

IWASAWA THEORY AND MOTIVIC L-FUNCTIONS

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ABSTRACT. We illustrate the use of Iwasawa theory in proving cases of the (equivariant) Tamagawa number conjecture.

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The present paper is a continuation of the survey article [19] on the equivariant Tamagawa number conjecture. We give a presentation of the Iwasawa theory of imaginary quadratic fields from the modern point of Kato and Fukaya [21] and shall take the opportunity to briefly mention developments which have occurred after the publication of [19], most notably the limit formula proved by Gealy [23], and work of Bley

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[4], Johnson [27] and Navilarekallu [32]. Unfortunately, we do not say as much about the history of the subject as we had wished. Although originally planned we also did not include a presentation of the full GL_2 -Iwasawa theory of elliptic modular forms following Kato and Fukaya (see Colmez' paper [14] for a thoroughly p -adic presentation).

We believe however that the point of view of this paper (due to Kato, Fontaine-Perrin-Riou, Burns-Flach et al), presenting both the theory of complex L-values as well as of p -adic L-functions as the construction of trivializations of determinant line bundles, holds great promise for future research, and might eventually also include complex L-functions via the idea of a Weil-etale topology due to Lichtenbaum [31] (see also [20]).

It is a pleasure to dedicate this paper to Jean-Pierre Serre who has contributed so much to modern arithmetic geometry, and also inspired the general theory of motivic L-functions with his definition in [39].

Part 1. Motivic L-functions

In this part we sketch the theory of motivic L-functions insofar as it does not involve Iwasawa theory.

1. REVIEW OF THE TAMAGAWA NUMBER CONJECTURE

We recall the (equivariant) Tamagawa number conjecture in the formulation of Fontaine and Perrin-Riou, referring to [19] and references therein for further details.

We consider pure motives M over \mathbb{Q} which one can think of as being given by (a direct summand of) $h^i(X)(j)$ where

$$X \rightarrow \text{Spec}(\mathbb{Q})$$

is a smooth projective variety and $i, j \in \mathbb{Z}$. The motive M gives rise to a "motivic structure" consisting of the realisations and the motivic cohomology groups of M together with comparison isomorphisms and exact sequences relating these groups. While the reader may want to think of M in a less formal way, e.g. as an object of Voevodsky's category $DM_{gm}(\mathbb{Q})$, it is only the motivic structure that enters into the definition of the L-function of M as well as into the conjectures on its leading term.

We shall now summarize this data. For a \mathbb{Q} -vector space W and a \mathbb{Q} -algebra R we put $W_R = W \otimes_{\mathbb{Q}} R$ and set $W_l = W_{\mathbb{Q}_l}$. Suppose a finite dimensional semisimple \mathbb{Q} -algebra A acts on M . In order not to overburden the notation we suppose that A is commutative in this summary. One has

- For any prime number l a continuous A_l -representation $M_l = H_{\text{et}}^i(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_l)(j)$ of the Galois group $G_{\mathbb{Q}}$.
- The characteristic polynomial $P_p(T) = \det_{A_l}(1 - \text{Fr}_p^{-1} \cdot T | M_l^{I_p}) \in A_l[T]$ where $\text{Fr}_p \in G_{\mathbb{Q}}$ is a Frobenius element.
- A finitely generated A -space $M_B = H^i(X(\mathbb{C}), \mathbb{Q})(j)$ which carries an action of complex conjugation and a Hodge structure.
- A finitely generated filtered A -space $M_{dR} = H_{dR}^i(X/\mathbb{Q})(j)$.
- Motivic cohomology A -spaces $H_f^0(M)$ and $H_f^1(M)$ for both M and its dual $M^*(1)$.
- An $A_{\mathbb{R}}$ -linear map

$$\alpha_M : M_{B, \mathbb{R}}^+ \rightarrow (M_{dR} / \text{Fil}^0 M_{dR})_{\mathbb{R}}$$

induced by the $A_{\mathbb{C}}$ -linear period isomorphism $M_{B, \mathbb{C}} \cong M_{dR, \mathbb{C}}$.

- Let S be a finite set of primes containing l , ∞ and primes of bad reduction of M . There are distinguished triangles in the derived category of A_l -spaces

$$(1.1) \quad \begin{aligned} & R\Gamma_c(\mathbb{Z}[\frac{1}{S}], M_l) \rightarrow R\Gamma(\mathbb{Z}[\frac{1}{S}], M_l) \rightarrow \bigoplus_{p \in S} R\Gamma(\mathbb{Q}_p, M_l) \\ & R\Gamma_c(\mathbb{Z}[\frac{1}{S}], M_l) \rightarrow R\Gamma_f(\mathbb{Q}, M_l) \rightarrow \bigoplus_{p \in S} R\Gamma_f(\mathbb{Q}_p, M_l). \end{aligned}$$

One expects the following conjectures to hold. Given the length of this list it might be worthwhile to point out that there are cases where all the conjectures formulated below are true (see [19]).

Conjecture 1. (Existence of $L({}_A M, s)$ at $s = 0$)

- (Independence on l): $P_p(T)$ lies in $A[T]$ and is independent of l . This is known if M has good reduction at p and if $A = \mathbb{Q}$.
- The $A_{\mathbb{C}}$ -valued L -function

$$L({}_A M, s) := \prod_p P_p(p^{-s})^{-1},$$

which is defined and analytic for s with sufficiently large real part, has a meromorphic continuation to $s = 0$.

The Taylor expansion

$$L({}_A M, s) = L^*({}_A M) s^{r({}_A M)} + \dots$$

at $s = 0$ defines $L^*({}_A M) \in A_{\mathbb{R}}^{\times}$ and $r({}_A M) \in H^0(\text{Spec}(A), \mathbb{Z})$ and the aim of the Tamagawa number conjecture is to describe these quantities.

Conjecture 2. (Vanishing Order)

$$r({}_A M) = \dim_A H_f^1(M^*(1)) - \dim_A H_f^0(M^*(1))$$

Conjecture 3. (Motivic cohomology with \mathbb{R} -coefficients) The spaces $H_f^i(M)$ and $H_f^i(M^*(1))$ are finitely generated over A and there exists an exact sequence of $A_{\mathbb{R}}$ -spaces

$$\begin{aligned} 0 \rightarrow H_f^0(M)_{\mathbb{R}} \xrightarrow{c} \ker(\alpha_M) \rightarrow H_f^1(M^*(1))_{\mathbb{R}}^* \xrightarrow{h} \\ H_f^1(M)_{\mathbb{R}} \xrightarrow{r} \operatorname{coker}(\alpha_M) \rightarrow H_f^0(M^*(1))_{\mathbb{R}}^* \rightarrow 0 \end{aligned}$$

where c is a cycle class map, h a height pairing, and r the Beilinson regulator.

The exact sequence in Conjecture 3 induces an isomorphism

$${}_A \vartheta_{\infty} : \operatorname{Det}_{A_{\mathbb{R}}}(0) \cong \Xi({}_A M) \otimes_A A_{\mathbb{R}}$$

where

$$\begin{aligned} \Xi({}_A M) := & \operatorname{Det}_A(H_f^0(M)) \otimes \operatorname{Det}_A^{-1}(H_f^1(M)) \\ & \otimes \operatorname{Det}_A(H_f^1(M^*(1))^*) \otimes \operatorname{Det}_A^{-1}(H_f^0(M^*(1))^*) \\ & \otimes \operatorname{Det}_A^{-1}(M_B^+) \otimes \operatorname{Det}_A(M_{dR}/\operatorname{Fil}^0) \end{aligned}$$

is the so called fundamental line of M . All determinants here and in the following are understood in the graded sense [5]. For example $\operatorname{Det}_R(0) = (R, 0)$ for any commutative ring R .

Conjecture 4. (Rationality)

$${}_A \vartheta_{\infty}(L^*({}_A M)^{-1}) \in \Xi({}_A M) \otimes 1$$

Viewing $L^*({}_A M)^{-1}$ as an element of $A_{\mathbb{R}}^{\times} = \operatorname{Aut}(\operatorname{Det}_{A_{\mathbb{R}}}(0))$ this conjecture is equivalent to the statement that the composite isomorphism

$$\operatorname{Det}_{A_{\mathbb{R}}}(0) \xrightarrow{L^*({}_A M)^{-1}} \operatorname{Det}_{A_{\mathbb{R}}}(0) \xrightarrow{{}_A \vartheta_{\infty}} \Xi({}_A M) \otimes_A A_{\mathbb{R}}$$

sends 1 to $\Xi({}_A M) \otimes 1$, i.e. is the scalar extension from A to $A_{\mathbb{R}}$ of an isomorphism

$$(1.2) \quad \zeta_A(M) : \operatorname{Det}_A(0) \cong \Xi({}_A M).$$

Conjecture 5. (Motivic cohomology with \mathbb{Q}_l -coefficients) There are natural isomorphisms $H_f^0(M)_{\mathbb{Q}_l} \cong H_f^0(\mathbb{Q}, M_l)$ (cycle class map) and $H_f^1(M)_{\mathbb{Q}_l} \cong H_f^1(\mathbb{Q}, M_l)$ (Chern class map).

Since one can construct an isomorphism $H_f^i(\mathbb{Q}, M_l) \cong H_f^{3-i}(\mathbb{Q}, M_l^*(1))^*$ for all i Conjecture 5 computes the cohomology of $R\Gamma_f(\mathbb{Q}, M_l)$ in all degrees.

The exact triangle (1.1) induces an isomorphism

$${}_A\vartheta_l : \Xi({}_A M) \otimes_A A_l \cong \text{Det}_{A_l} R\Gamma_c(\mathbb{Z}[\frac{1}{S}], M_l).$$

Let $\mathfrak{A} \subseteq A$ be any \mathbb{Z} -order so that for each prime l there exists a $G_{\mathbb{Q}}$ -stable projective \mathfrak{A}_l -lattice $\mathfrak{M}_l \subset M_l$.

Conjecture 6. (Integrality)

$$\mathfrak{A}_l \cdot {}_A\vartheta_l {}_A\vartheta_{\infty}(L^*({}_A M)^{-1}) = \text{Det}_{\mathfrak{A}_l} R\Gamma_c(\mathbb{Z}[\frac{1}{S}], \mathfrak{M}_l)$$

This conjecture is equivalent to the statement that the composite isomorphism

$$\zeta_{A_l}(M_l) : \text{Det}_{A_l}(0) \xrightarrow{\zeta_A(M) \otimes_A A_l} \Xi({}_A M) \otimes_A A_l \xrightarrow{{}_A\vartheta_l} \text{Det}_{\mathfrak{A}_l} R\Gamma_c(\mathbb{Z}[\frac{1}{S}], M_l)$$

is the scalar extension from \mathfrak{A}_l to A_l of an isomorphism

$$(1.3) \quad \zeta_{\mathfrak{A}_l}(\mathfrak{M}_l) : \text{Det}_{\mathfrak{A}_l}(0) \cong \text{Det}_{\mathfrak{A}_l} R\Gamma_c(\mathbb{Z}[\frac{1}{S}], \mathfrak{M}_l).$$

2. LIMIT FORMULAS

Before one can begin to apply Iwasawa theory to the conjectural picture outlined in the last section one has to verify Conjectures 1-5, if not for M then at least for motivic points in an l -adic family deforming M_l (see the next part for precise definitions). In other words, Iwasawa theory generally only pertains to Conjecture 6 although ideas related to Iwasawa theory can sometimes also be used to address Conjecture 2 (as in the Coates-Wiles theorem [9]). The verification of Conjectures 1, 2, 4 currently involves the identification of $L({}_A M, s)$ with a tuple of automorphic L-functions and the ensuing explicit integral or series representations. Conjecture 3 was formulated by Fontaine and Perrin-Riou, generalizing conjectures of Hodge and Beilinson. Known special cases include the Dirichlet unit theorem, the Borel regulator for K-groups of number fields and the Mordell-Weil theorem. In most other

examples Conjecture 3 is only known in a weak form where one replaces motivic cohomology groups by a subspace of explicit elements. Conjecture 5 is known in even fewer cases and amounts to a "Tate conjecture" or finiteness of a "Tate-Shafarevich group" or "Leopoldt conjecture".

In order to then address Conjecture 6 it is not sufficient to know Conjecture 4 as such (i.e. up to an unspecified factor in A^\times) but one needs to know it in a precise form. This is what we mean by a limit formula. This terminology originates with the Kronecker limit formula (see section 2.2 below) but we shall denote any precise description of $L^*({}_A M)$ in the fundamental line as a "limit formula". In all the examples a limit formula goes hand in hand with the fact that the spaces involved in the definition of $\Xi({}_A M)$ have A -rank ≤ 1 . This seems to be the basic limitation of the cases where proofs of Conjecture 6 are currently known (the only exception being the analytic class number formula).

2.1. Dirichlet L-functions. Let m be any positive integer and define

$$\zeta_m := e^{2\pi i/m}, \quad F_m = \mathbb{Q}(\zeta_m), \quad G_m = \text{Gal}(F_m/\mathbb{Q}) \cong (\mathbb{Z}/m\mathbb{Z})^\times$$

and

$$M = h^0(\text{Spec}(F_m)), \quad A = \mathbb{Q}[G_m] \cong \prod_x \mathbb{Q}(\chi), \quad \mathfrak{A} = \mathbb{Z}[G_m].$$

The L-function $L({}_A M, s)$ takes values in $A \otimes \mathbb{C} = \prod_{\eta \in \hat{G}_m} \mathbb{C}$, where $\hat{G}_m = \text{Hom}(G_m, \mathbb{C}^\times)$, and coincides with the tuple of Dirichlet L-functions $(L(\eta, s))_{\eta \in \hat{G}_m}$. Limit formulas are known for the leading coefficient of $L({}_A M, s)$ at any integer argument $s = j$. The article [19] already gives a comprehensive discussion of these limit formulas, including a description of $L^*({}_A M, 0)$ in the fundamental line and the ensuing Iwasawa theory. We shall therefore simply recall the trivialization of the fundamental line obtained at $s = 0$. Here one has the well known formulae

$$L(\eta, 0) = - \sum_{a=1}^{f_\eta} \left(\frac{a}{f_\eta} - \frac{1}{2} \right) \eta(a)$$

$$\frac{d}{ds} L(\eta, s)|_{s=0} = - \frac{1}{2} \sum_{a=1}^{f_\eta} \log |1 - e^{2\pi i a/f_\eta}| \eta(a) \quad \eta \neq 1 \text{ even,}$$

and it is known that $\text{ord}_{s=0} L(\eta, s) = 0$ (resp. 1) if $\eta = 1$ or η is odd (resp. $\eta \neq 1$ is even). For a number field F and set of places S denote by $Y_S = Y_S(F)$ (resp. $X_S = X_S(F)$) the free abelian group on S (resp. the kernel of the sum map on Y_S). For an abelian group G denote

by $\#$ the automorphism $g \mapsto g^{-1}$ as well as the induced functor on G -modules and the induced automorphism of A .

There is a canonical isomorphism

$$\begin{aligned} \Xi_{(AM)\#} &\xrightarrow{\sim} \mathrm{Det}_A^{-1}(\mathcal{O}_{L_m}^\times \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes \mathrm{Det}_A(X_{\{v|\infty\}} \otimes_{\mathbb{Z}} \mathbb{Q}) \\ &\xrightarrow{\sim} \prod_{\substack{\chi \neq 1 \\ \text{even}}} \mathrm{Det}_{\mathbb{Q}(\chi)}^{-1}(\mathcal{O}_{L_m}^\times \otimes_{\mathfrak{A}} \mathbb{Q}(\chi)) \otimes \mathrm{Det}_{\mathbb{Q}(\chi)}(X_{\{v|\infty\}} \otimes_{\mathfrak{A}} \mathbb{Q}(\chi)) \times \prod_{\text{other } \chi} \mathbb{Q}(\chi) \\ &\xrightarrow{\sim} \prod_{\substack{\chi \neq 1 \\ \text{even}}} (\mathcal{O}_{L_m}^\times \otimes_{\mathfrak{A}} \mathbb{Q}(\chi))^{-1} \otimes_{\mathbb{Q}(\chi)} (X_{\{v|\infty\}} \otimes_{\mathfrak{A}} \mathbb{Q}(\chi)) \times \prod_{\text{other } \chi} \mathbb{Q}(\chi) \end{aligned}$$

and in this description ${}_A\vartheta_\infty(L^*(AM)^{-1}) = (L^*(AM)^{-1})\# {}_A\vartheta_\infty(1)$ has components

$${}_A\vartheta_\infty(L^*(AM)^{-1})_\chi = \begin{cases} 2 \cdot [F_m : F_{f_x}][1 - \zeta_{f_x}]^{-1} \otimes \sigma_m & \chi \neq 1 \text{ even} \\ (L(\chi, 0)\#)^{-1} & \text{otherwise.} \end{cases}$$

Here $\sigma : F_m \rightarrow \mathbb{C}$ is the embedding with $\sigma_m(\zeta_m) = e^{2\pi i/m}$ and

$$L(\chi, 0)\# = \sum_{\eta \in \chi} L(\eta^{-1}, 0)e_\eta \in \sum_{\eta \in \chi} e_\eta A \cong \mathbb{Q}(\chi)$$

where we view a \mathbb{Q} -rational character χ as an $\mathrm{Aut}(\mathbb{C})$ -orbit of complex characters.

2.2. The Kronecker limit formula. Our exposition in this section closely follows the treatment in de Shalit's book [17, Ch.II] except for minor improvements involving the canonical choice of various 12-th roots. We also borrow from Johnson's thesis [27] and Bley's paper [4]. We first introduce certain (very classical) elliptic functions, then give the connection to elliptic curves and finally specialize to elliptic units. This leads to limit formulas for Artin L-functions at $s = 0$ similar to those discussed in the previous section. We then discuss Eisenstein series and logarithmic derivatives of elliptic functions which lead to limit formulas for CM elliptic curves at the central point. The interplay between these two sets of limit formulas lies at the heart of the Iwasawa theory of imaginary quadratic fields as discussed in section 4 below.

2.2.1. Elliptic functions. Let $L = \mathbb{Z} \cdot w_1 + \mathbb{Z} \cdot w_2$ be a lattice in \mathbb{C} with oriented basis w_1, w_2 , i.e. so that $\tau := w_1/w_2$ has positive imaginary

part. The Dedekind Eta-function is defined as

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty} (1 - q_{\tau}^n); \quad q_{\tau} := e^{2\pi i \tau}$$

and we put

$$\eta^{(2)}(w_1, w_2) = w_2^{-1} 2\pi i \eta(w_1/w_2)^2.$$

This function depends on the choice of basis but

$$\Delta(L) = \Delta(\tau) = (2\pi)^{12} \eta(\tau)^{24}$$

does not. Define a (non-holomorphic) Theta-function

$$\varphi(z, \tau) = e^{\pi i z \frac{z - \bar{z}}{\tau - \bar{\tau}}} q_{\tau}^{1/12} (q_z^{1/2} - q_z^{-1/2}) \prod_{n=1}^{\infty} (1 - q_z q_{\tau}^n) (1 - q_z^{-1} q_{\tau}^n)$$

where $q_z = e^{2\pi i z}$ and

$$\varphi(z; w_1, w_2) = \varphi(z/w_2, w_1/w_2).$$

The function φ has a simple zero at each lattice point $z \in \mathbb{Z} \cdot w_1 + \mathbb{Z} \cdot w_2$. At $z = 0$ we get

$$\frac{d}{dz} \varphi(z, \tau)|_{z=0} = \lim_{z \rightarrow 0} \frac{\varphi(z, \tau)}{z} = e^{\frac{2\pi i \tau}{12}} (2\pi i) \prod_{n=1}^{\infty} (1 - q_{\tau}^n)^2 = 2\pi i \eta(\tau)^2.$$

and hence

$$\frac{d}{dz} \varphi(z; w_1, w_2)|_{z=0} = \eta^{(2)}(w_1, w_2).$$

For any pair of lattices $L \subseteq \tilde{L}$ of index prime to 6 with oriented bases $\omega := (w_1, w_2)$ and $\tilde{\omega} := (\tilde{w}_1, \tilde{w}_2)$ Robert shows in [36, Thms. 1,2] that there exists a unique choice of 12-th root of unity $C(\omega, \tilde{\omega})$ so that the functions

$$\delta(L, \tilde{L}) := C(\omega; \tilde{\omega}) \eta^{(2)}(\omega)^{[\tilde{L}:L]} / \eta^{(2)}(z; \tilde{\omega})$$

and

$$\psi(z; L, \tilde{L}) = C(\omega; \tilde{\omega}) \varphi(z; \omega)^{[\tilde{L}:L]} / \varphi(z; \tilde{\omega}) = \delta(L, \tilde{L}) \prod_{u \in T} (\wp(z; L) - \wp(u; L))^{-1}$$

only depend on the lattices L, \tilde{L} and so that ψ satisfies the distribution relation

$$(2.1) \quad \psi(z; K, \tilde{K}) = \prod_{i=1}^{[K:L]} \psi(z + t_i; L, \tilde{L})$$

for any lattice $L \subseteq K$ so that $K \cap \tilde{L} = L$ (and where $\tilde{K} = K + \tilde{L}$). The $t_i \in K$ are a set of representatives of K/L . The set T is any set of representatives of $(\tilde{L} \setminus \{0\}) / (\pm 1 \times L)$ and \wp is the Weierstrass

\wp -function associated to L . In particular we see that $\psi(z; L, \tilde{L})$ is an elliptic function, i.e. a rational function on the elliptic curve $E = \mathbb{C}/L$ with divisor $[\tilde{L} : L](O) - \sum_{P \in \tilde{L}/L} (P)$.

2.2.2. *Elliptic curves.* Kato reproves Robert's result in a scheme theoretic context [29]. Again the key insight is that the distribution relation (or norm compatibility) suffices to canonically normalize the 12-th root. For elliptic curves over fields this insight goes back at least to John Coates' paper [11, Appendix].

Lemma 2.1. (*Kato*) *Let E/S be an elliptic curve over a base scheme S and $c : E \rightarrow \tilde{E}$ an S -isogeny of degree prime to 6. Then there is a unique function*

$${}_c\Theta_{E/S} \in \Gamma(E \setminus \ker(c), \mathcal{O}^\times)$$

satisfying

- (i) $\operatorname{div}({}_c\Theta_{E/S}) = \deg(c) \cdot (0) - \sum_{P \in \ker(c)} (P)$
- (ii) *For any morphism $g : S' \rightarrow S$ we have $g_E^*({}_c\Theta_{E/S}) = {}_{c'}\Theta_{E'/S}$ where $g_E : E' := E \times_S S' \rightarrow E$ and c' is the base change of c .*
- (iii) *For any S -isogeny $b : E \rightarrow E'$ of degree prime to $\deg(c)$ have $b_*({}_c\Theta_{E/S}) = {}_{c'}\Theta_{E'/S}$ where b_* is the norm map associated to the finite flat morphism $E \setminus \ker(c) \rightarrow E' \setminus \ker(c')$. Here c' is the isogeny $E' \rightarrow E'/b(\ker(c))$.*
- (iv) *For $S = \operatorname{Spec}(\mathbb{C})$, $E = \mathbb{C}/L$ and $c : \mathbb{C}/L \rightarrow \mathbb{C}/\tilde{L}$ for lattices $L \subseteq \tilde{L}$ we have*

$${}_c\Theta_{E/S}(z) = \psi(z; L, \tilde{L}).$$

Proof. This is [29, Prop. 1.3] or [38, Thm. 1.2.1] if c is multiplication by an integer. In general, one can follow the argument of [29] to prove existence and uniqueness of ${}_c\Theta_{E/S}$ satisfying (i) and (iii) with b the multiplication by an integer. For a general isogeny $b : E \rightarrow E'$ of degree prime to $\deg(c)$ the element $b_*({}_c\Theta_{E/S})$ then satisfies (i) for c' as well as (iii) for multiplication by an integer, hence must coincide with ${}_{c'}\Theta_{E'/S}$. A similar argument shows (ii). \square

2.2.3. *Elliptic Units.* Let K be an imaginary quadratic field. For any integral ideal \mathfrak{f} in K we denote by $K(\mathfrak{f})$ the ray class field of K of conductor \mathfrak{f} and by $w_{\mathfrak{f}}$ the number of roots of unity in K congruent to 1 modulo \mathfrak{f} (so $w_{\mathfrak{f}} \mid w_1 \mid 12$). We let h be the class number of K .

Given $\mathfrak{f} \neq 1$ and any (auxiliary) \mathfrak{a} which is prime to $6\mathfrak{f}$ we define an analog of the cyclotomic unit $1 - \zeta_{\mathfrak{f}}$ by

$${}_{\mathfrak{a}}z_{\mathfrak{f}} = \psi(1; \mathfrak{f}, \mathfrak{a}^{-1}\mathfrak{f})$$

and for $\mathfrak{f} = 1$ we define a family of elements indexed by *all* ideals \mathfrak{a} of K by

$$u(\mathfrak{a}) = \frac{\Delta(\mathcal{O}_K)}{\Delta(\mathfrak{a}^{-1})}.$$

Lemma 2.2. *The complex numbers ${}_{\mathfrak{a}}z_{\mathfrak{f}}$ and $u(\mathfrak{a})$ satisfy the following properties*

- a) (*Rationality*) ${}_{\mathfrak{a}}z_{\mathfrak{f}} \in K(\mathfrak{f})$, $u(\mathfrak{a}) \in K(1)$
b) (*Integrality*)

$${}_{\mathfrak{a}}z_{\mathfrak{f}} \in \begin{cases} \mathcal{O}_{K(\mathfrak{f})}^{\times} & \mathfrak{f} \text{ divisible by primes } \mathfrak{p} \neq \mathfrak{q} \\ \mathcal{O}_{K(\mathfrak{f}), \{v|\mathfrak{f}\}}^{\times} & \mathfrak{f} = \mathfrak{p}^n \text{ for some prime } \mathfrak{p} \end{cases}$$

$$u(\mathfrak{a}) \cdot \mathcal{O}_{K(1)} = \mathfrak{a}^{-12} \mathcal{O}_{K(1)}$$

- c) (*Galois action*) For $(\mathfrak{c}, \mathfrak{f}\mathfrak{a}) = 1$ with Artin symbol $\sigma_{\mathfrak{c}} \in \text{Gal}(K(\mathfrak{f})/K)$ we have

$${}_{\mathfrak{a}}z_{\mathfrak{f}}^{\sigma_{\mathfrak{c}}} = \psi(1; \mathfrak{c}^{-1}\mathfrak{f}, \mathfrak{c}^{-1}\mathfrak{a}^{-1}\mathfrak{f}); \quad u(\mathfrak{a})^{\sigma_{\mathfrak{c}}} = u(\mathfrak{a}\mathfrak{c})/u(\mathfrak{c}).$$

This implies (see also [29, 15.4.4])

$${}_{\mathfrak{a}}z_{\mathfrak{f}}^{N\mathfrak{c} - \sigma_{\mathfrak{c}}} = {}_{\mathfrak{c}}z_{\mathfrak{f}}^{N\mathfrak{a} - \sigma_{\mathfrak{a}}}; \quad u(\mathfrak{a})^{1 - \sigma_{\mathfrak{c}}} = u(\mathfrak{c})^{1 - \sigma_{\mathfrak{a}}}.$$

- d) (*Norm compatibility*) For a prime ideal \mathfrak{p} one has

$$N_{K(\mathfrak{p}\mathfrak{f})/K(\mathfrak{f})}({}_{\mathfrak{a}}z_{\mathfrak{p}\mathfrak{f}})^{w_{\mathfrak{f}}/w_{\mathfrak{p}\mathfrak{f}}} = \begin{cases} {}_{\mathfrak{a}}z_{\mathfrak{f}} & \mathfrak{p} \mid \mathfrak{f} \neq 1 \\ {}_{\mathfrak{a}}z_{\mathfrak{f}}^{1 - \sigma_{\mathfrak{p}}^{-1}} & \mathfrak{p} \nmid \mathfrak{f} \neq 1 \\ u(\mathfrak{p})^{(\sigma_{\mathfrak{a}} - N\mathfrak{a})/12} & \mathfrak{f} = 1 \end{cases}$$

- e) (*Kronecker limit formula*). Put $G_{\mathfrak{f}} = \text{Gal}(K(\mathfrak{f})/K)$ and let η be a complex character of $G_{\mathfrak{f}}$. If $\mathfrak{f} = 1$ and $\eta \neq 1$ choose any ideal \mathfrak{a} so that $\eta(\mathfrak{a}) \neq 1$. Then

$$L(\eta, 0) = \zeta_K(0) = -\frac{h}{w_1} \quad \eta = 1$$

$$\frac{d}{ds} L(\eta, s)|_{s=0} = -\frac{1}{1 - \eta(\mathfrak{a})} \frac{1}{12w_1} \sum_{\sigma \in G_1} \log |\sigma(u(\mathfrak{a}))|^2 \eta(\sigma) \quad \eta \neq 1, \quad \mathfrak{f} = 1$$

$$\frac{d}{ds} L(\eta, s)|_{s=0} = -\frac{1}{N\mathfrak{a} - \eta(\mathfrak{a})} \frac{1}{w_{\mathfrak{f}}} \sum_{\sigma \in G_{\mathfrak{f}}} \log |\sigma({}_{\mathfrak{a}}z_{\mathfrak{f}})|^2 \eta(\sigma) \quad \mathfrak{f} \neq 1.$$

Proof. As in [17, Ch.II.2]. Note that in d) we may choose any 12-th root of $u(\mathfrak{p})$ since $\sigma_{\mathfrak{a}} - N\mathfrak{a}$ annihilates any root of unity. \square

Remarks. The relations in c) show the auxiliary nature of \mathfrak{a} . In $\mathbb{Q}[G_{\mathfrak{f}}]$ the element $N\mathfrak{a} - \sigma_{\mathfrak{a}}$ becomes invertible and

$$z_{\mathfrak{f}} = (N\mathfrak{a} - \sigma_{\mathfrak{a}})^{-1} {}_{\mathfrak{a}}z_{\mathfrak{f}} \in \mathcal{O}_{K(\mathfrak{f}), \{v|\mathfrak{f}\}}^{\times} \otimes_{\mathbb{Z}} \mathbb{Q}$$

is independent of \mathfrak{a} . The last item in b) shows that $u(\mathfrak{c})^{1-\sigma_{\mathfrak{a}}} \in \mathcal{O}_{K(1)}^{\times}$ is a unit. However, $1 - \sigma_{\mathfrak{a}}$ is not invertible in $\mathbb{Q}[G_1]$, only in direct factors $\mathbb{Q}(\chi)$ where $\chi(\sigma_{\mathfrak{a}}) \neq 1$. For such χ we obtain an element

$$u(\mathfrak{a})^{(1-\sigma_{\mathfrak{a}})^{-1}} \in \mathcal{O}_{K(1), \{v|\mathfrak{a}\}}^{\times} \otimes_{\mathbb{Z}[G_1]} \mathbb{Q}(\chi)$$

independent of \mathfrak{a} .

The Galois action in c) together with the relation

$$\psi(\lambda z; \lambda L, \lambda \tilde{L}) = \psi(z; L, \tilde{L})$$

for any $\lambda \in \mathbb{C}^{\times}$ shows that the Galois conjugates of ${}_{\mathfrak{a}}z_{\mathfrak{f}}$ are the numbers ${}_{\mathfrak{a}}\Theta_{E/\mathbb{C}}(\alpha)$ where (E, α) runs through all isomorphism classes of pairs with $E/\mathbb{C} \cong \mathbb{C}/L$ an elliptic curve with CM by \mathcal{O}_K , $\mathfrak{a} : \mathbb{C}/L \rightarrow \mathbb{C}/\mathfrak{a}^{-1}L$ and $\alpha \in E(\mathbb{C})$ a primitive \mathfrak{f} -division point. In fact ${}_{\mathfrak{a}}z_{\mathfrak{f}}$ is the value of ${}_{\mathfrak{a}}\Theta_{\mathcal{E}/K(1)}$ at a single closed point with residue field $K(\mathfrak{f})$ on an elliptic curve $\mathcal{E}/K(1)$.

We now fix an ideal \mathfrak{m} of \mathcal{O}_K and set

$$M = h^0(\text{Spec}(K(\mathfrak{m}))), \quad A = \mathbb{Q}[G_{\mathfrak{m}}], \quad \mathfrak{A} = \mathbb{Z}[G_{\mathfrak{m}}].$$

We view $K(\mathfrak{m})$ as a subfield of \mathbb{C} and denote by $\tau_{\mathfrak{m}}$ the resulting archimedean place of $K(\mathfrak{m})$.

For $\mathfrak{f} \neq 1$ the image of ${}_{\mathfrak{a}}z_{\mathfrak{f}} \in K(\mathfrak{m})$ under the Dirichlet regulator map is

$${}_{\mathfrak{a}}z_{\mathfrak{f}} \mapsto - \sum_{\sigma \in G_{\mathfrak{m}}} \log |\sigma({}_{\mathfrak{a}}z_{\mathfrak{f}})|^2 \sigma^{-1}(\tau_{\mathfrak{m}})$$

and hence for any η of conductor \mathfrak{f}

$$e_{\eta} \cdot {}_{\mathfrak{a}}z_{\mathfrak{f}} \mapsto (N\mathfrak{a} - \eta(\mathfrak{a})^{-1})L'(\eta^{-1}, 0) \cdot w_{\mathfrak{f}} \cdot [K(\mathfrak{m}) : K(\mathfrak{f})] \cdot e_{\eta}(\tau_{\mathfrak{m}}).$$

Note here that $e_{\eta}\sigma = \eta(\sigma)e_{\eta}$, and that $\tau_{\mathfrak{m}}$ (resp. ${}_{\mathfrak{a}}z_{\mathfrak{f}}$ if \mathfrak{f} is a prime power) lies in the larger \mathfrak{A} -module $Y_{\{v|\infty\}} \supset X_{\{v|\infty\}}$ (resp. $\mathcal{O}_{K(\mathfrak{f})}[\frac{1}{\mathfrak{f}}]^{\times} \supset \mathcal{O}_{K(\mathfrak{f})}^{\times}$) but application of e_{η} turns this inclusion into an equality since η has conductor $\mathfrak{f} \neq 1$.

For a nontrivial character η of conductor $\mathfrak{f} = 1$ we pick an ideal \mathfrak{a} with $\eta(\mathfrak{a}) \neq 1$. Then $e_{\eta}\mathcal{O}_{K(1)}[\frac{1}{\mathfrak{a}}]^{\times} = e_{\eta}\mathcal{O}_{K(1)}^{\times}$, and

$$e_{\eta} \cdot u(\mathfrak{a}) \mapsto (N\mathfrak{a} - \eta(\mathfrak{a})^{-1})L'(\eta^{-1}, 0) \cdot 12w_1 \cdot [K(\mathfrak{m}) : K(1)] \cdot e_{\eta}(\sigma_{\mathfrak{m}}).$$

There is a canonical isomorphism

$$\begin{aligned} \Xi({}_A M)^\# &\xrightarrow{\sim} \mathrm{Det}_A^{-1}(\mathcal{O}_{K(\mathfrak{m})}^\times \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes \mathrm{Det}_A(X_{\{v|\infty\}} \otimes_{\mathbb{Z}} \mathbb{Q}) \\ &\xrightarrow{\sim} \mathbb{Q} \times \prod_{\chi \neq 1} \mathrm{Det}_{\mathbb{Q}(\chi)}^{-1}(\mathcal{O}_{K(\mathfrak{m})}^\times \otimes_{\mathfrak{A}} \mathbb{Q}(\chi)) \otimes \mathrm{Det}_{\mathbb{Q}(\chi)}(X_{\{v|\infty\}} \otimes_{\mathfrak{A}} \mathbb{Q}(\chi)) \end{aligned}$$

and in this description ${}_A \vartheta_\infty(L^*({}_A M, 0)^{-1}) = (L^*({}_A M, 0)^{-1})^\# {}_A \vartheta_\infty(1)$ has components

(2.2)

$$\begin{aligned} &{}_A \vartheta_\infty(L^*({}_A M, 0)^{-1})_\chi = \\ &= \begin{cases} (N\mathfrak{a} - \chi(\mathfrak{a})) \cdot w_{\mathfrak{f}} \cdot [K(\mathfrak{m}) : K(\mathfrak{f})] \cdot [{}_a z_{\mathfrak{f}}]^{-1} \otimes \tau_{\mathfrak{m}} & \mathfrak{f} \neq 1 \\ (1 - \chi(\mathfrak{a})) \cdot 12w_1 \cdot [K(\mathfrak{m}) : K(1)] \cdot [u(\mathfrak{a})]^{-1} \otimes \tau_{\mathfrak{m}} & \mathfrak{f} = 1, \chi \neq 1 \\ -\frac{w_1}{h} & \chi = 1. \end{cases} \end{aligned}$$

2.2.4. *CM elliptic curves at the central point.* As in [17, Ch.II 1.4] we fix an abelian extension F/K and an elliptic curve E/F with CM by \mathcal{O}_K and so that the Weil restriction $B := \mathrm{Res}_K^F E$ is an abelian variety of CM-type [24, Thm. 4.1]. Put

$$A = \mathrm{End}_K B \otimes \mathbb{Q}$$

a semisimple K -algebra isomorphic to a product of CM-fields. If E/F is the base change of an elliptic curve E/K (which can only happen if K has class number one) then $A = K[G]$ where $G = \mathrm{Gal}(F/K)$. In general one can think of A as a twisted form of $K[G]$. Considering E as a scheme over K , the motive

$$M = h^1(E)(1)$$

over K as well as the spaces

$$M_B^+ = H_1(E^{\tau_1}(\mathbb{C}), \mathbb{Q}), \quad M_{dR}/\mathrm{Fil}^0 = t_E := \mathrm{Hom}_F(H^0(E, \Omega_{E/F}^1), F)$$

have rank one over A . Here E^{τ_1} is the base change via our fixed embedding $\tau_1 : K \rightarrow \mathbb{C}$. Note that $E^{\tau_1} \cong \prod_{\tau} E^\tau$ where τ runs through all embeddings of F restricting to τ_1 . We have

$$\begin{aligned} \Xi({}_A M) &= H_1(E^{\tau_1}(\mathbb{C}), \mathbb{Q})^{-1} \otimes_A \mathrm{Hom}_F(H^0(E, \Omega_{E/F}^1), F) \\ &\quad \otimes_A \mathrm{Det}_A^{-1} E(F)_{\mathbb{Q}} \otimes_A \mathrm{Det}_A E(F)_{\mathbb{Q}}^* \end{aligned}$$

and

$$L({}_A M, s) = (L(\phi_\varepsilon^{-1}, s))_{\varepsilon \in J} \in \prod_{\varepsilon \in J} \mathbb{C} \cong A_{\mathbb{C}}$$

where ϕ is the A -valued Serre-Tate character associated to B and $J = \mathrm{Hom}_{\mathbb{Q}}(A, \mathbb{C})$.

In order to describe $L(A\mathcal{M}, 0)$ we introduce Eisenstein numbers following [17, Ch.II .3]. The (non-holomorphic) Eisenstein series

$$E_1(z, L) := \frac{\partial}{\partial z} \log \varphi(z, w_1, w_2) - \frac{\bar{z}}{2A(L)}$$

only depends on the lattice $L = \mathbb{Z} \cdot w_1 + \mathbb{Z} \cdot w_2$ with volume $\pi A(L) = (2i)^{-1}(w_1 \bar{w}_2 - \bar{w}_1 w_2)$. For a pair $L \subseteq \tilde{L}$ of lattices of index prime to 6 we set

$$(2.3) \quad E_1(z, L, \tilde{L}) := [\tilde{L} : L]E_1(z, L) - E_1(z, \tilde{L}) = \frac{d}{dz} \log \psi(z, L, \tilde{L}).$$

Fix an F -basis ω of $H^0(E, \Omega_{E/F}^1)$ and an integral ideal \mathfrak{m} divisible by the conductor of both F/K and ϕ . Choose an embedding $\tau : F \rightarrow \mathbb{C}$ extending τ_1 on K so that the period lattice L of ω^τ on E^τ equals $\Omega\mathfrak{m}$ for some $\Omega = \Omega_{\mathfrak{m}} \in \mathbb{C}^\times$. Then

$$\Omega = \int_{\gamma} \omega^\tau$$

for a unique $\gamma = \gamma_{\mathfrak{m}} \in H_1(E^\tau(\mathbb{C}), \mathbb{Q})$. Choose a set \mathfrak{C} of ideals of \mathcal{O}_K so that the Artin symbol $\mathfrak{c} \rightarrow \sigma_{\mathfrak{c}}$ gives a bijection $\mathfrak{C} \cong G$. Considering E over the field F we have the isogenies

$$\lambda(\mathfrak{c}) : E \rightarrow E^{\sigma_{\mathfrak{c}}}$$

as in [17, Ch.II.1.5] and define $\Lambda(\mathfrak{c}) \in F$ by

$$(2.4) \quad \phi(\mathfrak{c})^* \omega = \lambda(\mathfrak{c})^* \omega^{\sigma_{\mathfrak{c}}} = \Lambda(\mathfrak{c})\omega.$$

Note that $H^0(E, \Omega_{E/F}^1)$ is free of rank 1 over both F and A but the two actions do not commute since for $\lambda \in F$ and $\mathfrak{a} \in \mathfrak{C}$

$$\phi(\mathfrak{a})^* \lambda \omega = \lambda(\mathfrak{a})^* \lambda^{\sigma_{\mathfrak{a}}} \omega^{\sigma_{\mathfrak{a}}} = \lambda^{\sigma_{\mathfrak{a}}} \phi(\mathfrak{a})^* \omega.$$

Lemma 2.3. *Setting $L = \Omega\mathfrak{m}$ the following hold.*

a) *(Homogeneity) For $\lambda \in \mathbb{C}^\times$ one has*

$$E_1(\lambda z, \lambda L) = \lambda^{-1} E_1(z, L)$$

b) *(Rationality) If $z \in \Omega\mathcal{O}_K$ then*

$$E_1(z, L) \in F(E[\mathfrak{m}])$$

c) *(Galois Action) If $z \in \Omega\mathcal{O}_K$ and $\mathfrak{c} \in \mathfrak{C}$ then*

$$E_1(z, L)^{\sigma_{\mathfrak{c}}} = E_1(\Lambda(\mathfrak{c})z, \Lambda(\mathfrak{c})\mathfrak{c}^{-1}L)$$

d) *(Kronecker limit formula)*

$$L_{\mathfrak{m}}(\phi_{\varepsilon}^{-1}, 0) = \Omega \cdot \sum_{\mathfrak{c} \in \mathfrak{C}} \Lambda(\mathfrak{c}) \phi_{\varepsilon}^{-1}(\mathfrak{c}) E_1(\Omega, L)^{\sigma_{\mathfrak{c}}}$$

Proof. See [17, Ch.II.3]. \square

Corollary 2.1. *Assume in addition that the \mathfrak{m} -torsion points of E are rational over F so that $E_1(\Omega, L) \in F$. If $L(\phi_\varepsilon^{-1}, 0) \neq 0$ for all $\varepsilon \in J$ then*

$$\Xi({}_A M) = H_1(E^{\tau_1}(\mathbb{C}), \mathbb{Q})^{-1} \otimes_A \mathrm{Hom}_F(H^0(E, \Omega_{E/F}^1), F)$$

and

$${}_A \vartheta_\infty(L^*({}_A M, 0)^{-1}) = {}_A \vartheta_\infty(L({}_A M, 0)^{-1}) = \prod_{\mathfrak{p}|\mathfrak{m}} (1 - \phi^{-1}(\mathfrak{p})) \gamma^{-1} \otimes \omega'.$$

where ω' is the F -linear form whose A -dual coincides with

$$[F : K]E_1(\Omega, L)\omega \in H^0(E, \Omega_{E/F}^1).$$

Proof. By the theorem of Coates-Wiles (and Arthaud) [9], the non-vanishing of $L(\phi_\varepsilon^{-1}, 0)$ implies $E(F)_\mathbb{Q} = 0$ and hence the formula for $\Xi({}_A M)$. We have a chain of $A_\mathbb{R}$ -linear isomorphisms

$$\begin{aligned} H_1(E^{\tau_1}(\mathbb{C}), \mathbb{Q})_\mathbb{R} &\cong \mathrm{Hom}_{F_\mathbb{R}}(H^0(E, \Omega^1)_\mathbb{R}, F_\mathbb{R}) \\ &\cong \mathrm{Hom}_\mathbb{C}(H^0(E, \Omega^1)_\mathbb{R}, \mathbb{C}) \\ &\cong \mathrm{Hom}_{A_\mathbb{R}}(H^0(E, \Omega^1)_\mathbb{R}, A_\mathbb{R}) \end{aligned}$$

induced by, respectively, the integration pairing

$$(\gamma_\rho, \eta^\rho) \mapsto \left(\int_{\gamma_\rho} \eta^\rho \right)_\rho \in \prod_{\rho \in \mathrm{Hom}_{K, \tau_1}(F, \mathbb{C})} \mathbb{C} = F_\mathbb{R},$$

the trace map $F_\mathbb{R} \rightarrow K_\mathbb{R} = \mathbb{C}$ and the trace map $A_\mathbb{R} \rightarrow K_\mathbb{R} = \mathbb{C}$. The value

$$(a_\varepsilon)_{\varepsilon \in J_K} \in \prod_{\varepsilon \in J_K} \mathbb{C} \cong A_\mathbb{R}$$

on a given pair $(\gamma_\rho, \omega^\rho)$ is determined by the set of equations

$$\mathrm{Tr}_{F_\mathbb{C}/\mathbb{C}} \left(\int_{\gamma_\rho} (\phi(\mathfrak{b})^* \eta)^\rho \right)_\rho = \sum_{\varepsilon \in J_K} \varepsilon(\phi(\mathfrak{b})) a_\varepsilon$$

as \mathfrak{b} runs through \mathfrak{C} and where $J_K = \mathrm{Hom}_{K, \tau_1}(A, \mathbb{C})$. Taking $\gamma_\tau = \gamma$ and $\gamma_\rho = 0$ for $\rho \neq \tau$, and $\eta = [F : K]E_1(\Omega, L)\omega$, we find

$$\sum_{\varepsilon \in J_K} \varepsilon(\phi(\mathfrak{b})) a_\varepsilon = \Lambda(\mathfrak{b}) [F : K] E_1(\Omega, L)^{\sigma_\mathfrak{b}} \int_\gamma \omega = \Lambda(\mathfrak{b}) [F : K] E_1(\Omega, L)^{\sigma_\mathfrak{b}} \Omega.$$

Now set

$$a_\varepsilon = L_\mathfrak{m}(\phi_\varepsilon^{-1}, 0) = \Omega \cdot \sum_{\mathfrak{c} \in \mathfrak{C}} \Lambda(\mathfrak{c}) \phi_\varepsilon^{-1}(\mathfrak{c}) E_1(\Omega, L)^{\sigma_\mathfrak{c}}$$

and note that $\varepsilon(\phi(\mathbf{b})) = \phi_\varepsilon(\mathbf{b})$. Then

$$\begin{aligned} \sum_{\varepsilon \in J_K} \varepsilon(\lambda(\mathbf{b}))a_\varepsilon &= \sum_{\varepsilon \in J_K} \sum_{\mathbf{c} \in \mathfrak{C}} \phi_\varepsilon(\mathbf{b})\phi_\varepsilon(\mathbf{c})^{-1} \Lambda(\mathbf{c})E_1(\Omega, L)^{\sigma_{\mathbf{c}}} \Omega \\ &= \sum_{\mathbf{c} \in \mathfrak{C}} \left(\sum_{\varepsilon \in J_K} \phi_\varepsilon\left(\frac{\mathbf{b}}{\mathbf{c}}\right) \right) \Lambda(\mathbf{c})E_1(\Omega, L)^{\sigma_{\mathbf{c}}} \Omega \\ &= [F : K] \Lambda(\mathbf{b})E_1(\Omega, L)^{\sigma_{\mathbf{b}}} \Omega \end{aligned}$$

since the set of characters ϕ_ε coincides with the set of twists of any of its members by $\text{Hom}(G, \mathbb{C}^\times)$ [24, Lemme 4.8]. We conclude that $\gamma \otimes (\omega')^{-1} \in \Xi_{(A)M}^{-1}$ is sent to $L_{\mathfrak{m}}(A)M, 0 = (L_{\mathfrak{m}}(\phi_\varepsilon^{-1}, 0))_{\varepsilon \in J_K} \in A_{\mathbb{R}}$. \square

Remark. One could make a slightly finer statement here assuming only that $L(\phi_\varepsilon^{-1}, 0) \neq 0$ for all $\varepsilon : A' \rightarrow \mathbb{C}$ where A' is a direct factor of A . This would imply $E(F) \otimes_A A' = 0$ and a corresponding description of the fundamental line of the motive $M \otimes_A A'$ over A' .

Lemma 2.4. *Let ψ be an algebraic Hecke character of K of infinity type $(1, 0)$. Then there is a finite extension F/K , an elliptic curve E/F with $B = \text{Res}_K^F(E)$ of CM-type and a homomorphism $\varepsilon : A \rightarrow \mathbb{C}$ so that $\psi = \phi_\varepsilon$.*

Proof. See [17, Ch. II, Lemma 1.4 (i)]. \square

Remark. This Lemma shows that all L-values $L(\psi, 0)$ where ψ is an algebraic Hecke character of an imaginary quadratic field of infinity type $(-1, 0)$ are covered by the discussion in this section. All Hecke characters of infinity type $(0, 0)$ are covered by Lemma 2.2e). We remark that there are limit formulas for $L^*(\psi, 0)$ for ψ a Hecke character of any infinity type (k, j) . If ψ is critical (modulo replacing ψ by $\bar{\psi}$ this is the range $k < 0, j \geq 0$) these go back to Kronecker and Damerell and can be found in [17, Ch.II, Prop. 3.5]. In the noncritical range these formulas are due to Deninger [16] building on Beilinson's Eisenstein symbol.

2.3. Other Limit Formulas. We recall our convention that a limit formula is an exact formula without any unspecified factors in A^\times . The list of such formulas (known to the author) is indeed quite limited. Conjecture 4 is known in many more cases of motives attached to automorphic form (see also the survey article [19]).

Formulas of Gross-Zagier type. The original Gross-Zagier formula concerns the motive $M(f)(1)$ where f is an elliptic modular form of

weight 2 base changed to a suitable imaginary quadratic field K [25]. As the formula in Corollary 2.1 they deal with the L-function of an abelian variety at the central point but where now the group of points has A -rank 1 and hence occurs in the fundamental line. The Gross-Zagier formula has been generalized to forms of higher weight [45] and to Hilbert modular forms [46], [47] by Zhang.

Formulas of Beilinson-Gealy. These concern the motive $M(f)(j)$ where f is an elliptic modular form of weight $k \geq 2$ and $j \leq 0$. These examples (or rather their duals under the functional equation) were instrumental in Beilinson's generalization of Conjecture 4 from the critical to the general case [1] but a precise limit formula has only been recently proven by Gealy [23]. As in the case of the Gross-Zagier formula this required the exact evaluation of certain Rankin Selberg integrals and is the basis for attacking Conjecture 6. For j in the critical range $1 \leq j \leq k - 1$ limit formulas are also known and are much more elementary.

The adjoint motive of a modular form. This is an example of a critical motive. The exact limit formula was proven by Diamond, Flach and Guo [18] building on work of Shimura and Hida.

Part 2. Iwasawa Theory

Fix a prime number l . Throughout this part we consider pairs (Λ, T) where Λ is a pro- l -ring such that

$$\Lambda = \varprojlim_n \Lambda/J^n,$$

with J the Jacobson radical of Λ and Λ/J^n is a finite ring of l -power order, and where T is a finitely generated projective Λ -module with a continuous action of $G_{\mathbb{Q}, S}$ for some finite set of primes S containing l . We call such a pair an *l -adic family*. In order not to overburden notation we assume that Λ is commutative with the exception of section 6 below.

A *motivic point* of the family (Λ, T) is a tuple

$$(K, A, M) = (K, A, M, \phi, \psi, \tau)$$

where M is a motive with an action of the semisimple algebra A , K is a finite extension of \mathbb{Q}_l , $\phi : \Lambda \rightarrow K$, $\psi : A_l \rightarrow K$ are ring homomorphisms and $\tau : M_l \otimes_{A_l} K \cong T \otimes_{\Lambda} K$ is an isomorphism of $G_{\mathbb{Q}}$ -representations. By a *dense set of points* of the family (Λ, T) we simply mean a set of homomorphisms $\phi_l : \Lambda \rightarrow K_l$ so that the set of prime ideals $\ker(\phi_l)$ is

dense in the Zariski topology of $\text{Spec}(\Lambda)$. If $\{\phi_\iota\}_{\iota \in I}$ is a dense set of points and Λ is reduced then

$$\Lambda \rightarrow \prod_{\iota \in I} K_\iota$$

is injective.

Suppose (Λ, T) is an l -adic family with a dense set of motivic points $(K_\iota, A_\iota, M_\iota)$ for which Conjecture 4 is known. The isomorphisms

$$\zeta_{A_\iota, l}(M_\iota, l) \otimes_{A_\iota, l} K_\iota : \text{Det}_{K_\iota}(0) \cong \text{Det}_{K_\iota} R\Gamma_c(\mathbb{Z}[\frac{1}{S}], M_\iota, l \otimes_{A_\iota, l} K_\iota)$$

combine to give an isomorphism

$$\zeta_{\prod K_\iota}(T \otimes_\Lambda \prod K_\iota) : \text{Det}_{\prod K_\iota}(0) \cong \text{Det}_{\prod K_\iota} R\Gamma_c(\mathbb{Z}[\frac{1}{S}], T \otimes_\Lambda \prod K_\iota)$$

over the ring $\prod_{\iota \in I} K_\iota$. The strategy to prove Conjecture 6 for any such point $(K_\iota, A_\iota, M_\iota)$ in the family (Λ, T) is to descend the isomorphism $\zeta_{\prod K_\iota}(T \otimes_\Lambda \prod K_\iota)$ to an isomorphism

$$\zeta_\Lambda(T) : \text{Det}_\Lambda(0) \cong \text{Det}_\Lambda R\Gamma_c(\mathbb{Z}[\frac{1}{S}], T).$$

If in addition we have $T \otimes_\Lambda \mathfrak{A}_\iota \cong \mathfrak{M}_\iota$ then Conjecture 6 simply follows by base change. That this strategy should be feasible using *any* l -adic family is the content of Conjecture 7 below. Formulated in this generality, our strategy includes taking $\Lambda = \mathfrak{A}_\iota$ and does not seem to offer any progress over the original problem of proving Conjecture 6 in a case where Conjecture 4 is known. The technical advantage of passing to an l -adic family is that Λ can be a regular ring where \mathfrak{A}_ι is not, and if Λ is regular then the verification of the existence of $\zeta_\Lambda(T)$ often reduces to an equality of Fitting ideals. Such an equality is then in turn equivalent to two inverse divisibilities, proven by rather different arguments: Euler systems for one direction and congruences for Galois representations associated to automorphic forms for the other (or in simple cases the analytic class number formula).

The descent from $\prod K_\iota$ to Λ proceeds via intermediate rings

$$\prod_{\iota \in I} K_\iota \supset Q(\Lambda) \supset \Lambda$$

where $Q(\Lambda)$ is the localisation of Λ at the set of elements s so that $\phi_\iota(s) \neq 0$ for all ι . The descent from $\prod_{\iota \in I} K_\iota$ to $Q(\Lambda)$ amounts to the "existence of an l -adic L-function" because it expresses l -adic continuity properties of the leading coefficients $L^*(A_\iota M_\iota)$. The descent from $Q(\Lambda)$ to Λ is traditionally called a "main conjecture" and is equivalent to the equality of Fitting ideals mentioned above (if Λ is regular).

Other rings $\mathcal{H}(\Lambda)$ and \mathbb{K} are involved in the case where (Λ, T) is the cyclotomic deformation of a motive with good reduction. We refer to section 5 below for details.

3. THE ZETA ISOMORPHISM OF KATO AND FUKAYA

We recall the central conjecture of Kato and Fukaya in [21, Conj. 2.3.2] which goes back to Kato [28, Conj. 3.2.2] if Λ is commutative. In essence, this is a generalization of Conjecture 6 to any l -adic family. Our notation is different in that our $\zeta_\Lambda(T)$ is the $\zeta_\Lambda(T)^{-1}$ of [21] where $P \mapsto P^{-1} = \text{Hom}_R(P, R)$ is the functor sending a graded invertible R -module to its inverse and a morphism $\phi : P \rightarrow Q$ to its contragredient $\text{Hom}_R(\phi, R)^{-1}$.

Conjecture 7. There exists a unique way to associate an isomorphism

$$\zeta_\Lambda(T) : \text{Det}_\Lambda(0) \cong \text{Det}_\Lambda R\Gamma_c(\mathbb{Z}[\frac{1}{S}], T)$$

to any l -adic family (Λ, T) so that the following conditions hold.

- (i) For any exact sequence of l -adic families

$$0 \rightarrow T' \rightarrow T \rightarrow T'' \rightarrow 0$$

with common Λ there is a commutative diagram

$$\begin{array}{ccc} \text{Det}_\Lambda R\Gamma_c(\mathbb{Z}[\frac{1}{S}], T) & \xrightarrow{\sim} & \text{Det}_\Lambda R\Gamma_c(\mathbb{Z}[\frac{1}{S}], T') \otimes \text{Det}_\Lambda R\Gamma_c(\mathbb{Z}[\frac{1}{S}], T'') \\ \zeta_\Lambda(T) \uparrow & & \zeta_\Lambda(T') \otimes \zeta_\Lambda(T'') \uparrow \\ \text{Det}_\Lambda(0) & \xrightarrow{\text{can}} & \text{Det}_\Lambda(0) \otimes \text{Det}_\Lambda(0). \end{array}$$

- (ii) Let (Λ, T) and (Λ', T') be l -adic families and Y a $\Lambda' \text{-}\Lambda$ -bimodule, finitely generated and projective over Λ' and so that $T' = Y \otimes_\Lambda T$. Then there is a commutative diagram

$$\begin{array}{ccc} Y \otimes_\Lambda \text{Det}_\Lambda R\Gamma_c(\mathbb{Z}[\frac{1}{S}], T) & \xrightarrow{\sim} & \text{Det}_{\Lambda'} R\Gamma_c(\mathbb{Z}[\frac{1}{S}], T') \\ Y \otimes_\Lambda \zeta_\Lambda(T) \uparrow & & \zeta_{\Lambda'}(T') \uparrow \\ Y \otimes_\Lambda \text{Det}_\Lambda(0) & \xrightarrow{\text{can}} & \text{Det}_{\Lambda'}(0). \end{array}$$

In particular, for any homomorphism $\Lambda \rightarrow \Lambda'$ we have

$$\zeta_\Lambda(T) \otimes_\Lambda \Lambda' = \zeta_{\Lambda'}(T \otimes_\Lambda \Lambda').$$

- (iii) For any motivic point (K, A, M) of the l -adic family (Λ, T) we have

$$\zeta_\Lambda(T) \otimes_\Lambda K = \zeta_{A_l}(M_l) \otimes_{A_l} K$$

where $\zeta_{A_l}(M_l)$ is the isomorphism (1.2) arising from Conjecture 4.

The sweeping generality of this conjecture leads to some interesting problems. For example, as explained in the introduction, $\zeta_\Lambda(T)$ can often be constructed using a dense set of motivic points but it is expected to induce the isomorphism $\zeta_{A_l}(M_l)$ for *any* motivic point. A limit formula has to be available for such a point, not only for the complex L-function but also for the l -adic L-function (here we use the term l -adic L-function synonymous for the isomorphism $\zeta_\Lambda(T)$ constructed from the dense set of motivic points). Much work in the theory of l -adic L-functions can be subsumed under this problem. An example is our discussion in section 4 below.

Also note that Λ can be a finite ring. This expresses the fact that l -adic families are related by congruences, and it can be used to show uniqueness of any collection of isomorphisms $\zeta_\Lambda(T)$ satisfying (ii), (iii) [21, 2.3.5]. While it is not true that any l -adic family (Λ, T) has a dense set of motivic points, or indeed any motivic point, any given (Λ, T) arises by base change via property (ii) from a family with a dense set of (Artin) motivic points (see section 6 below).

The content of the Fontaine-Mazur conjecture [22] is that a pair (\mathbb{Z}_l, T) is motivic if and only if it is "motivic at l ", i.e. the restriction of the $G_{\mathbb{Q}}$ -representation T to $G_{\mathbb{Q}_l}$ is a potentially semistable representation. This would imply that an arbitrary l -adic family (Λ, T) has a dense set of motivic points if the restriction of the family to $G_{\mathbb{Q}_l}$ has a dense set of potentially semistable points.

4. IWASAWA THEORY OF IMAGINARY QUADRATIC FIELD

In some sense the Iwasawa theory of imaginary quadratic fields began with the Coates-Wiles theorem (see [17, IV.2] for a "properly" Iwasawa theoretic proof). A major subsequent success was the work of Rubin [37] giving the first examples of elliptic curves over number fields with finite Tate-Shafarevich group and proving a main conjecture. From the point of view of this paper, the main conjecture is the construction of a basis $\zeta_\Lambda(T)$ for a certain l -adic family.

4.1. The main conjecture for imaginary quadratic fields. Let K be an imaginary quadratic field, \mathfrak{m} an ideal of \mathcal{O}_K and l a prime number. We shall define an l -adic family (Λ, T) depending on these parameters and show how the limit formulas of section 2.2.3 lead to a candidate for $\zeta_\Lambda(T)$.

Resume the notation introduced in section 2.2. Put

$$\Lambda = \varprojlim_n \mathbb{Z}_l[G_{\mathfrak{m}^n}] \cong \mathbb{Z}_l[G_{\mathfrak{m}^\infty}^{\text{tor}}][[S_1, S_2]]$$

where $G_{\mathfrak{m}l^\infty}^{\text{tor}}$ is the torsion subgroup of $G_{\mathfrak{m}l^\infty} = \varprojlim_n G_{\mathfrak{m}l^n}$. The Iwasawa algebra Λ is a finite product of complete local 3-dimensional Cohen-Macaulay (even complete intersection) rings. However, Λ is regular if and only if $l \nmid \#G_{\mathfrak{m}l^\infty}^{\text{tor}}$. The elements $S_i = \gamma_i - 1 \in \Lambda$ depend on the choice of a complement $\Gamma \cong \mathbb{Z}_l^2$ of $G_{\mathfrak{m}l^\infty}^{\text{tor}}$ in $G_{\mathfrak{m}l^\infty}$ and of a choice of topological generators γ_1, γ_2 of Γ . Define

$$T = \varprojlim_n H^0(\text{Spec}(K(\mathfrak{m}l^n) \otimes_K \bar{\mathbb{Q}}), \mathbb{Z}_l)$$

which is a free, rank one Λ -module. We may regard (Λ, T) as an l -adic family over K (or $(\Lambda, \text{Ind}_K^{\mathbb{Q}}(T))$ as an l -adic family in the above sense). The motivic points of (Λ, T) are given by (motives associated to) algebraic Hecke characters of K of conductor dividing $\mathfrak{m}l^n$ for some n . However already the Artin motives $h^0(\text{Spec}(K(\mathfrak{m}l^n)))$ with their action of $A = \mathbb{Q}[G_{\mathfrak{m}l^n}]$ form a dense set of motivic points. These correspond to the Hecke characters of infinity type $(0, 0)$.

Define a perfect complex of Λ -modules

$$\Delta^\infty = R\text{Hom}_\Lambda(R\Gamma_c(\mathcal{O}_K[\frac{1}{\mathfrak{m}l}], T), \Lambda)^\#[-3].$$

Then $H^i(\Delta^\infty) = 0$ for $i \neq 1, 2$ and there is a canonical isomorphism

$$H^1(\Delta^\infty) \cong U_{\{v|\mathfrak{m}l\}}^\infty := \varprojlim_n \mathcal{O}_{K(\mathfrak{m}_0l^n)}[\frac{1}{\mathfrak{m}l}]^\times \otimes_{\mathbb{Z}} \mathbb{Z}_l$$

and a short exact sequence

$$0 \rightarrow P_{\{v|\mathfrak{m}l\}}^\infty \rightarrow H^2(\Delta^\infty) \rightarrow X_{\{v|\mathfrak{m}l^\infty\}}^\infty \rightarrow 0$$

where

$$P_{\{v|\mathfrak{m}l\}}^\infty := \varprojlim_n \text{Pic}(\mathcal{O}_{K(\mathfrak{m}l^n)}[\frac{1}{\mathfrak{m}l}]) \otimes_{\mathbb{Z}} \mathbb{Z}_l$$

$$X_{\{v|\mathfrak{m}l^\infty\}}^\infty := \varprojlim_n X_{\{v|\mathfrak{m}l^\infty\}}(K(\mathfrak{m}l^n)) \otimes_{\mathbb{Z}} \mathbb{Z}_l.$$

All limits are taken with respect to Norm maps (on $X_S \subset Y_S$ this is the map sending a place to its restriction).

Let \mathfrak{m}_0 be the prime to l -part of \mathfrak{m} . For $d \mid \mathfrak{m}_0$ put

$$\mathfrak{a}z_{dl^\infty} := (\mathfrak{a}z_{dl^n})_{n \gg 0} \in U_{\{v|\mathfrak{m}l\}}^\infty$$

$$\tau := (\tau_{\mathfrak{m}l^n})_{n \gg 0} \in Y_{\{v|\mathfrak{m}l^\infty\}}^\infty$$

We fix an embedding $\bar{\mathbb{Q}}_l \rightarrow \mathbb{C}$ and identify \hat{G}_k with the set of $\bar{\mathbb{Q}}_l$ -valued characters. The total ring of fractions

$$(4.1) \quad Q(\Lambda) \cong \prod_{\psi \in G_{\mathfrak{m}l^\infty}^{\text{tor}}{}^{\mathbb{Q}_l}} Q(\psi)$$

of Λ is a product of fields indexed by the \mathbb{Q}_l -rational characters of $G_{\mathfrak{m}l^\infty}^{\text{tor}}$. Since for any place w of \mathbb{Q} the $\mathbb{Z}[G_{\mathfrak{m}l^n}]$ -module $Y_{\{v|w\}}(K(\mathfrak{m}l^n))$ is induced from the trivial module \mathbb{Z} on the decomposition group $D_w \subseteq G_{\mathfrak{m}l^n}$, and for $w = \infty$ (resp. nonarchimedean w) we have $[G_{\mathfrak{m}l^n} : D_w] = [K(\mathfrak{m}l^n) : K]$ (resp. the index $[G_{\mathfrak{m}l^n} : D_w]$ is bounded as $n \rightarrow \infty$) one computes easily

$$(4.2) \quad \dim_{Q(\psi)}(U_{\{v|\mathfrak{m}l\}}^\infty \otimes_\Lambda Q(\psi)) = \dim_{Q(\psi)}(Y_{\{v|\mathfrak{m}l^\infty\}}^\infty \otimes_\Lambda Q(\psi)) = 1$$

for all characters ψ . Note that the inclusion $X_{\{v|\mathfrak{m}l^\infty\}}^\infty \subseteq Y_{\{v|\mathfrak{m}l^\infty\}}^\infty$ becomes an isomorphism after tensoring with $Q(\psi)$ and that $e_\psi(\mathfrak{a}z_{\mathfrak{m}l^\infty}^{-1} \otimes \sigma)$ is a $Q(\psi)$ -basis of

$$\begin{aligned} & \text{Det}_{Q(\psi)}^{-1}(U_{\{v|\mathfrak{m}l\}}^\infty \otimes_\Lambda Q(\psi)) \otimes \text{Det}_{Q(\psi)}(X_{\{v|\mathfrak{m}l^\infty\}}^\infty \otimes_\Lambda Q(\psi)) \\ & \cong \text{Det}_{Q(\psi)}(\Delta^\infty \otimes_\Lambda Q(\psi)). \end{aligned}$$

Hence we obtain a $Q(\Lambda)$ -basis

$$\mathcal{L} := (N\mathfrak{a} - \sigma_{\mathfrak{a}})_{\mathfrak{a}z_{\mathfrak{m}l^\infty}^{-1}} \otimes \tau$$

of $\text{Det}_{Q(\Lambda)}(\Delta^\infty \otimes_\Lambda Q(\Lambda))$.

Lemma 4.1. *Let F/K be a subextension of $K(\mathfrak{m}l^\infty)/K$ with group G and put*

$$M = h^0(\text{Spec}(F)); \quad A = \mathbb{Q}[G].$$

Denote by χ any \mathbb{Q}_l -rational character of G , by $\chi : \Lambda \rightarrow \mathbb{Q}_l(\chi)$ the corresponding ring homomorphism, by \mathfrak{q} its kernel and set $V = M_l \otimes_{\mathbb{Q}_l[G]} \mathbb{Q}_l(\chi)$. There is an isomorphism of perfect complexes

$$\Delta_{\Lambda_{\mathfrak{q}}}^\infty \otimes_{\Lambda_{\mathfrak{q}}} \mathbb{Q}_l(\chi) \cong R\Gamma_c(\mathcal{O}_K[\frac{1}{\mathfrak{m}l}], V)^*[-3]$$

and an isomorphism of determinants

$$\text{Det}_{\Lambda_{\mathfrak{q}}} \Delta_{\Lambda_{\mathfrak{q}}}^\infty \otimes_{\Lambda_{\mathfrak{q}}} \mathbb{Q}_l(\chi) \cong \text{Det}_{\mathbb{Q}_l(\chi)} R\Gamma_c(\mathcal{O}_K[\frac{1}{\mathfrak{m}l}], V)^\# \cong \Xi_{(A)M}^\# \otimes_A \mathbb{Q}_l(\chi).$$

If $\chi(l) \neq 1$ then the element \mathcal{L} is already a $\Lambda_{\mathfrak{q}}$ -basis of $\Delta_{\Lambda_{\mathfrak{q}}}^\infty$ and maps to the basis of $\Xi_{(A)M}^\# \otimes_A \mathbb{Q}_l(\chi)$ described in (2.2).

Proof. This is a straightforward descent computation analogous to the case of even characters χ with $\chi(l) \neq 1$ in the proof of [19, Thm. 5.1]. \square

Remark. The Artin motivic points described in this Lemma form a dense set in those direct factors of the family (Λ, T) corresponding to characters χ of $G_{\mathfrak{m}l^\infty}^{\text{tor}}$ with $\chi(l) \neq 1$. In order to extend the Lemma to characters χ with $\chi(l) = 1$ one needs to prove results for elliptic units

analogous to those of Solomon [40] for cyclotomic units. This has been done by Bley [3] if l is split in K/\mathbb{Q} .

Conjecture 8. (Iwasawa Main Conjecture, see also [28, Conj. III.1.2.3]) There is an identity of invertible Λ -submodules

$$\Lambda \cdot \mathcal{L} = \text{Det}_\Lambda \Delta^\infty$$

of $\text{Det}_{Q(\Lambda)}(\Delta^\infty \otimes_\Lambda Q(\Lambda))$.

Remarks. a) Following the example of Burns and Greither [7] (see also [19, Thm. 5.2]) one shows that this conjecture is in many, but not all, cases implied by Rubin's 2-variable Iwasawa main conjecture [37, Thm. 4.1] together with the vanishing of the μ -invariant of a 2-variable Iwasawa module. The cases not covered by Rubin are those where either l divides $6h \cdot \#G_{\mathfrak{m}l^\infty}^{\text{tor}}$ or where one considers a χ -component for characters vanishing on the decomposition group of l in $G_{\mathfrak{m}l^\infty}^{\text{tor}}$ if l is ramified or inert in K/\mathbb{Q} .

b) Conjecture 8 together with Lemma 4.1 (for *all* characters χ of G) immediately imply Conjecture 6 for the pair $(h^0(\text{Spec}(F)), \mathbb{Z}[G])$ and the prime l in accordance with the strategy outlined at the beginning of this section. This has again been carried out by Bley if l is split in K/\mathbb{Q} [4]. If \mathfrak{l} is a prime dividing l , Bley also proves a one-variable main conjecture over $G_{\mathfrak{m}l^\infty}$ where $l \nmid 2h$ is allowed to divide $G_{\mathfrak{m}l^\infty}^{\text{tor}}$.

c) In order to show that the trivialization of $\text{Det}_\Lambda \Delta^\infty$ given by \mathcal{L} agrees with the isomorphism $\zeta_\Lambda(T)$ of Kato and Fukaya one must show that property (iii) of Conjecture 7 holds for *all* motivic points of (Λ, T) , not only for Artin motivic points. This is a rather delicate question which we shall address for the motives attached to CM elliptic curves in the remainder of this section.

4.2. The explicit reciprocity law. In this paragraph we sketch the descent computations relating the l -adic L-function \mathcal{L} introduced above with the L-value described in Corollary 2.1. The existence of such a relationship was anticipated in the seminal work of Coates and Wiles [9] and then formulated precisely in the reciprocity law of Wiles [44]. We use the equivalent formulation of this law by Kato [28, Thm II.2.1.7].

Resume the notation of section 2.2.4, in particular F/K is an abelian extension and E/F is an elliptic curve with CM by \mathcal{O}_K so that the Weil restriction B of E to K is of CM type. The motive $M = h^1(E)(1)$ over K has an action of $A = \mathfrak{A} \otimes \mathbb{Q}$ where $\mathfrak{A} = \text{End}_K B$. We let \mathfrak{m} be the conductor of B , fix a prime \mathfrak{l} of \mathcal{O}_K dividing l and set

$$\mathfrak{m} = \tilde{\mathfrak{m}}_0 \mathfrak{l}^m$$

with $\mathfrak{l} \nmid \tilde{\mathfrak{m}}_0$. So $\tilde{\mathfrak{m}}_0 = \mathfrak{m}_0$ unless l is split in K/\mathbb{Q} and \bar{l} divides \mathfrak{m} . Since M is a direct summand of $M' = h^1(E \otimes_F F')(1)$ for any extension F'/F , and we have $M' \otimes_{A'} A \cong M$ if F'/K is also abelian, we may always replace F by such a larger extension F' and prove Conjecture 6 for the pair (M', A') .

Lemma 4.2. *After possibly replacing \mathfrak{m} by a multiple we may assume that $F = K(\mathfrak{m})$ and that in addition E is defined over $K(\tilde{\mathfrak{m}}_0)$ and has good reduction at primes dividing \mathfrak{l} .*

Proof. After replacing $\tilde{\mathfrak{m}}_0$ by a multiple so that $w_{\tilde{\mathfrak{m}}_0} = 1$ there is a Hecke character ϕ' of K of conductor dividing $\tilde{\mathfrak{m}}_0$ and infinity type $(1, 0)$ by [17, Ch. II, Lemma 1.4 (ii)]. By [17, Ch.II, Lemma 1.4 (i)] there exists then an elliptic curve over $E'/K(\tilde{\mathfrak{m}}_0)$ with j -invariant $j(E)$ and with Serre-Tate character $\phi' \circ N_{K(\tilde{\mathfrak{m}}_0)/K}$, and hence with good reduction outside primes dividing $\tilde{\mathfrak{m}}_0$. Since the conductors of both ϕ and ϕ' divide \mathfrak{m} , so does the conductor of the finite order character ϕ/ϕ' . Again by [17, Ch.II, Lemma 1.4 (i)] the curves E and E' then become isomorphic over $K(\mathfrak{m})$ since they have the same j -invariant and the same Serre-Tate character (namely $\phi \circ N_{K(\mathfrak{m})/K} = \phi' \circ N_{K(\mathfrak{m})/K}$). \square

We henceforth assume that the assumptions of Lemma 4.2 are in effect and that ω is also defined over $K(\tilde{\mathfrak{m}}_0)$. We put $F_n = K(\tilde{\mathfrak{m}}_0 \mathfrak{l}^n) = F_0(E[\mathfrak{l}^n])$ [17, Ch. II, Prop. 1.6] and shall in the following use the notation $\Omega, \gamma, \Lambda, \phi$ etc. introduced in section 2.2.4 referring to E/F_0 and $\tilde{\mathfrak{m}}_0$ rather than E/F and \mathfrak{m} . In particular $\tau_0 : F_0 \rightarrow \mathbb{C}$ is an embedding so that the pair $(E^{\tau_0}, \omega^{\tau_0})$ is isomorphic to $(\mathbb{C}/\Omega\tilde{\mathfrak{m}}_0, dz)$. For any $n \geq 0$ we have the isogeny

$$\lambda(\mathfrak{l})^{\sigma_{\bar{\mathfrak{l}}}^{-n}} : E^{\sigma_{\bar{\mathfrak{l}}}^{-n}} \rightarrow E^{\sigma_{\bar{\mathfrak{l}}}^{-(n-1)}}$$

isomorphic under τ_0 to

$$(4.3) \quad \mathbb{C}/\Lambda(\mathfrak{l}^{-n})\mathfrak{l}^n\Omega\tilde{\mathfrak{m}}_0 \xrightarrow{\Lambda(\mathfrak{l})^{\sigma_{\bar{\mathfrak{l}}}^{-n}}} \mathbb{C}/\Lambda(\mathfrak{l}^{-(n-1)})\mathfrak{l}^{n-1}\Omega\tilde{\mathfrak{m}}_0.$$

The points $\Lambda(\mathfrak{l}^{-n})\Omega \in \mathbb{C}$ define a sequence of primitive $\mathfrak{l}^n\tilde{\mathfrak{m}}_0$ -division points p_n on $E^{\sigma_{\bar{\mathfrak{l}}}^{-n}}$ which are compatible under the isogenies $\lambda(\mathfrak{l})^{\sigma_{\bar{\mathfrak{l}}}^{-n}}$. Let $p_n = \zeta_n + q_n$ be the unique decomposition with ζ_n (resp. q_n) a primitive \mathfrak{l}^n (resp. $\tilde{\mathfrak{m}}_0$)-division point. Then

$$\lambda(\mathfrak{l})^{\sigma_{\bar{\mathfrak{l}}}^{-n}}(\zeta_n) = \zeta_{n-1}; \quad \lambda(\mathfrak{l})^{\sigma_{\bar{\mathfrak{l}}}^{-n}}(q_n) = q_n^{\sigma_{\bar{\mathfrak{l}}}} = q_{n-1}$$

where we have used [17, Prop.1.5]. In particular

$$(4.4) \quad q_n = q_0^{\sigma_{\bar{\mathfrak{l}}}^{-n}}.$$

Let d be the order of σ_{Γ} in $G_{\mathfrak{m}_0}$ and set $\xi = \Lambda(l^d) \in \mathcal{O}_K$ so that the composite morphism

$$E = E^{\sigma_{\Gamma}^{-d}} \rightarrow E$$

is multiplication with ξ . Let

$$T_{\xi}(E) = \varprojlim_{n=kd} E[l^n]$$

be the Tate module of E formed with respect to multiplication by ξ . Then $T_{\xi}(E)$ is free of rank 1 over $\mathcal{O}_{K_{\Gamma}}$ with basis $\zeta = (\zeta_{kd})_{k \geq 0}$ and $\mathfrak{M}_{\Gamma} = \text{Ind}_F^K T_{\Gamma}(E)$ is a G_K -stable projective \mathfrak{A}_{Γ} -lattice in M_{Γ} . The action of G_K on \mathfrak{M}_{Γ} factors through a character

$$\kappa : G_{\mathfrak{m}^{\infty}} \rightarrow \mathfrak{A}_{\Gamma}^{\times}$$

and we also denote by $\kappa : \Lambda \rightarrow \mathfrak{A}_{\Gamma}$ the corresponding ring homomorphism. We denote by $*$ the $\mathcal{O}_{K_{\Gamma}}$ or K_{Γ} -dual and by ζ^{-1} the dual basis of ζ .

Lemma 4.3. *a) There is a natural isomorphism*

$$\Delta^{\infty} \otimes_{\Lambda, \kappa}^{\mathbb{L}} \mathfrak{A}_{\Gamma} \cong R\Gamma_c(\mathcal{O}_K[\frac{1}{\mathfrak{m}l}], \mathfrak{M}_{\Gamma})^*[-3].$$

b) The image of an element

$$u = (u_n)_{n \geq 0} \in \varprojlim_n H^1(\mathcal{O}_{F_n}[\frac{1}{\mathfrak{m}l}], \mathbb{Z}/l^n \mathbb{Z}(1)) \cong U_{\{v|\mathfrak{m}l\}}^{\infty} = H^1(\Delta^{\infty})$$

under the induced isomorphism

$$H^1(\Delta^{\infty}) \otimes_{\Lambda, \kappa} \mathfrak{A}_{\Gamma} \cong H^1(\mathcal{O}_K[\frac{1}{\mathfrak{m}l}], \mathfrak{M}_{\Gamma}^*(1)) \cong H^1(\mathcal{O}_F[\frac{1}{\mathfrak{m}l}], T_{\Gamma}(E)^*(1))$$

is given by

$$(4.5) \quad \text{Tr}_{F_n/F}(u_n \cup \zeta_n^{-1})_{n=kd \geq 0}.$$

c) The image of an element

$$s = (s_n)_{n \geq 0} \in \varprojlim_n \mathbb{Z}/l^n \mathbb{Z}[G_{\mathfrak{m}_0 l^n}] \cdot \tau = Y_{\{v|\infty\}}^{\infty}$$

under the isomorphism $Y_{\{v|\infty\}}^{\infty} \otimes_{\Lambda, \kappa} A_{\Gamma} \cong H^0(\text{Spec}(K \otimes_{\mathbb{Q}} \mathbb{R}), M_{\Gamma}^) = M_{\Gamma}^*$ is given by*

$$(s_n \cup \zeta_n^{-1})_{n=kd \geq 0}.$$

Proof. Set $\mathfrak{A}_n = \mathfrak{A}/\Gamma^n$, $\mathfrak{M}_n = \mathfrak{M}_l/\Gamma^n$, $\Lambda_{\mathfrak{A},n} = \mathfrak{A}_n[G_{\tilde{\mathfrak{m}}_0^n}]$ and denote by $\kappa_n : G_{\tilde{\mathfrak{m}}_0^n} \rightarrow \mathfrak{A}_n^\times$ the action on \mathfrak{M}_n . We also denote by κ_n the automorphism of $\Lambda_{\mathfrak{A},n}$ induced by the character $g \mapsto \kappa_n(g)g$ of $G_{\tilde{\mathfrak{m}}_0^n}$. Then $\kappa_\infty = \varprojlim_n \kappa_n$ is an automorphism of

$$\Lambda_{\mathfrak{A}} := \mathfrak{A}_l[[G_{\tilde{\mathfrak{m}}_0^\infty}]].$$

Note here that the notational change from Λ to $\Lambda_{\mathfrak{A}}$ also involves a projection from $G_{\mathfrak{m}_0^\infty}$ to $G_{\tilde{\mathfrak{m}}_0^\infty}$. The sheaf $\mathcal{F}_n := f_{n,*}f_n^*\mathfrak{A}_n$ (where $f_n : \text{Spec}(\mathcal{O}_{F_n}[\frac{1}{\mathfrak{m}_l}]) \rightarrow \text{Spec}(\mathcal{O}_K[\frac{1}{\mathfrak{m}_l}])$ is the natural map and \mathfrak{A}_n denotes the constant sheaf) is free of rank one over $\Lambda_{\mathfrak{A},n}$ with G_K -action given by the *inverse* of the natural projection $G_K \rightarrow G_{\tilde{\mathfrak{m}}_0^n} \subset \Lambda_{\mathfrak{A},n}^\times$. There is a $\Lambda_{\mathfrak{A},n}$ - κ_n^{-1} -semilinear isomorphism $\text{tw} : \mathcal{F}_n \rightarrow \mathcal{F}_n \otimes_{\mathfrak{A}_n} \mathfrak{M}_n$ sending 1 to $1 \otimes \zeta_n$. Shapiro's lemma gives a commutative diagram of isomorphisms

$$(4.6) \quad \begin{array}{ccc} R\Gamma_c(\mathcal{O}_K[\frac{1}{\mathfrak{m}_l}], \mathcal{F}_n) & \xrightarrow{\text{tw}} & R\Gamma_c(\mathcal{O}_K[\frac{1}{\mathfrak{m}_l}], \mathcal{F}_n \otimes_{\mathfrak{A}_n} \mathfrak{M}_n) \\ \downarrow & & \downarrow \\ R\Gamma_c(\mathcal{O}_{F_n}[\frac{1}{\tilde{\mathfrak{m}}_0^l}], \mathfrak{A}_n) & \xrightarrow{\cup \zeta_n} & R\Gamma_c(\mathcal{O}_{F_n}[\frac{1}{\tilde{\mathfrak{m}}_0^l}], \mathfrak{M}_n) \end{array}$$

where the horizontal arrows are $\Lambda_{\mathfrak{A},n}$ - κ_n^{-1} -semilinear. Taking the \mathcal{O}_K/Γ^n -dual (with contragredient $G_{\tilde{\mathfrak{m}}_0^n}$ -action) we obtain a $\# \circ \kappa_n^{-1} \circ \# = \kappa_n$ -semilinear isomorphism

$$R\Gamma_c(\mathcal{O}_K[\frac{1}{\tilde{\mathfrak{m}}_0^l}], \mathcal{F}_n \otimes_{\mathfrak{A}_n} \mathfrak{M}_n)^*[-3] \rightarrow R\Gamma_c(\mathcal{O}_K[\frac{1}{\tilde{\mathfrak{m}}_0^l}], \mathcal{F}_n)^*[-3].$$

After passage to the limit this gives a κ_∞ -semilinear isomorphism

$$R\Gamma_c(\mathcal{O}_K[\frac{1}{\tilde{\mathfrak{m}}_0^l}], \mathcal{F}_\infty \otimes_{\mathfrak{A}_l} \mathfrak{M}_l)^*[-3] \rightarrow R\Gamma_c(\mathcal{O}_K[\frac{1}{\tilde{\mathfrak{m}}_0^l}], \mathcal{F}_\infty)^*[-3] \cong \Delta^\infty \otimes_\Lambda \Lambda_{\mathfrak{A}}.$$

where $\mathcal{F}_\infty = \varprojlim_n \mathcal{F}_n \cong T \otimes_\Lambda \Lambda_{\mathfrak{A}}$. Hence a Λ -linear isomorphism

$$(\Delta^\infty \otimes_\Lambda \Lambda_{\mathfrak{A}}) \otimes_{\Lambda_{\mathfrak{A}}, \kappa_\infty} \Lambda_{\mathfrak{A}} \cong R\Gamma_c(\mathcal{O}_K[\frac{1}{\tilde{\mathfrak{m}}_0^l}], \mathcal{F}_\infty \otimes_{\mathfrak{A}_l} \mathfrak{M}_l)^*[-3].$$

Part a) follows by noting that κ coincides with the composite

$$\Lambda \rightarrow \Lambda_{\mathfrak{A}} \xrightarrow{\kappa_\infty} \Lambda_{\mathfrak{A}} \rightarrow \mathfrak{A}_l$$

where the last map is the augmentation map, and that $\mathcal{F}_\infty \otimes_{\Lambda_{\mathfrak{A}}} \mathfrak{A}_l \cong \mathfrak{A}_l$ with trivial G_K -action. The \mathcal{O}_K/Γ^n -dual of the H^2 of the inverse map in the lower row in (4.6) coincides with

$$H^1(\mathcal{O}_{F_n}[\frac{1}{\tilde{\mathfrak{m}}_0^l}], \mathfrak{M}_n^*(1)) \xleftarrow{\cup \zeta_n^{-1}} H^1(\mathcal{O}_{F_n}[\frac{1}{\tilde{\mathfrak{m}}_0^l}], \mathfrak{A}_n(1))$$

by Poitou-Tate duality. This gives b). Similarly to the lower row in (4.6) we have a κ^{-1} -semilinear isomorphism

$$\mathcal{F}_n = H^0(F_n \otimes \mathbb{R}, \mathfrak{A}_n) \xrightarrow{\cup \zeta_n} H^0(F_n \otimes \mathbb{R}, \mathcal{F}_n \otimes_{\mathfrak{A}_n} \mathfrak{M}_n) = \mathcal{F}_n \otimes_{\mathfrak{A}_n} \mathfrak{M}_n,$$

the $\mathcal{O}_K/\mathfrak{l}^n$ -dual of the inverse of which is the κ -semilinear isomorphism $\Lambda_{\mathfrak{A}_n} \cdot \tau \rightarrow \Lambda_{\mathfrak{A}_n} \cdot \tau \cup \zeta_n^{-1}$ given by cup product with ζ_n^{-1} . Passing to the limit and tensoring over $\Lambda_{\mathfrak{A}}$ with A_l we deduce c). \square

For the next proposition we introduce some notation from [17, Ch.I]. Let L/K_l be a finite unramified extension of degree d and \mathcal{E}/L a relative Lubin-Tate group in the sense of [17, Thm. I.1.3] with respect to a given element $\xi \in \mathcal{O}_{K_l}$ of valuation d . Let $\varphi \in \text{Gal}(L/K_l)$ be the Frobenius and

$$f : \mathcal{E} \rightarrow \mathcal{E}^\varphi$$

an isogeny so that $f^{(d)}$ is multiplication by ξ where

$$f^{(n)} = f^{\varphi^{n-1}} \circ \dots \circ f^\varphi \circ f : \mathcal{E} \rightarrow \mathcal{E}^{\varphi^n}.$$

Denote by ζ_n a sequence of primitive \mathfrak{l}^n -division points on $\mathcal{E}^{\varphi^{-n}}$ so that $f^{\varphi^{-n}}(\zeta_n) = \zeta_{n-1}$ and set $L_n = L(\zeta_n)$. Finally, let $u_n \in \varprojlim L_n^\times$ be a norm compatible system with Coleman power series $g_{u,\zeta}$, i.e. so that

$$g_{u,\zeta}^{\varphi^{-n}}(\zeta_n) = u_n.$$

We view $g_{u,\zeta}$ as a function on the formal group \mathcal{E} . For a fixed $m \geq 1$ define

$$c_m(u) = \text{Tr}_{L_n/L_m}(u_n \cup \zeta_n^{-1})_{n=kd \geq m} \in H^1(L_m, T_\xi(\mathcal{E})^*(1))$$

where $T_\xi(\mathcal{E}) = \varprojlim_{n=kd} \mathcal{E}[\mathfrak{l}^n]$ is the Tate-module of \mathcal{E} with respect to multiplication by ξ .

Proposition 4.1. (*Explicit reciprocity law*) For $m \geq 1$ the dual exponential map of the Galois representation $T_\xi(\mathcal{E})^*(1)$

$$\exp^* : H^1(L_m, T_\xi(\mathcal{E})^*(1)) \rightarrow D_{dR}^0(V_\xi(\mathcal{E})^*(1)) \cong H^0(\mathcal{E}, \Omega^1) \otimes_L L_m$$

sends $c_m(u)$ to

$$\pi_m^{-\varphi^{-m}} \omega \frac{\text{dlog}(g_{u,\zeta}^{\varphi^{-m}})}{\omega}(\zeta_m)$$

where ω is an L -basis of $H^0(\mathcal{E}, \Omega^1)$ and $\pi_m \in L$ is defined by $f^{(m)*} \omega^{\varphi^n} = \pi_m \omega$.

Proof. The proof of [28, Thm. II.2.1.7] is only for the case where \mathcal{E} is a base change to L of a Lubin-Tate group over K_l but extends easily to the case of a relative Lubin-Tate group. \square

After fixing a place \mathfrak{l} of K^{ab} extending \mathfrak{l} we let \mathcal{E} be the formal group of E over $L = F_{0,\mathfrak{l}}$. Then \mathcal{E} is a relative Lubin-Tate group with respect to the element $\xi = \Lambda(\mathfrak{l}^d) \in \mathcal{O}_K$ since E/F_0 has good reduction at \mathfrak{l} , and

$$L_n = F_{n,\mathfrak{l}} = K(\tilde{\mathfrak{m}}_0 \mathfrak{l}^n)_{\mathfrak{l}}, \quad \varphi = \sigma_{\mathfrak{l}}, \quad f = \lambda(\mathfrak{l}), \quad \pi_m = \Lambda(\mathfrak{l}^m).$$

The system $({}_a z_{\tilde{\mathfrak{m}}_0 \mathfrak{l}^n} \in L_n^\times)_{n \geq 1}$ is norm compatible.

Lemma 4.4. *The Coleman power series of the system $u_n = {}_a z_{\tilde{\mathfrak{m}}_0 \mathfrak{l}^n}$ with respect to the system of torsion points ζ_n is induced by the function*

$$g(x) = g_{u,\zeta}(x) = {}_a \Theta_{E/F_0}(x + q_0)$$

on the elliptic curve E/F_0 .

Proof. (see also [17, Ch. II, Prop. 4.9]). The function $g^{\varphi^{-n}}(x)$ on $\mathcal{E}^{\varphi^{-n}}$ is induced by the function

$${}_a \Theta_{E/F_0}(x + q_0)^{\sigma_{\mathfrak{l}}^{-n}} = {}_a \Theta_{E^{\sigma_{\mathfrak{l}}^{-n}}/F_0}(x + q_0^{\sigma_{\mathfrak{l}}^{-n}}) = {}_a \Theta_{E^{\sigma_{\mathfrak{l}}^{-n}}/F_0}(x + q_n)$$

on $E^{\sigma_{\mathfrak{l}}^{-n}}$. By (4.3) under the embedding τ_0 we have

$$(4.7) \quad {}_a \Theta_{E^{\sigma_{\mathfrak{l}}^{-n}}/F_0}(z) = \psi(z, \Lambda(\mathfrak{l}^{-n}) \mathfrak{l}^n \Omega \tilde{\mathfrak{m}}_0, \mathfrak{a}^{-1} \Lambda(\mathfrak{l}^{-n}) \mathfrak{l}^n \Omega \tilde{\mathfrak{m}}_0)$$

and hence

$$\begin{aligned} g^{\varphi^{-n}}(\zeta_n) &= {}_a \Theta_{E^{\sigma_{\mathfrak{l}}^{-n}}/F_0}(p_n) = \psi(\Lambda(\mathfrak{l}^{-n}) \Omega, \Lambda(\mathfrak{l}^{-n}) \mathfrak{l}^n \Omega \tilde{\mathfrak{m}}_0, \mathfrak{a}^{-1} \Lambda(\mathfrak{l}^{-n}) \mathfrak{l}^n \Omega \tilde{\mathfrak{m}}_0) \\ &= \psi(1, \mathfrak{l}^n \tilde{\mathfrak{m}}_0, \mathfrak{a}^{-1} \mathfrak{l}^n \tilde{\mathfrak{m}}_0) = {}_a z_{\tilde{\mathfrak{m}}_0 \mathfrak{l}^n}. \end{aligned}$$

□

The following Proposition is an analogue of Lemma 4.1 for the pair (M, A) considered in this section.

Proposition 4.2. *Let R be a direct factor of $A_{\mathfrak{l}}$ which is a field, \mathfrak{q} the kernel of the map $\kappa : \Lambda \rightarrow A_{\mathfrak{l}} \rightarrow R$ and $\Lambda_{\mathfrak{q}}$ the localization of Λ at \mathfrak{q} . If $L(\phi_{\varepsilon}^{-1}, 0) \neq 0$ for all $\varepsilon \in J$ then the element \mathcal{L} is a basis of $\text{Det}_{\Lambda_{\mathfrak{q}}} \Delta_{\mathfrak{q}}^{\infty}$. Denote by $\mathcal{L} \otimes 1$ the image of \mathcal{L} under the determinant*

$$\text{Det}_{\Lambda_{\mathfrak{q}}} \Delta_{\mathfrak{q}}^{\infty} \otimes_{\Lambda_{\mathfrak{q}}, \kappa} A_{\mathfrak{l}} \cong \text{Det}_{A_{\mathfrak{l}}} R\Gamma_c(\mathcal{O}_K[\frac{1}{\mathfrak{ml}}], M_{\mathfrak{l}})^*[-3].$$

of the isomorphism of Lemma 4.3 a). Then the image of $\mathcal{L} \otimes 1$ under

$$(4.8) \quad \begin{aligned} \text{Det}_{A_{\mathfrak{l}}} R\Gamma_c(\mathcal{O}_K[\frac{1}{\mathfrak{ml}}], M_{\mathfrak{l}})^*[-3] &\cong \text{Det}_{A_{\mathfrak{l}}} R\Gamma_c(\mathcal{O}_K[\frac{1}{\mathfrak{ml}}], M_{\mathfrak{l}})^{\#} \\ &\cong \Xi_{(A)M}^{\#} \otimes_A A_{\mathfrak{l}} \end{aligned}$$

coincides with the element described in Corollary 2.1.

Proof. As already indicated in Corollary 2.1 the non-vanishing of $L(\phi_\varepsilon^{-1}, 0)$ implies that $E(F) = 0$ does not occur in the fundamental line. The isomorphism

$$\Xi({}_A M) \otimes_A A_{\mathfrak{l}} \cong \text{Det}_{A_{\mathfrak{l}}} R\Gamma_c(\mathcal{O}_K[\frac{1}{\mathfrak{ml}}], M_{\mathfrak{l}})$$

is then induced by passing to cohomology as well as the isomorphisms

$$H_1(E^{\tau_1}(\mathbb{C}), \mathbb{Q}) \otimes_A A_{\mathfrak{l}} \cong M_{\mathfrak{l}} \cong H^0(K \otimes_{\mathbb{Q}} \mathbb{R}, M_{\mathfrak{l}}) \xrightarrow{\sim} H_c^1(\mathcal{O}_K[\frac{1}{\mathfrak{ml}}], M_{\mathfrak{l}})$$

and

$$\text{Hom}_F(H^0(E, \Omega_{E/F}^1), F) \otimes_A A_{\mathfrak{l}} \cong H_{dR}^1(E/F)/F^0 \otimes_A A_{\mathfrak{l}} \cong D_{dR}(M_{\mathfrak{l}})/D_{dR}^0(M_{\mathfrak{l}})$$

and

$$D_{dR}(M_{\mathfrak{l}})/D_{dR}^0(M_{\mathfrak{l}}) \xrightarrow{\text{exp}} H_f^1(K_{\mathfrak{l}}, M_{\mathfrak{l}}) \xrightarrow{\sim} H_c^2(\mathcal{O}_K[\frac{1}{\mathfrak{ml}}], M_{\mathfrak{l}}).$$

The isomorphism (4.8) is induced by the dual maps

(4.9)

$$H_1(E^{\tau_1}(\mathbb{C}), \mathbb{Q})^* \otimes_A A_{\mathfrak{l}} \cong M_{\mathfrak{l}}^* \cong H^0(K \otimes_{\mathbb{Q}} \mathbb{R}, M_{\mathfrak{l}})^* \xleftarrow{\sim} H_c^1(\mathcal{O}_K[\frac{1}{\mathfrak{ml}}], M_{\mathfrak{l}})^*$$

and

$$(4.10) \quad D_{dR}^0(M_{\mathfrak{l}}^*(1)) \xleftarrow{\text{exp}^*} H_{/f}^1(K_{\mathfrak{l}}, M_{\mathfrak{l}}^*(1)) \\ \xleftarrow{\sim} H^1(\mathcal{O}_K[\frac{1}{\mathfrak{ml}}], M_{\mathfrak{l}}^*(1)) \xleftarrow{\sim} H_c^2(\mathcal{O}_K[\frac{1}{\mathfrak{ml}}], M_{\mathfrak{l}})^*.$$

Lemma 4.5. *The image of $(N\mathfrak{a} - \sigma_{\mathfrak{a}})^{-1} \mathfrak{a} z_{\mathfrak{m}_0 l^\infty}$ under the isomorphism of Lemma 4.3 b) composed with (4.10) is*

$$(4.11) \quad \prod_{\mathfrak{p}|l, \mathfrak{p} \nmid \mathfrak{m}} (1 - \phi(\mathfrak{p})^{-1}) E_1(\Omega, \Omega \mathfrak{m}) \omega.$$

Proof. Assume $m \geq 1$. Then by [17, II.1.4, (17)]

$$\lambda := \pi_m^{-\phi^{-m}} = \Lambda(\mathfrak{l}^m)^{-\sigma_{\mathfrak{l}}^{-m}} = \Lambda(\mathfrak{l}^{-m})$$

and Prop. 4.1, Lemma 4.4 and equations (4.7) and (2.3) show that the image of $\mathfrak{a} z_{\tilde{\mathfrak{m}}_0 l^\infty}$ under the isomorphism of Lemma 4.3 b) is given by

$$\lambda \omega \frac{d}{dz} \log \psi(z + q_m, \lambda \mathfrak{l}^m \Omega \tilde{\mathfrak{m}}_0, \mathfrak{a}^{-1} \lambda \mathfrak{l}^m \Omega \tilde{\mathfrak{m}}_0)|_{z=\zeta_m} \\ = \lambda \omega E_1(\lambda \Omega, \lambda \mathfrak{l}^m \Omega \tilde{\mathfrak{m}}_0, \mathfrak{a}^{-1} \lambda \mathfrak{l}^m \Omega \tilde{\mathfrak{m}}_0) \\ = E_1(\Omega, \Omega \mathfrak{m}, \mathfrak{a}^{-1} \Omega \mathfrak{m}) \omega.$$

If $m = 0$ (i.e. $\tilde{\mathfrak{m}}_0 = \mathfrak{m}$) the image is

$$\begin{aligned}
 & \mathrm{Tr}_{F_1/F} E_1(\Omega, \mathfrak{l}\Omega\tilde{\mathfrak{m}}_0, \mathfrak{a}^{-1}\mathfrak{l}\Omega\tilde{\mathfrak{m}}_0)\omega \\
 &= \sum_{c \in 1 + \tilde{\mathfrak{m}}_0 / 1 + \tilde{\mathfrak{m}}_0 \mathfrak{l}} E_1(\Omega, \mathfrak{l}\Omega\tilde{\mathfrak{m}}_0, \mathfrak{a}^{-1}\mathfrak{l}\Omega\tilde{\mathfrak{m}}_0)^{\sigma(c)}\omega \\
 &= \sum_{c \in 1 + \tilde{\mathfrak{m}}_0 / 1 + \tilde{\mathfrak{m}}_0 \mathfrak{l}} E_1(\Lambda((c))\Omega, \Lambda((c))c^{-1}\mathfrak{l}\Omega\tilde{\mathfrak{m}}_0, \mathfrak{a}^{-1}\Lambda((c))c^{-1}\mathfrak{l}\Omega\tilde{\mathfrak{m}}_0)\omega \\
 &= \sum_{c \in 1 + \tilde{\mathfrak{m}}_0 / 1 + \tilde{\mathfrak{m}}_0 \mathfrak{l}} E_1(c\Omega, \mathfrak{l}\Omega\tilde{\mathfrak{m}}_0, \mathfrak{a}^{-1}\mathfrak{l}\Omega\tilde{\mathfrak{m}}_0)\omega \\
 &= \sum_{t \in \tilde{\mathfrak{m}}_0 / \tilde{\mathfrak{m}}_0 \mathfrak{l}} E_1(\Omega + \Omega t, \mathfrak{l}\Omega\tilde{\mathfrak{m}}_0, \mathfrak{a}^{-1}\mathfrak{l}\Omega\tilde{\mathfrak{m}}_0)\omega - E_1(c_0\Omega, \mathfrak{l}\Omega\tilde{\mathfrak{m}}_0, \mathfrak{a}^{-1}\mathfrak{l}\Omega\tilde{\mathfrak{m}}_0)\omega \\
 &= E_1(\Omega, \Omega\tilde{\mathfrak{m}}_0, \mathfrak{a}^{-1}\Omega\tilde{\mathfrak{m}}_0)\omega - E_1(c_0\Omega, \mathfrak{l}\Omega\tilde{\mathfrak{m}}_0, \mathfrak{a}^{-1}\mathfrak{l}\Omega\tilde{\mathfrak{m}}_0)\omega \\
 &= (1 - \phi(\mathfrak{l})^{-1})E_1(\Omega, \Omega\mathfrak{m}, \mathfrak{a}^{-1}\Omega\mathfrak{m})\omega.
 \end{aligned}$$

Here we have used the distribution relation (2.1) and $c_0 \in \mathcal{O}_K$ is such that $c_0 \equiv 1 \pmod{\tilde{\mathfrak{m}}_0}$, $c_0 \equiv 0 \pmod{\mathfrak{l}}$. Since E is defined over $F = K(\tilde{\mathfrak{m}}_0)$ and the conductor of ϕ divides $\tilde{\mathfrak{m}}_0$ we have $\Lambda((c)) = \phi((c)) = c$ for $c \in 1 + \tilde{\mathfrak{m}}_0$. The last equality follows since

$$\begin{aligned}
 \phi(\mathfrak{l})E_1(c_0\Omega, \Omega\mathfrak{l}\mathfrak{m}, \mathfrak{a}^{-1}\Omega\mathfrak{l}\mathfrak{m})\omega &= E_1(c_0\Omega, \Omega\mathfrak{l}\mathfrak{m}, \mathfrak{a}^{-1}\Omega\mathfrak{l}\mathfrak{m})^{\sigma_{\mathfrak{l}}}\Lambda(\mathfrak{l})\omega \\
 &= E_1(\Lambda(\mathfrak{l})c_0\Omega, \Lambda(\mathfrak{l})\Omega\mathfrak{m}, \Lambda(\mathfrak{l})\mathfrak{a}^{-1}\Omega\mathfrak{m})\Lambda(\mathfrak{l})\omega \\
 &= E_1(c_0\Omega, \Omega\mathfrak{m}, \mathfrak{a}^{-1}\Omega\mathfrak{m})\omega \\
 &= E_1(\Omega, \Omega\mathfrak{m}, \mathfrak{a}^{-1}\Omega\mathfrak{m})\omega.
 \end{aligned}$$

using [17, II, Prop. 3.3] and the fact that $E_1(z, \Omega\mathfrak{m}, \mathfrak{a}^{-1}\Omega\mathfrak{m})$ is $\Omega\mathfrak{m}$ -periodic.

Using Lemma 2.3 a) and c) as well as (2.4) we find

$$\begin{aligned}
 & E_1(\Omega, \Omega\mathfrak{m}, \mathfrak{a}^{-1}\Omega\mathfrak{m})\omega \\
 &= (N\mathfrak{a} E_1(\Omega, \Omega\mathfrak{m}) - E_1(\Omega, \mathfrak{a}^{-1}\Omega\mathfrak{m}))\omega \\
 &= (N\mathfrak{a} E_1(\Omega, \Omega\mathfrak{m}) - \Lambda(\mathfrak{a})E_1(\Lambda(\mathfrak{a})\Omega, \Lambda(\mathfrak{a})\mathfrak{a}^{-1}\Omega\mathfrak{m}))\omega \\
 &= (N\mathfrak{a} E_1(\Omega, \Omega\mathfrak{m}) - \Lambda(\mathfrak{a})E_1(\Omega, \Omega\mathfrak{m})^{\sigma_{\mathfrak{a}}})\omega \\
 &= N\mathfrak{a} E_1(\Omega, \Omega\mathfrak{m})\omega - E_1(\Omega, \Omega\mathfrak{m})^{\sigma_{\mathfrak{a}}}\lambda(\mathfrak{a})^*\omega^{\sigma_{\mathfrak{a}}} \\
 &= N\mathfrak{a} E_1(\Omega, \Omega\mathfrak{m})\omega - \phi(\mathfrak{a})^*(E_1(\Omega, \Omega\mathfrak{m})\omega) \\
 &= (N\mathfrak{a} - \phi(\mathfrak{a})^*)E_1(\Omega, \Omega\mathfrak{m})\omega.
 \end{aligned}$$

Since $\kappa(\sigma_{\mathfrak{p}}) = \phi(\mathfrak{p})$ for $\mathfrak{p} \nmid \mathfrak{m}$ and since the image of ${}_a z_{\mathfrak{m}_0 l^\infty}$ is

$$\begin{cases} {}_a z_{\tilde{\mathfrak{m}}_0 l^\infty}^{1-\sigma_l^{-1}} & \text{if } \mathfrak{m}_0 = \tilde{\mathfrak{m}}_0 \text{ and } l \text{ is split} \\ {}_a z_{\tilde{\mathfrak{m}}_0 l^\infty} & \text{otherwise,} \end{cases}$$

the image of the element $(N\mathfrak{a} - \sigma_{\mathfrak{a}})^{-1} {}_a z_{\mathfrak{m}_0 l^\infty}$ in $H^0(E, \Omega_{E/F}^1)$ is given by (4.11) for any $m \geq 0$. \square

The point $\Omega = \Omega_{\tilde{\mathfrak{m}}_0} \in \mathbb{C}$ is a primitive $\tilde{\mathfrak{m}}_0$ -division point corresponding to

$$\gamma = \gamma_{\tilde{\mathfrak{m}}_0} \in H_1(E^{\tau_0}(\mathbb{C}), \mathbb{Q}) \subseteq H_1(E^{\tau_0}(\mathbb{C}), \mathbb{R}) \cong \mathbb{C}.$$

In $H_1(E^{\tau_0}(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_l \cong V_{\xi}(E)$ the element γ corresponds to the system

$$(\xi^{-k} \Omega)_{k \geq 0} = (\Lambda(\iota^{-kd}) \Omega)_{k \geq 0} = (p_{kd})_{k \geq 0} = (\zeta_{kd})_{k \geq 0}$$

where we have used the fact that $q_n = p_n - \zeta_n$ is prime-to- l -torsion. Recalling that passage to cohomology on $\text{Det}_{A_t} R\Gamma_c(\mathcal{O}_K[\frac{1}{ml}], M_t)$ introduces a factor $\prod_{\mathfrak{p}|m} (1 - \phi(\mathfrak{p})^{-1})$ Lemma 4.5 then shows that \mathcal{L} is mapped to the element

$$(4.12) \quad \prod_{\mathfrak{p}|m} (1 - \phi^{-1}(\mathfrak{p})) \gamma_{\tilde{\mathfrak{m}}_0}^* \otimes_A (E_1(\Omega_{\tilde{\mathfrak{m}}_0}, \Omega_{\tilde{\mathfrak{m}}_0} \mathfrak{m}) \omega)^{-1}.$$

For simplicity we now also assume that $\mathfrak{m} = \tilde{\mathfrak{m}}_0 l^m$ has been increased so that m is a multiple of d so that $l^m = (\xi^{m/d})$ is principal. By definition $\Omega_{\tilde{\mathfrak{m}}_0} \mathfrak{m}_0 = \Omega_{\mathfrak{m}} \mathfrak{m}$ is the period lattice of ω and hence $\Omega_{\tilde{\mathfrak{m}}_0} = \xi^{-m/d} \Omega_{\mathfrak{m}}$ and $\gamma_{\tilde{\mathfrak{m}}_0} = \xi^{-m/d} \gamma_{\mathfrak{m}}$. Hence (4.12) equals

$$\begin{aligned} & \prod_{\mathfrak{p}|m} (1 - \phi^{-1}(\mathfrak{p})) \xi^{m/d} (\gamma_{\mathfrak{m}}^*) \otimes_A \xi^{-m/d} (E_1(\Omega_{\mathfrak{m}}, \Omega_{\mathfrak{m}} \mathfrak{m}) \omega)^{-1} \\ &= \prod_{\mathfrak{p}|m} (1 - \phi^{-1}(\mathfrak{p})) (\gamma_{\mathfrak{m}}^*) \otimes_A (E_1(\Omega_{\mathfrak{m}}, \Omega_{\mathfrak{m}} \mathfrak{m}) \omega)^{-1} \\ &= \prod_{\mathfrak{p}|m} (1 - \phi^{-1}(\mathfrak{p})) [F : K] (\gamma_{\mathfrak{m}}^*) \otimes_A ([F : K] E_1(\Omega_{\mathfrak{m}}, \Omega_{\mathfrak{m}} \mathfrak{m}) \omega)^{-1} \\ &= \prod_{\mathfrak{p}|m} (1 - \phi^{-1}(\mathfrak{p})) \gamma_{\mathfrak{m}}^{-1} \otimes_A ([F : K] E_1(\Omega_{\mathfrak{m}}, \Omega_{\mathfrak{m}} \mathfrak{m}) \omega)^{-1} \end{aligned}$$

since $\gamma_{\mathfrak{m}}^{-1} = [F : K] \gamma_{\mathfrak{m}}^*$ (the identification of the A -dual with the K -dual is made via the trace map). So we do indeed find the element of Corollary 2.1. Since this element is a A_t -basis of (4.8) \mathcal{L} is a basis of $\text{Det}_{\Lambda_{\mathfrak{q}}} \Delta_{\mathfrak{q}}^{\infty}$ by Nakayama's Lemma. \square

We remark that the computations of the present section have been (essentially) extended to infinity type (k, j) where $k \leq -1$, $j = 0$ by Kato [28, Thm. III.1.2.6], to $0 \geq -j > k \leq -1$ by Tsuji [41], to $j = k + 1 > 0$ by Kings [30], to $k, j > 0$ by Chida [8] (all of these only if K has class number one) and to $j = k > 0$ by Johnson [27] (in general).

5. THE CYCLOTOMIC DEFORMATION AND THE EXPONENTIAL OF PERRIN-RIOU

Let M_l be a finite dimensional \mathbb{Q}_l -vector space with a continuous action of $G_{\mathbb{Q}, S}$ for a finite set of primes S containing l , and such that M_l is crystalline as a representation of $G_{\mathbb{Q}_l}$. In this section we do not assume that M_l is the l -adic realisation of a motive. We shall briefly sketch the ideas of Perrin-Riou [34] of how to construct, under some weak assumptions, a trivialisation of $\text{Det}_{\mathbb{K}} R\Gamma_c(\mathbb{Z}[\frac{1}{S}], T) \otimes \mathbb{K}$ where T is the cyclotomic deformation of M_l and \mathbb{K} is a rather large coefficient ring containing the classical Iwasawa algebra. The key ingredients are the crystalline comparison isomorphism and the exponential of Perrin-Riou, both purely local constructions over \mathbb{Q}_l .

We let $F_n = \mathbb{Q}(\zeta_l^n)$, $F_\infty = \bigcup F_n$ and set $G_\infty = \text{Gal}(F_\infty/\mathbb{Q}) \cong \mathbb{Z}_l^\times$ and $\Delta = G_\infty^{\text{tor}}$. Define

$$\Lambda := \mathbb{Z}_l[[G_\infty]] \cong \mathbb{Z}_l[\Delta][[X]]; \quad T^{\text{cyclo}} = \varprojlim_n H^0(\text{Spec}(F_n \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}), \mathbb{Z}_l)$$

so that T^{cyclo} is a free rank one Λ -module upon which $G_{\mathbb{Q}}$ acts via the inverse of the tautological character $G_{\mathbb{Q}} \rightarrow G_\infty \subseteq \Lambda^\times$. Define a free Λ -module

$$T = \mathfrak{M}_l \otimes_{\mathbb{Z}_l} T^{\text{cyclo}}$$

where \mathfrak{M}_l is any $G_{\mathbb{Q}}$ -stable \mathbb{Z}_l -lattice in M_l and $G_{\mathbb{Q}}$ acts diagonally on T . We call the resulting l -adic family (Λ, T) the cyclotomic deformation of \mathfrak{M}_l .

Set $D = D_{\text{crys}}(M_l)$ and let

$$(5.1) \quad B_{\text{cris}} \otimes_{\mathbb{Q}_l} M_l \cong B_{\text{cris}} \otimes_{\mathbb{Q}_l} D$$

be the l -adic period isomorphism of the crystalline representation M_l . Let $\mathcal{H}(X)$ be the set of power series in $\mathbb{Q}_l[[X]]$ which converges on the unit disc $\{X \in \mathbb{C}_l \mid |X|_l < 1\}$ and set $\mathcal{H}(\Lambda) = \mathbb{Z}_l[\Delta] \otimes_{\mathbb{Z}_l} \mathcal{H}(X)$ where $X = \gamma - 1$. Set $D^\infty = D \otimes_{\mathbb{Z}_l} \mathbb{Z}_l[[X]]^{\psi=0}$, let

$$\text{Exp} : D^\infty \rightarrow \mathcal{H}(\Lambda) \otimes_{\Lambda} H^1(\mathbb{Q}_l, T)/T^{\text{Gal}(\bar{\mathbb{Q}}_l/F_{\infty, l})}$$

be the exponential of Perrin-Riou [33] and

$$\text{Per} : B_{cris} \otimes_{\mathbb{Z}_l} T \cong B_{cris} \otimes_{\mathbb{Z}_l} D^\infty$$

the isomorphism induced by (5.1), as well as a choice of basis of the (free, rank one) Λ -modules T^{cyclo} and $\mathbb{Z}_l[[X]]^{\psi=0}$. Let

$$\mathbb{K} = \text{Frac}(B_{cris} \otimes \mathcal{H}(\Lambda))$$

be the total ring of fractions of $B_{cris} \otimes \mathcal{H}(\Lambda)$, a finite product of fields indexed by the \mathbb{Q}_l -rational characters of Δ . The maps Exp and Per induce an isomorphism

$$(5.2) \quad H^1(\mathbb{Q}_l, T) \otimes_\Lambda \mathbb{K} \cong T \otimes_\Lambda \mathbb{K}.$$

Assume that

- a) (Weak Leopoldt) $H^2(\mathbb{Z}[\frac{1}{S}], T)$ is a torsion Λ -module
- b) The composite map

$$\rho : H^1(\mathbb{Z}[\frac{1}{S}], T) \rightarrow H^1(\mathbb{Q}_l, T) \rightarrow H^1(\mathbb{Q}_l, T) \otimes_\Lambda \mathbb{K} \cong T \otimes_\Lambda \mathbb{K}$$

is injective and we have

$$T \otimes_\Lambda \mathbb{K} \cong H^0(\mathbb{R}, T) \otimes_\Lambda \mathbb{K} \oplus \text{im}(\rho) \otimes_\Lambda \mathbb{K}$$

for a given choice of decomposition groups $G_{\mathbb{R}}, G_{\mathbb{Q}_l} \subseteq G_{\mathbb{Q}}$.

By standard formulae for the Euler characteristic the class in $K_0(\mathbb{K})$ of both sides in b) agree. Under these conditions one has an isomorphism

$$\zeta_{\mathbb{K}}^{\text{alg}}(T) : \text{Det}_{\mathbb{K}}(0) \cong \text{Det}_{\Lambda} R\Gamma_c(\mathbb{Z}[\frac{1}{S}], T) \otimes_\Lambda \mathbb{K}$$

arising from the triangle

$$R\Gamma_c(\mathbb{Z}[\frac{1}{S}], T) \rightarrow R\Gamma(\mathbb{Z}[\frac{1}{S}], T) \rightarrow \bigoplus_{p \in S} R\Gamma(\mathbb{Q}_p, T),$$

the isomorphisms

$$\begin{aligned} R\Gamma(\mathbb{Z}[\frac{1}{S}], T) \otimes_\Lambda \mathbb{K} &\cong H^1(\mathbb{Z}[\frac{1}{S}], T) \otimes_\Lambda \mathbb{K} \\ R\Gamma(\mathbb{Q}_l, T) \otimes_\Lambda \mathbb{K} &\cong H^1(\mathbb{Q}_l, T) \otimes_\Lambda \mathbb{K} \\ R\Gamma(\mathbb{R}, T) \otimes_\Lambda \mathbb{K} &\cong H^0(\mathbb{R}, T) \otimes_\Lambda \mathbb{K}, \end{aligned}$$

and the acyclicity of $R\Gamma(\mathbb{Q}_p, T) \otimes_\Lambda \mathbb{K}$ for $p \neq l$. The "algebraic" l -adic L-function of \mathfrak{M}_l is any element $\mathcal{L} \in \mathbb{K}^\times = \text{Aut}(\text{Det}_{\mathbb{K}}(0))$ so that the composite isomorphism

$$(5.3) \quad \text{Det}_{\mathbb{K}}(0) \xrightarrow{\mathcal{L}} \text{Det}_{\mathbb{K}}(0) \xrightarrow{\zeta_{\mathbb{K}}^{\text{alg}}(T)} \text{Det}_{\Lambda} R\Gamma_c(\mathbb{Z}[\frac{1}{S}], T) \otimes_\Lambda \mathbb{K}$$

is the scalar extension of an isomorphism

$$\mathrm{Det}_\Lambda(0) \cong \mathrm{Det}_\Lambda R\Gamma_c(\mathbb{Z}[\frac{1}{S}], T).$$

Of course \mathcal{L} is only determined up an element in $\Lambda^\times = \mathrm{Aut}(\mathrm{Det}_\Lambda(0))$. The "analytic" l -adic L-function of \mathfrak{M}_l is the unique element $\mathcal{L} = \mathcal{L}^{an}$ of \mathbb{K}^\times so that the composite isomorphism (5.3) is the scalar extension of the isomorphism $\zeta_\Lambda(T)$ of Kato and Fukaya (and hence specializes to the motivic isomorphisms $\zeta_{A_l}(M_l)$).

The isomorphism $\zeta_{\mathbb{K}}^{alg}(T)$ can be constructed without any reference to motivic L-functions and only requires the rather weak assumptions a) and b) on global Galois cohomology. However, it only exists over the large ring \mathbb{K} and no general techniques seem to be available to descend it to Λ (i.e. to specify any element \mathcal{L}).

Perrin-Riou has generalized her theory to the case of a semi-stable representation in [35]. The Exponential of Perrin-Riou, together with the theory of (ϕ, Γ) -modules has been used by Berger and Benois [2] to show the compatibility of Conjecture 6 with the functional equation of $L(A, s)$ in case M_l is crystalline and $A = \mathbb{Q}$.

6. NONCOMMUTATIVE IWASAWA THEORY

There has been much activity in noncommutative Iwasawa theory recently, dealing with either the module theory over non-commutative Iwasawa algebras (see e.g. [43], [42] and other papers by Venjakob) or the conjectural existence of an l -adic L-function and a formulation of a main conjecture for ordinary elliptic curves over the GL_2 -extension generated by the torsion points [12], [13]. We have kept non-commutative Iwasawa theory in the background in this paper because, due to the lack of limit formulas, it has not led to the proof of new cases of Conjectures 2 or 6 so far (there is exciting recent progress, however, in the case of so called false Tate-curve extensions by Darmon and Tian [15]).

The conjectural picture, on the other hand, extends easily to the non-commutative situation. The determinant functor with values in graded invertible modules has to be replaced by the universal determinant functor (with values in virtual objects) which exists for any ring R [19],[5], [21]. This step is justified by the fact that for a ring R which is a product of commutative local rings (such as the coefficient rings A , A_l , \mathfrak{A}_l , Λ) the functor to graded invertible modules is already universal [5, Lemma 3b)].

With this convention the conjectures in part 1 almost literally apply to the situation where A is a non-commutative semisimple finite dimensional \mathbb{Q} -algebra (with some minor modifications if $A_{\mathbb{R}}$ has quaternionic

Wedderburn components, see [19, Part 3]). Similarly, the restriction to commutative Λ in Conjecture 7 is unnecessary. As demonstrated in [21, §4] the nonabelian conjectures of [13] (and also of [26]) are consequences of Conjecture 7. It is also shown in [21, §4] that the earlier conjectures of Coates and Perrin-Riou on the cyclotomic deformation of ordinary motives [10] follow from Conjecture 7.

The functoriality in (ii) includes Morita equivalence by taking $\Lambda = M_n(\Lambda')$ and Y the space of row vectors of length n . Another application of (ii) is the following reduction to Artin motives (see [21, 2.3.5]) which was also observed by Huber and Kings [26, §3.3]. Given any l -adic family (Λ', T') let G_∞ be the image of $G_\mathbb{Q}$ in $\text{Aut}_{\Lambda'}(T')$. Then $G_\infty = \text{Gal}(F_\infty/\mathbb{Q})$ where F_∞/\mathbb{Q} is the union of finite Galois extensions F_n/\mathbb{Q} , say. If we define

$$\Lambda = \mathbb{Z}_l[[G_\infty]]; \quad T = \varprojlim_n H^0(\text{Spec}(F_n \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}), \mathbb{Z}_l)$$

then T is a free rank one Λ -module (via the first factor in $F_n \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}$) with Λ -linear $G_\mathbb{Q}$ -action (via the second factor in $F_n \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}$). Setting $Y = T'$ with right Λ -action via the inverse of $G_\infty \rightarrow \text{Aut}_{\Lambda'}(T')$ there is an isomorphism of l -adic families $T' \cong Y \otimes_\Lambda T$ over Λ' . The isomorphism $\zeta_\Lambda(T)$ can be constructed if Conjecture 3 is known for the Artin motives $h^0(\text{Spec}(F_n))$ over $\mathfrak{A} = \mathbb{Z}[\text{Gal}(F_n/\mathbb{Q})]$. Using (ii) one can then construct $\zeta_{\Lambda'}(T')$ without however knowing (iii) for all motivic points in (Λ', T') . The only examples where the necessary limit formulas are available to carry out this program (and verify (iii) for all motivic points!) is when F_∞ is abelian over \mathbb{Q} or an imaginary quadratic field.

We remark that the only examples where Conjecture 3 is proven for an Artin motive $h^0(\text{Spec}(F))$ over a noncommutative $\mathfrak{A} = \mathbb{Z}[\text{Gal}(F/\mathbb{Q})]$ (or indeed for any motive M with non-commutative non-maximal \mathfrak{A}) is an infinite family of quaternion extensions [6], some dihedral extensions F/\mathbb{Q} and a single example of an $A4$ -extension F/\mathbb{Q} due to Navilarekallu [32].

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