \pi_{F,p}$.

Let χ_1, χ_2 and σ be unramified characters of $\mathbb{Q}_p^\times$. Denote by $\chi_1 \times \chi_2 \rtimes \sigma$ the representation of $\mathrm{G}(\mathbb{Q}_p)$ induced from the character of the Borel subgroup of $\mathrm{G}(\mathbb{Q}_p)$ given by

$$\begin{pmatrix} a_1 & * & * & * \\ & a_2 & * & * \\ & & b_1 a_1^{-1} & \\ & & * & b_1 a_2^{-1} \end{pmatrix} \mapsto \chi_1(a_1)\chi_2(a_2)\sigma(b_1).$$

Adopting the notation of [ST], for $p \nmid N$ we have that $\pi_{F,p}$ is isomorphic to the Langlands quotient of an induced representation of the form $\chi_1 \times \chi_2 \rtimes \sigma$.

We can attach a degree 4 Spinor L-function to F as either the Langlands Spinor L-function or the Andrianov Spinor L-function. The only difference is at the primes $p \mid N$. When we wish to refer to the Langlands L-function we will always use the notation $L(s, \pi_F, \mathrm{Spin})$ and for the Andrianov L-function we will write $L(s, F, \mathrm{Spin})$. For $p \nmid N$, the pth Euler factor is given by

$$\begin{aligned} L_p(s, \pi_F, \mathrm{Spin}) &= L_p(s, F, \mathrm{Spin}) \\ &= [(1 - \alpha_{p,0} p^{-s})(1 - \alpha_{p,0}\alpha_{p,1} p^{-s})(1 - \alpha_{p,0}\alpha_{p,2} p^{-s})(1 - \alpha_{p,0}\alpha_{p,1}\alpha_{p,2} p^{-s})]^{-1} \end{aligned}$$

where $\alpha_{p,1} = \chi_1(p)$, $\alpha_{p,2} = \chi_2(p)$, and $\alpha_{p,0} = \sigma(p)$ are the pth Satake parameters of F. We have normalized the L-function here is a somewhat non-standard manner. It amounts to substituting $s + k - 3/2$ for s in Andrianov's normalization. Note that by our choice of normalization here we have $\alpha_{p,0}^2 \alpha_{p,1} \alpha_{p,2} = 1$. For $p \mid N$, the pth Euler factors defining the Andrianov Spinor L-function are given by

$$L_p(s, F, \mathrm{Spin}) = (1 - \lambda_F(p) p^{-s})^{-1}$$

where $\lambda_F(p)$ now refers to the eigenvalue of the Frobenius operator acting on F as defined in [A01]. Again, our normalization of the L-function here means our $\lambda_F(p)$ differs from Andrianov's by a factor of $p^{k-3/2}$. The Andrianov Spinor L-function satisfies the functional equation given by

$$\Lambda_F(s) = (-1)^k \Lambda_F(1 - s)$$

where

$$\Lambda_F(s) = (2\pi)^{-2(s+k-3/2)}\Gamma(s+k-3/2)\Gamma(s+1/2)L(s,F,\mathrm{Spin}).$$

3. CAP forms

In this section we give the relevant definitions and results generalizing those of $F \in \mathcal{S}_k^{\mathrm{M}}(\mathrm{Sp}_4(\mathbb{Z}))$ to the case of level $\Gamma_0^2(N)$ for $N > 1$. For a more detailed exposition of the material in this section the reader is urged to consult [PS08] or [PS].

Let $\mathrm{P} = \mathrm{M}\,\mathrm{N}$ be a proper parabolic subgroup of $\mathrm{G}(\mathbb{A})$ where M is the Levi subgroup. Let τ be an irreducible cuspidal automorphic representation of M. A cuspidal automorphic represetation π of $\mathrm{G}(\mathbb{A})$ is said to be CAP (cuspidal associated to parabolic) if there is an irreducible component π' of $\mathrm{Ind}_{\mathrm{P}(\mathbb{A})}^{\mathrm{G}(\mathbb{A})}\tau$ so that $\pi_p \cong \pi'_p$ for almost all places p. Our interest in CAP forms is that they provide the natural generalization of Saito-Kurokawa lifts when we consider $N > 1$. In particular, if π_F is CAP then it must be CAP to the Siegel parabolic ([PS], Corollary 4.5). If $N = 1$ then π_F is CAP if and only if F is a classical Saito-Kurokawa lifting. Suppose now that $N > 1$. In general one has π is CAP if and only if it is a theta lift or a theta lift twisted by an idele class character. We know from [P83] that if $F \in \mathcal{S}_k^{\mathrm{M}}(\Gamma_0^2(N))$ then it is a theta lift and so CAP forms are a generalization of Saito-Kurokawa lifts. In general one has that π is a theta lift if and only if $L(s,\pi,\mathrm{Spin})$ has a pole ([P83].) If π is a twist of a theta lift by a non-trivial character then $L(s,\pi,\mathrm{Spin})$ has no poles. One should observe that since we are assuming F is without character, we can say that π_F is CAP if and only if it is either a theta lift or a twist of a theta lift by a quadratic character. In such a case, we have the following characterization of the local representations $\pi_{F,p}$ for $p \nmid N$ (see [PS08].)

1. If π_F is a theta lift, then for $p \nmid N$ the local representation $\pi_{F,p}$ is the spherical constituent of the induced representation of the form $\nu^{1/2}\chi \times \nu^{1/2}\chi^{-1} \rtimes \nu^{-1/2}$ with $|\chi| = 1$ and ν the normalized absolute value from $\mathbb{Q}_p^\times \to \mathbb{C}^\times$.
2. If π_F is the twist of a theta lift by a quadratic character $\sigma_0 = \otimes\sigma_{0,p}$, then for each $p \nmid N$ for which $\sigma_{0,p}$ is unramified, the local representation is the spherical constituent of the induced representation of the form $\nu^{1/2}\chi \times \nu^{1/2}\chi^{-1} \rtimes \nu^{-1/2}\sigma_{0,p}$ with $|\chi| = 1$.

The following theorem is essentially Theorem 3.1 of [PS08]. The second part of the theorem is not stated there, but it is easily deduced from their arguments. We include a proof for the reader's convenience.

Theorem 3.1. *Let N and k be positive integers with $k > 2$. Let $F \in \mathcal{S}_k(\Gamma_0^2(N))$ be a non-zero Hecke eigenform with eigenvalues $\lambda_F(n)$ for all n with $\gcd(n,N) = 1$. Let $\pi_F = \otimes\pi_{F,p}$ be the corresponding irreducible cuspidal automorphic representation of $\mathrm{G}(\mathbb{A})$.*

(1) *If π_F is a theta lift, then for all $p \nmid N$ and all $n > 0$ we have $\lambda_F(p^n) > 0$.*
(2) *Suppose π_F is a twist of a theta lift by a non-trivial quadratic character σ_0. For those primes where $\sigma_0(p) = 1$ we have that $\lambda_F(p^n) > 0$ for all $n > 0$. For those primes where $\sigma_0(p) = -1$, we have that $\lambda_F(p^n) > 0$ for n even and $\lambda_F(p^n) < 0$ for n odd.*

Proof. Proposition 4.1 of [PS] shows that if $\pi_{F,p}$ is given by $\chi_1 \times \chi_2 \rtimes \sigma$, then for $n > 0$ one has

$$\frac{\lambda_F(p^n)}{(p^n)^{k-3/2}} = A_{a,b}(n) + (1 - 1/p) \sum_{j=1}^{[n/2]} A_{a,b}(n - 2j) \tag{1}$$

where

$$A_{a,b}(m) = \left(\sum_{j=0}^{m} a^{m-j} b^j \right) \left(\sum_{j=0}^{m} (ab)^{-j} \right)$$

where $a = \sigma(p)$ and $b = \sigma(p)\chi_1(p)$.

In the situation of π_F being a theta lift, using the characterization of $\pi_{F,p}$ given above we have that $a = \nu(p)^{-1/2} = p^{1/2}$ and $b = \chi(p)$. Thus,

$$A_{a,b}(m) = p^{m/2} \left| \sum_{j=0}^{m} (p^{1/2}\chi(p))^{-j} \right|^2 > 0.$$

Hence, we see that $\lambda_F(p^n) > 0$ for all $n > 0$ and $p \nmid N$.

Suppose now that π_F is the twist of a theta lift by a non-trivial quadratic character σ_0. In this case we obtain that $a = p^{1/2}\sigma_{0,p}(p)$ and $b = \chi(p)\sigma_{0,p}(p)$. From this we calculate that

$$A_{a,b}(m) = (p^{1/2}\sigma_{0,p}(p))^m \left| \sum_{j=0}^{m} (p^{1/2}\chi(p))^{-j} \right|^2 .$$

From this it is clear that if $\sigma_{0,p}(p) = 1$ then $A_{a,b}(m)$ is positive for all m used in equation (1) and so $\lambda_F(p^n) > 0$. If $\sigma_{0,p}(p) = -1$, then we see that $A_{a,b}(m) > 0$ for m even and $A_{a,b}(m) < 0$ for m odd. Equation (1) then clearly gives the result. □

Remark 3.2. One should note that it is necessary to remove the eigenvalues $\lambda_F(p)$ for $p \mid N$ in the above theorem. For example, if $F \in \mathcal{S}_k^{\mathrm{M}}(\Gamma_0^2(N))$ is a Saito-Kurokawa lift of $f \in S_{2k-2}(\Gamma_0(N))$, then $\lambda_F(p) = \lambda_f(p)$ for all $p \mid N$. In this case it is entirely possible that $\lambda_F(p) < 0$ for $p \mid N$. For example, if $N = 11$ and $k = 3$, then the dimension of $S_4(\Gamma_0(11))$ is 2 and each newform has $\lambda_f(11) < 0$. For $N = 7$ and $k = 7$ we have that the dimension of $S_7(\Gamma_0(7))$ is dimension 7 and one has newforms with $\lambda_f(7) > 0$ and newforms with $\lambda_f(7) < 0$.

4. Non-CAP forms

Let $F \in \mathcal{S}_k(\Gamma_0^2(N))$ be such that π_F is not CAP. Let F have Satake parameters $\alpha_{p,0}, \alpha_{p,1}$ and $\alpha_{p,2}$ as in § 2. The Ramanujan-Petersson conjecture states that

$$|\alpha_{p,1}| = |\alpha_{p,2}| = 1$$

for all $p \nmid N$. A proof of this conjecture has been announced in [W93] and we assume its validity throughout this section. We assume F is a newform, where we take the definition of newform given in [AP]. We follow the arguments of [KS07] in this section. The goal of this section is to prove the following theorem.

Theorem 4.1. *Let $F \in \mathcal{S}_k(\Gamma_0^2(N))$ be a non-zero Siegel newform such that π_F is not CAP. There exists a positive integer n with*

$$n \ll k^2 \log^{20} k$$

such that $\lambda_F(n) < 0$.

Write

$$L(s, F, \mathrm{Spin}) = \sum_{n \geq 1} a_F(n) n^{-s}$$

and

$$L(s, F, \mathrm{Spin}) = \left(\prod_{p \mid N} (1 - \lambda_F(p) p^{-s})^{-1} \right) \sum_{n \geq 1} b_F(n) n^{-s}.$$

From our normalization of the Satake parameters along with the Ramanujan-Petersson conjecture we see that

$$|b_F(n)| \leq d_4(n)$$

where

$$\zeta^4(s) = \sum_{n \geq 1} d_4(n) n^{-s}.$$

Theorem 18 of [AP] gives $|\lambda_F(p)| = p^{-1/2}$ for $p \mid N$ with our normalization and choice that F be a newform and so $\prod_{p \mid N}(1 - \lambda_F(p)p^{-1})^{-1}$ is bounded. We combine this with the fact that $\zeta^4(s)$ has a pole of order 4 at $s = 1$ to conclude by a standard Tauberian argument that

$$\sum_{x_0 \leq n \leq x} |a_F(n)| \ll_{x_0} x \log^3 x$$

for $x_0 > 1$. From this estimate we conclude exactly as in [KS07] that we have:

Proposition 4.2. *For $c > 1$ we have*

$$|L(c + it, F, \mathrm{Spin})| \ll 1 + \frac{c}{(c-1)^4}$$

for all $t \in \mathbb{R}$.

One can see Proposition 1 of [KS06] for a proof of this type of result in a slightly different setting. We can now apply the same argument as in (Page 56-57, [KS07]) to conclude the following theorem.

Proposition 4.3. *Let $F \in \mathcal{S}_k(\Gamma_0^2(N))$ be a non-zero newform such that π_F is not CAP. Let $0 < \delta < 1/2$. Then for all $t \in \mathbb{R}$ we have*

$$|L(\delta + it, F, \mathrm{Spin})| \ll k^{1-\delta} \log^4 k \left| 1 + \frac{1}{2\log k} + \delta + it \right|^{2-2\delta+1/\log k}. \tag{2}$$

We can use Proposition 4.3 and the same proof as in [KS07] to obtain the following generalization of Proposition 2 of [KS07].

Proposition 4.4. *Let $F \in \mathcal{S}_k(\Gamma_0^2(N))$ be a non-zero newform such that π_F is not CAP. We have*

$$\sum_{n \le x} \lambda_F(n) \log^2\left(\frac{x}{n}\right) \ll (k \log^8 k) x^{\frac{2}{3\log k}}.$$

Finally, we give a lower bound for the sum of eigenvalues.

Proposition 4.5. *Let $F \in \mathcal{S}_k(\Gamma_0^2(N))$ be a non-zero newform such that π_F is not CAP. Suppose that $\lambda_F(n) \ge 0$ for $1 \le n \le x$. Then we have*

$$\sum_{n \le x} \lambda_F(n) \log^2\left(\frac{x}{n}\right) \gg \frac{\sqrt{x}}{\log^2 x}.$$

Proof. It is straightforward to show that

$$\sum_{n \le x} \lambda_F(n) \log^2\left(\frac{x}{n}\right) \gg \sum_{n \le x/2} \lambda_F(n).$$

Thus, if we can show that

$$\sum_{n \le x} \lambda_F(n) \gg \frac{\sqrt{x}}{\log^2 x}$$

we will be done.

For $p \nmid N$ we have

$$L_p(X, F, \mathrm{Spin})^{-1} = 1 - \lambda_F(p)X + (\lambda_F(p)^2 - \lambda_F(p^2) - 1/p)X^2 - \lambda_F(p)X^3 + X^4$$

and for $p \mid N$ we have

$$L_p(X, F, \mathrm{Spin})^{-1} = 1 - \lambda_F(p)X.$$

We can use the fact that

$$\zeta(2s+1)^{-1} L(s, F, \mathrm{Spin}) = \sum_{n \ge 1} \lambda_F(n) n^{-s}$$

to conclude that for all p we have

$$(1 - X^2/p) L_p(X, F, \mathrm{Spin}) = \sum_{n \ge 0} \lambda_F(p^n) X^n.$$

Thus, for $p \nmid N$ we have

$$\begin{aligned}\lambda_F(p^n) =&\lambda_F(p)\lambda_F(p^{n-1}) - (\lambda_F(p)^2 - \lambda_F(p^2) - 1/p)\lambda_F(p^{n-2}) \\ &+ \lambda_F(p)\lambda_F(p^{n-3}) - \lambda_F(p^{n-4})\end{aligned} \tag{3}$$

where we put $\lambda_F(p^n) = 0$ for $n < 0$. For $p \mid N$ we have

$$\lambda_F(p^2) = \lambda_F(p)^2 - 1/p,$$
$$\lambda_F(p^n) = \lambda_F(p)\lambda_F(p^{n-1}) \quad (n > 2).$$

With our normalization of the Satake parameters, the Ramanujan-Petersson conjecture, and equation (3) we have

$$|\lambda_F(p)|, |\lambda_F(p^2)|, |\lambda_F(p^3)| \ll 1. \tag{4}$$

Let $S = \{p : p \nmid N, p \leq \sqrt[4]{x}\}$. Since we are assuming that $\lambda_F(n) \geq 0$ for $n \leq x$, we have

$$\sum_{n \leq x} \lambda_F(n) \geq \sum_{p,q \in S} \lambda_F(p^2 q^2) + \sum_{p,q \in S} \lambda_F(p^2 q) + \sum_{p,q \in S} \lambda_F(pq). \tag{5}$$

As is shown in [KS07], for $p \in S$ we have

$$\lambda_F(p^4) \gg \lambda_F(p^2)^2 - c_1$$
$$\lambda_F(p^3) \gg \lambda_F(p)\lambda_F(p^2) - c_2$$
$$\lambda_F(p^2) \gg \lambda_F(p)^2 - c_3$$

where c_1, c_2 and c_3 are absolute constants each greater then 0. Let $\pi(x)$ denote the number of primes $p \leq x$ for any $x > 1$. Then we have

$$\sum_{p,q \in S} \lambda_F(p^2 q^2) \gg \left(\sum_{p \in S} \lambda_F(p^2)\right)^2 - c_1 \pi(\sqrt[4]{x}),$$
$$\sum_{p,q \in S} \lambda_F(p^2 q) \gg \left(\sum_{p \in S} \lambda_F(p^2)\right)\left(\sum_{p \in S} \lambda_F(p)\right) - c_2 \pi(\sqrt[4]{x}),$$
$$\sum_{p,q \in S} \lambda_F(pq) \gg \left(\sum_{p \in S} \lambda_F(p)\right)^2 - c_3 \pi(\sqrt[4]{x}).$$

Combining these equations with equation (5) we obtain

$$\sum_{n \leq x} \lambda_F(n) \gg \left(\sum_{p \in S} \lambda_F(p^2) + \sum_{p \in S} \lambda_F(p)\right)^2 - c\pi(\sqrt[4]{x}) \tag{6}$$

with $c > 0$ an absolute constant.

We claim that there exists an absolute constant $d > 0$ so that for any $p \in S$ we have

$$\lambda_F(p^2) + \lambda_F(p) \geq d.$$

Suppose not. By assumption $\lambda_F(p^2)$ and $\lambda_F(p)$ are both greater then or equal to 0, we must have that $\lambda_F(p^2)$ and $\lambda_F(p)$ are both small. Equation (3) gives

$$\lambda_F(p^3) = \lambda_F(p)\lambda_F(p^2) - (\lambda_F(p)^2 - \lambda_F(p^2) - 1/p)\lambda_F(p) + \lambda_F(p),$$

and so $\lambda_F(p^3)$ must be small as well. However, equation (3) also shows that $\lambda_F(p^4)$ is given by

$$\lambda_F(p^4) = (\lambda_F(p^2)^2 + \lambda_F(p)\lambda_F(p^3) + \lambda_F(p^2)(1/p - \lambda_F(p)^2) + \lambda_F(p)^2) - 1.$$

This contradicts $\lambda_F(p^4) \geq 0$ if $\lambda_F(p^2)$ and $\lambda_F(p)$ are arbitrarily small. Thus, such a $d > 0$ exists. We combine this fact with equation (6) along with the prime number theorem to conclude that

$$\sum_{n \geq x} \lambda_F(n) \gg \frac{\sqrt{x}}{\log^2 x}.$$

□

Combining the previous two propositions we see that if $F \in \mathcal{S}_k(\Gamma_0^2(N))$ is a newform such that π_F is not CAP and $\lambda_F(n) \geq 0$ for $n \leq x$ we have

$$\frac{\sqrt{x}}{\log^2 x} \ll (k \log^8 k) x^{\frac{2}{3 \log k}}. \tag{7}$$

However, this equation cannot hold for large enough x. In particular, following [KS07] we see that for equation (7) to hold we must have

$$x \ll k^2 \log^{20} k,$$

which finishes the proof of Theorem 4.1.

Remark 4.6. For $F \in \mathcal{S}_k(\mathrm{Sp}_4(\mathbb{Z}))$ a Hecke eigenform, it was shown in [K07] that $\lambda_F(n)$ changes sign infinitely often. This result has been generalized in [PS08] to the case $N > 1$. Their result shows that if F is a Hecke eigenform for all $T(n)$ with $\gcd(n, N) = 1$ such that π_F is not CAP, then there exists an infinite set S_F of primes numbers $p \nmid N$ such that if $p \in S_F$, then there are infinitely many r such that $\lambda_F(p^r) > 0$ and infinitely many r such that $\lambda_F(p^r) < 0$.

References

[A74] A. Andrianov, *Euler products corresponding to Siegel modular forms of genus 2*, Russian Math. Surveys 29, 45-116 (1974).

[A01] A. Andrianov, *On functional equations satisfied by Spinor Euler products for Siegel modular forms of genus 2 with characters*, Abh. Math. Sem. Univ. Hamburg 71, 123-142 (2001).

[AP] A. Andrianov and A. Panchishkin, *Singular Frobenius operators on Siegel modular forms with characters, and zeta functions*, St. Petersburg Math. J. Vol 12, no 2, 233-257 (2001).

[B99] S. Breulmann, *On Hecke eigenforms in the Maass space*, Math Z. 232, 527-530 (1999).

[E81] S.A. Evdokimov, *A characterization of the Maass space of Siegel cusp forms of second degree*, Math USSR Sb. 40, 541-558 (1981).

[K07] W. Kohnen, *Sign changes of Hecke eigenvalues of Siegel cusp forms of genus two*, Proc. Amer. Math. Soc. 135, 997-999 (2007).

[KS06] W. Kohnen and J. Sengupta, *On the first sign change of Hecke eigenvalues of newforms*, Math. Z. 254, 173-184 (2006).

[KS07] W. Kohnen and J. Sengupta, *The first negative Hecke eigenvalue of a Siegel cusp form of genus two*, Acta Arith. 129.1, 53-62 (2007).

[P83] I.I. Piatetski-Shapiro, *On the Saito-Kurokawa lifting*, Invent. Math. 71, 309-338 (1983).

[PS08] A. Pitale and R. Schmidt, *Sign changes of Hecke eigenvalues of Siegel cusp forms of degree 2*, Proc. Amer. Math. Soc., posted June 2, 2008, S 0002-9939(08)09364-7(to appear in print).

[PS] A. Pitale and R. Schmidt, *Ramanujan-type results for Siegel cusp forms of degree 2*, preprint.

[ST] P. Sally and M. Tadić, *Induced representations and classifications for* $\mathrm{GSp}(2, F)$ *and* $\mathrm{Sp}(2, F)$, Bull. Soc. Math. France 121, 209-240 (2005).

[W93] R. Weissauer, *The Ramanujan conjecture for genus 2 Siegel modular forms (an application of the trace formula)*, preprint , Mannheim, (1993).

Department of Mathematics, Clemson University, Clemson, SC 29634
E-mail address: jimlb@clemson.edu