A THEOREM ON GL(n) À LÀ TCHEBOTAREV

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Abstract

Let K/F be a finite Galois extension of number fields. It is well known that the Tchebotarev density theorem implies that an irreducible, finitely ramified p-adic representation ρ of the absolute Galois group of K is determined (up to equivalence) by the characteristic polynomials of Frobenius elements Fr_v at any set of primes v of K of degree 1 over F. Here we prove an analogue for GL(n), namely that a cuspidal automorphic representation π of $\mathrm{GL}(n, \mathbb{A}_K)$ is determined up by the knowledge of its local components at the degree one primes. The method uses, besides the Rankin-Selberg theory of L-functions and the Luo-Rudnick-Sarnak bound for the Hecke roots of π, certain consequences of class field theory via Galois cohomology, which allow us to avoid assuming the presence of sufficiently many roots of unity. We make use of suitable solvable base changes π_M to $M = EK$ relative to certain auxiliary solvable Galois extensions E/F, first deduce that $\pi_M \simeq \pi'_M$ and then descend this isomorphism to one over K. In fact we prove the main result for *isobaric* automorphic representations, which are analogues of *semisimple* Galois representations.

Introduction

Let F be a number field with adele ring $\mathbb{A}_F$. The object of this article is to prove the following:

Theorem A *Let K/F be a finite Galois extension, with $\Sigma^1(K/F)$ denoting the set of primes of K which are of degree 1 over F. Suppose, for $n \geq 1$, π, π' are isobaric automorphic representations of $GL_n(\mathbb{A}_K)$ such that $\pi_v \simeq \pi'_v$ for all but a finite number of v in $\Sigma^1(K/F)$. Then $\pi \simeq \pi'$.*

This result is analogous to, and inspired by, the well known consequence of the Tchebotarev density theorem that a semisimple, finitely ramified, n-dimensional $\overline{\mathbb{Q}}_p$-representation ρ of the absolute Galois group of K is determined, up to isomorphism, by the collection of its restrictions ρ_v to the (decomposition groups of) primes v of degree 1 over F. Note that our main result applies in particular to cuspidal automorphic representations π of $\mathrm{GL}_n(\mathbb{A}_K)$, which are the building blocks of isobaric representations (cf. [7], [5]), and are expected, when algebraic, to be associated to irreducible, potentially semistable p-adic Galois representations of dimension n which are unramified outside a finite set of primes. Even if one is interested only in the case when π, π' are cuspidal, the fact that they may become Eisensteinian when we apply base change to suitable solvable extensions necessitates working in a larger framework.

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On the other hand, it is essential to stick to the isobaric setup of Langlands for otherwise one can produce (non-cuspidal) counterexamples using non-isobaric forms.

A stronger result than Theorem A is known for GL(2), where a cusp form is determined by its components at any set of primes of density $> 7/8$ (cf. [12]), which is the best possible in that setting. When $n = 1$, automorphic forms are just Grossencharacters χ, and a theorem of Hecke asserts that χ is determined by the knowledge of χ_v at a set of primes v of density $> 1/2$. Even more is true in that case: If $\chi_v = \chi'_v$ for all v in a set of positive density, then χ' is a product of χ with a finite order character ν; it is easy to see that such a strong result cannot hold for $n \geq 2$, since even two non-twist equivalent dihedral cusp forms π, π' of $\mathrm{GL}(2)/K$ of a fixed central character will, for example, agree at a quarter of the primes (or more).

A milder version of Theorem A was established for GL(n) by the author in 2010 ([14]), which dealt only with cyclic extensions K/F and also determined π only up to twist equivalence (unless $[K : F] = 2$).

The *main difficulty* arises from the fact that one does not know the truth of the generalized Ramanujan conjecture for unitary cusp forms π on $\mathrm{GL}(n)/K$. Indeed, if one could assume it, which predicts that the (unitarily) normalized inverse roots $\alpha_{j,v}, 1 \leq j \leq n$, of π_v at any unramified v satisfy $|\alpha_{j,v}| = 1$ (purity), then Theorem A would follow with very little work. In fact one can even deduce in this case the natural analogue of the GL(2) (and GL(1)) result, namely that a cuspidal π is determined by its components π_v at any set of primes v of (upper) density $> 1 - \frac{1}{2n^2}$ (see Lemma 3.2 below), which is the best possible.

An elegant Galois analogue was established for ℓ-adic representations by Rajan in [10]. To be precise, he showed that given continuous semisimple ℓ-adic representations ρ, ρ' of $\mathrm{Gal}(\overline{K}/K)$ which are unramified outside a finite number of places, if $\mathrm{tr}(\rho(\mathrm{Fr}_v)) = \mathrm{tr}(\rho'(\mathrm{Fr}_v))$ for all v outside a set of v of lower density $< 1 - \frac{1}{2n^2}$, then ρ, ρ' are isomorphic. Life is unfortunately more complicated in the automorphic world.

An analytic substitute for the generalized Ramanujan conjecture which is useful for automorphic forms on $\mathrm{GL}(n)/K$ is the result of Luo, Rudnick and Sarnak ([8]) yielding the bound $|\alpha_{j,v}| < q_v^{\frac{1}{2}-\delta_n}$ with $\delta_n = \frac{1}{n^2+1}$; this was already used for similar purposes in [11]. (Here q_v denotes as usual the absolute norm of v.)

What is important to our proof of Theorem A is not the exact value of δ_n furnished by [8], but rather that it is positive and *independent of the number field* K. The standard Rankin-Selberg theory, developed for GL(n) in [4] and [16], would give the weaker bound $|\alpha_{j,v}| < q_v^{\frac{1}{2}}$, which does not suffice for us. This is analogous to applications of subconvexity of L-functions, where the exact subconvex bound is not important, but only that it is better than the convexity bound.

A starting point, as in the case of GL(2) ([12]), is the use of a suitable positive Dirichlet series, as well as the ratio $L(s, \pi \otimes \pi')/L(s, \pi \times \pi)$, to get a contradiction if π, π' satisfying the hypothesis of Theorem A are not isomorphic. The use of the bound from [8] shows, as already noticed in [11], that one can effectively ignore all but the set $\Sigma = \Sigma(\pi, \pi')$ of finite places v of degree j over F with $2 \leq j \leq n^2 + 1$ where $\pi_v \not\simeq \pi'_v$. Still, the places in Σ present a serious difficulty, and the main point of this paper is its resolution. The basic idea here is to move to a bigger Galois extension L of F containing K with L/K solvable, and with $[L:F]$ being divisible by the same primes as $[K:F]$, such that the divisors $\tilde{v}$ in L of those in Σ have higher degree over F. Roughly speaking, we use an inductive argument and at each stage the relevant Σ is partitioned into a finite union of subsets Σ_j, with all but finitely many places in each Σ_j acquiring a higher degree in a suitable Galois extension L_j/F containing K (and solvable over K). There will of course be new places u of L_j with low degree, but they will be arising from degree one places in K and so we would know that π_{L_j} and π'_{L_j} agree at such u.

It may be instructive to look at the simplest case, namely when K/F is a cyclic extension of prime degree p, with F containing a primitive p^2-th root of unity, and such that $p < \frac{n^2+1}{2} \leq p^2$. Here, after writing K as $F[\alpha^{1/p})]$ for some $\alpha \in F^* - F^{*p}$, we may consider $L = F[\alpha^{1/p^2}]$, in which the places of F which become inert in K stay inert and hence acquire degree p^2 (over F), allowing us to conclude that the base changes (cf. [1]) π_L, π'_L to GL(n)$/L$ are isomorphic. Suppose for simplicity of exposition here that π, π' are cuspidal. Then $\pi' \simeq \pi \otimes \delta$, for a character δ of K becoming trivial on L (when pulled back by norm). If δ is trivial, there is nothing to prove, so we may take δ to cut out L/K. Besides, we may replace L by $L' = F[(\alpha\beta)^{1/p^2}]$ with $\beta \in F^{*p} - F^{*p^2}$, which furnishes a similar isomorphism $\pi' \simeq \pi \otimes \delta'$. Putting them together, we may assume that π admits a non-trivial self-twist under $\lambda := \delta'\delta^{-1}$, implying in particular that $p \mid n$, in turn forcing $n = p$ since $2p^2 \geq n^2 + 1$. By varying β, we get many different such self-twists, one for each element of the image of F^* in K^*/K^{*p}, which is huge. To get a contradiction, we check (see Lemma B in section 4) that the number of such self-twists for any cusp form on GL(n)$/K$ is bounded above by n^2.

The general case is more involved, and the results and methods of class field theory play a key role in our proof. We repeatedly make use of auxiliary cyclic extensions and auxiliary places, and we also appeal to the solvable base change for GL(n) ([1]), to deduce that base changes of π and π' to suitable solvable Galois extensions R of K, with R/F Galois, are isomorphic. Then we vary R suitably by making an auxiliary finite set of places split completely in R, allowing us to eventualy descend the isomorphism to on over K.

If F were to contain sufficiently many roots of unity, then by using Kummer theory, the representations π, π' satisfying the hypothesis of Theorem A can be shown to be isomorphic by a finer version of the argument given above for p-extensions. When the roots of unity are not present, the most one can show by base changing to appropriate cyclotomic fields is that π, π' are twist equivalent. (This is what we achieved in [14] for K/F cyclic, but the proof here is different, and self-contained.) To do better and eliminate this twisting ambiguity, we have to work without worrying about the presence of roots of unity. Then the difficulty of writing certain Galois characters as p-th powers, and this leads us to contend with some obstructions in Galois cohomology (in degree 2). For every p dividing $|\mathrm{Gal}(K/F)|$, we appeal to the description of $H^2(F, \mathbb{Z}/p^m)$ in [2] via the Tate duality concerning the global-to-local kernel. ($H^2(F, \mathbb{Z}/p^m)$ is the p^m-part of the Brauer group of F when the p^m-th roots of unity are in F, but not otherwise.) The obstructions appear over intermediate fields K'

with $[K : K'] = p$, which we trivialize over auxiliary extensions M' in $M = KE$, with $[M : M'] = p$ and E/F abelian. This way we avoid assuming that F contains any root of unity, and the choice of E/F is sufficiently flexible to force, in addition, any given finite set T of good places v in K to split completely in M. This allows us to descend later to K the isomorphism of (the base changes of) π and π' over M. It is important to note that at no point do we assume that K/F is solvable.

We also have the following consequence of Theorem A (when combined with solvable base change cf. [1] for $\mathrm{GL}(n)$):

Corollary B *Let K/F be a finite non-normal extension of number fields, but with its normal closure $\tilde{K}$ (over F) being solvable over K. If π, π' are isobaric automorphic representations of $GL_n(\mathbb{A}_K)$ such that $\pi_v \simeq \pi'_v$ for all but a finite number of primes v of K of degree one over F, then π, π' become isomorphic upon base change to $\tilde{K}$.*

Note that in this Corollary, $\tilde{K}/F$ need not be solvable, only $\tilde{K}/K$ needs to be so.

The first few sections are brief and of a preliminary nature, which the experts can skip. They are nevertheless needed since, for one, we need to do everything for isobaric representations, and a bit of care is needed for the extension; even if one is interested only in cuspidal automorphic representations, they could become non-cuspidal at an intermediate stage. The key parts of the proof come later, starting in section 5; the important sections are 6 and 8.

As it will be apparent to many, Theorem A is a strengthening of the celebrated, and oft-used *strong multiplicity one theorem* due to Jacquet, Shalika, and Piatetski-Shapiro, which deals with π, π' agreeing outside a *finite* number of places.

We thank David Whitehouse and Jayce Getz for asking for our main result (Theorem A) when K/F is quadratic, which is already non-trivial, for potential applications to (two different instances of) functoriality via the relative trace formula, as our result allows one to reduce to a comparison of certain geometric integrals only at split primes. (David's work is now being continued by B. Feigon and K. Martin.) Recently, we have also come to know of a very nice application of our result in Wei Zhang's proof of the global Gan-Gross-Prasad conjecture for unitary groups under some local restrictions ([18]). We expect that there will be other instances where Theorem A will be useful, potentially for K/F non-solvable.

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0. Basic Facts: A Brief Review

Let F be a global field with adèle ring $\mathbb{A}_F$. Let Σ_F denote the set of all places of F. If $v \in \Sigma_F$ is finite, let q_v denote the cardinality of the residue field at v. For $n \geq 1$, let $\mathcal{A}_0(n, F)$ denote the set of isomorphism classes irreducible, cuspidal automorphic representations of $\mathrm{GL}(n, \mathbb{A}_F)$. Every π representing a class in $\mathcal{A}(n, F)$ is (isomorphic to) a tensor product $\otimes_v, \pi_v$, where v runs over all the places of F, such that each π_v is an irreducible generic representation of $\mathrm{GL}(n, F_v)$ and such that π_v is unramified at almost all v. The strong multiplicity one theorem ([5]) asserts that, for any *finite* subset S of Σ_F, π is determined up to isomorphism by the collection $\{\pi_v \,|\, v \notin S\}$.

For any irreducible, automorphic representation π of $GL(n, \mathbb{A}_F)$, denote by $L(s, \pi) = L(s, \pi_\infty)L(s, \pi_f)$ the associated *standard* $L-$function of π; it has an Euler product expansion

$$L(s, \pi) \;=\; \prod_v L(s, \pi_v),$$

convergent in a right-half plane. If v is a finite place where π_v is unramified, there is a corresponding semisimple (Langlands) conjugacy class $A_v(\pi)$ (or $A(\pi_v)$) in $\mathrm{GL}(n, \mathbb{C})$ such that

$$L(s, \pi_v) \;=\; \det(1 - A_v(\pi)T)^{-1}|_{T=q_v^{-s}}.$$

One may find a diagonal representative $\mathrm{diag}(\alpha_{1,v}(\pi), ..., \alpha_{n,v}(\pi))$ for $A_v(\pi)$, which is unique up to permutation of the diagonal entries. Let $[\alpha_{1,v}(\pi), ..., \alpha_{n,v}(\pi)]$ denote the resulting unordered $n-$tuple. One knows (by Godement-Jacquet) that for any non-trivial cuspidal representation π of $\mathrm{GL}(n, \mathbb{A}_F)$, $L(s, \pi)$ is entire.

If $L(s) = \prod_{v \in \sum_\infty \cap \sum_f} L_v(s)$ is any global L-function with Euler product, and Y a set of places of F, then we will set

$$L_Y(s) = \prod_{v \in Y} L_v(s), \;\text{ and }\; L^Y(s) = \prod_{v \notin Y} L_v(s).$$

By Langlands's theory of Eisenstein series, one has a sum operation $\boxplus$, called the isobaric sum ([5]): Given any $m-$tuple of cuspidal automorphic representations $\pi_1, ..., \pi_m$ of $\mathrm{GL}(n_1, \mathbb{A}_F), ..., \mathrm{GL}(n_m, \mathbb{A}_F)$ respectively, there exists an irreducible, automorphic representation $\pi_1 \boxplus ... \boxplus \pi_m$ of $\mathrm{GL}(n, \mathbb{A}_F)$, $n = n_1 + ... + n_m$, which is unique up to equivalence, such that for any finite set S of places,

$$L^S(s, \boxplus_{j=1}^m \pi_j) \;=\; \prod_{j=1}^m L^S(s, \pi_j).$$

Call such a (Langlands) sum $\pi \simeq \boxplus_{j=1}^m \pi_j$, with each π_j cuspidal, an *isobaric* representation.

Denote by $\mathcal{A}(n, F)$ the set, up to equivalence, of isobaric automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$, and by $\mathcal{A}_u(n, F)$ the subset of isobaric sums of *unitary* cuspidal automorphic representations. If $\pi = \boxplus_{i=1}^m \pi_i$, resp. $\pi' = \boxplus_{j=1}^r \pi'_j$, is in $\mathcal{A}_u(n,F)$, resp. $\mathcal{A}_u(n',F)$, with π_i, π'_j unitary cuspidal, we will need to consider the associated Rankin-Selberg L-function

$$L(s, \pi \times \pi') = \prod_{i,j} L(s, \pi_i \times \pi_j),$$

with

$$L(s, \pi_{i,v} \times \pi'_{j,v}) = \det(1 - A_v(\pi_i) \otimes A_v(\pi'_j)T)^{-1}|_{T=q_v^{-s}}.$$

We have the following basic result ([5]):

Theorem 0.1 (Jacquet–Piatetski-Shapiro–Shalika, Shahidi) *Let $\pi = \boxplus_{i=1}^m \pi_i$, $\pi' = \boxplus_{j=1}^r \pi'_j$ be in $\mathcal{A}_u(n,F)$, with π_i, π'_j unitary cuspidal . Suppose Y is a finite set of places of F containing the archimedean places such that π, π' are unramified outside Y. Then $L^S(s, \pi \times \overline{\pi}')$ has a pole at $s = 1$ iff for some (i,j), π_i is isomorphic to π'_j, in which case the pole is simple.*

Here $\overline{\pi}'$ denotes the complex conjugate representation of π', which, by unitarity, is the contragredient of π'.

The general Ramanujan conjecture predicts that for any $\pi \in \mathcal{A}_u(F)$, π_v is tempered at all v. In particular, if v is a finite place where π is unramified, the unordered n-tuple $\{\alpha_{1,v}(\pi), ..., \alpha_{n,v}(\pi)\}$ representing $A_v(\pi)$ should satisfy $|\alpha_{i,v}| = 1$ for every i. This is far from being proved, and the best known bound to date (for general n) is given by the following:

Theorem 0.2 (Luo–Rudnick–Sarnak [8]) *Let $\pi \in \mathcal{A}_u(n,F)$, and v a finite place where π is unramified, with $A_v(\pi) = \{\alpha_{1,v}(\pi), ..., \alpha_{n,v}(\pi)\}$. Then for every $j \leq n$, one has*

$$|\alpha_{j,v}| < q_v^{\frac{1}{2} - \frac{1}{n^2+1}}.$$

It is crucial to us that this bound is independent of the number field F.

To be precise, Luo, Rudnick and Sarnak only address the case of cusp forms. But for $\pi \in \mathcal{A}_u(n,F)$, any $\alpha_j(\pi)$ must be associated to a cuspidal isobaric constituent π_i on $\mathrm{GL}(n_i)/F$ with $n_i \leq n$, and so the assertion above follows immediately from [8].

We will also need the following (weak) version of the base change theorem for $\mathrm{GL}(n)$:

Theorem 0.3 (Arthur–Clozel [1]) *Let M/F be a finite extension of number fields obtained as a succession of cyclic extensions. Then for every $\pi \in \mathcal{A}_u(n,F)$, there exists a corresponding $\pi_M \in \mathcal{A}_u(n,M)$ such that for every finite place v of F where π and M are unramified, and for all places w of M dividing v, we have*

$$A_v(\pi) = \{\alpha_{1,v}, ..., \alpha_{n,v}\} \implies A_w(\pi_M) = \{\alpha_{1,v}^{f_v}, ..., \alpha_{n,v}^{f_v}\},$$

where $f_v = [M_w : F_v]$.

A word of explanation may be helpful. In [1], it is proved that for every cuspidal π, the base change π_M is equivalent to an isobaric sum of unitary cuspidal automorphic representations; when M/F is cyclic of prime degree p, for example, π_M is either cuspidal or of the form

$\boxplus_{j=0}^{p-1}(\eta \circ \tau^j)$, where τ is a generator of $\mathrm{Gal}(M/F)$. Since base change is additive relative to isobaric sums, it follows that for any π in $\mathcal{A}_u(n,F)$, π_M lies in $\mathcal{A}_u(n,M)$.

1. Reduction to isobaric sums of unitary cuspidals

Lemma 1.1 *Preserving the hypotheses of Theorem A, it suffices to prove the assertion for π, π' which are isobaric sums of unitary cuspidal automorphic representations. In other words, we may assume that π, π' lie in $\mathcal{A}_u(n,K)$.*

Proof. If η is an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, then put, for each $t \in \mathbb{R}$,

$$\eta(t) := \eta \otimes |\det(\cdot)|^t.$$

If η is cuspidal, then $\eta(t)$ is, for some $t \in \mathbb{R}$, a *unitary* cuspidal automorphic representation. Hence we may, for any π in $\mathcal{A}_n(K)$, write

$$\pi \simeq \boxplus_{i=1}^m \pi_i(t_i),$$

with π_i in $\mathcal{A}_u(n_i,K)$, $t_i \in \mathbb{R}$, and $\sum_{i=1}^m n_i$. Similarly, we may decompose π' as $\boxplus_{j=1}^{m'} \pi'_j(t'_j)$, with π'_j in $\mathcal{A}_u(n'_j,K)$, $t'_j \in \mathbb{R}$, and $\sum_j^{m'} n'_j$. By hypothesis (of Theorem A), π_v and π'_v are isomorphic for all (but a finite number of) primes v of degree 1 over F. Since the absolute values of the (inverse) Hecke roots α_i (resp. α'_j) of π_i (resp. π'_j) are all strictly bounded above by $q_v^{1/2}$ (by Theorem 0.2), it follows immediately that $m = m'$ and for each $i \leq m$, $n_i = n'_i$ and $t_i = t'_i$. Moreover, we must have $\pi_{i,v} \simeq \pi'_{i,v}$ for almost all primes v of degree 1 over F ($\forall\, i \leq m$). If we can show that $\pi_i \simeq \pi'_i$ for each i, then it follows that $\pi \simeq \pi'$. Done!

Thus we may, and we will, assume from here on till the end of the paper that π, π' in Theorem A are isobaric sums of unitary cuspidal automorphic representations.

2. A Preliminary Step

Proposition 2.1 *Let K be a number field and $n \geq 1$ an integer. Suppose $\pi, \pi' \in \mathcal{A}_u(n,K)$ are such that for every positive integer $m \leq (n^2+1)/2$, and for all but a finite number of primes v of K of degree m, we have $\pi_v \simeq \pi'_v$. Then π and π' are isomorphic.*

This is essentially an immediate consequence of the bound of Luo-Rudnick-Sarnak. For completeness, we quickly go through the relevant points of [12] to make it evident that they carry over, modulo the basic results cited in section 1 and induction on the number of cuspidal isobaric summands, from ($n = 2$; π, π' cuspidal) to (n arbitrary; π, π' isobaric sums of unitary cuspidal automorphic representations).

Proof. Denote by X the complement in Σ_K of the union of the archimedean places and the finite places where π or π' is ramified. Given any subset Y of X we set (as in [12]):

$$Z_Y(s) = \frac{L_Y(\bar\pi \times \pi, s) L_Y(\bar\pi' \times \pi', s)}{L_Y(\bar\pi \times \pi', s) L_Y(\bar\pi' \times \pi, s)}. \tag{2.1}$$

Write

$$\pi = \boxplus_{i=1}^{\ell} m_i \pi_i, \;\; \pi' = \boxplus_{j=1}^{r} m_j' \pi_j',$$

with $m_i, m_j' \in \mathbb{N}$, and π_i, π_j' unitary cuspidal, with $\pi_i \not\simeq \pi_a$ if $i \neq a$ and $\pi_j' \not\simeq \pi_b'$ if $j \neq b$.

Suppose $\pi_i \not\simeq \pi_j'$ for all i, j. Then, using Theorem 0.1, we see that $Z_X(s)$ is holomorphic at every $s \neq 1$, with

$$-\mathrm{ord}_{s=1} Z_X(s) \;=\; \mu + \mu', \tag{2.2 − a}$$

where

$$\mu = \sum_{i=1}^{\ell} m_i^2, \;\; \mu' = \sum_{j=1}^{r} {m_j'}^2. \tag{2.2 − b}$$

As the subproduct of an absolutely convergent Euler product is absolutely convergent, we have the following

Lemma 2.3 *Let S denote the subset of X consisting of finite places v of degree $> \frac{n^2+1}{2}$. Then the incomplete Euler products $L_S(\bar{\pi} \times \pi, s)$ and $L_S(\bar{\pi} \times \pi', s) L_s(\bar{\pi}' \times \pi, s)$ converge absolutely in $\{s \in \mathbb{C} \,|\, \Re(s) > 1\}$.*

We may write

$$\log(L_Y(\bar{\pi} \otimes \pi, s)) = \sum_{m \geqq 1} c_m(Y) m^{-s} \tag{2.4 − a}$$

for all subsets Y of X. Then $c_m(Y) = 0$ unless m is of the form Nv^r for some $v \in Y$ and $r \in \mathbb{N}$, and when m is of this form,

$$c_m(Y) = \sum_{M} \frac{1}{r} \sum_{1 \leqq i,j \leqq n} \overline{\alpha_{i,v}^r} \alpha_{j,v}^r. \tag{2.4 − b}$$

where M is the set of pairs $(v, r) \in Y \times \mathbb{N}$ such that $m = Nv^r$.

When $v \in S$, as $Nv > \frac{n^2+1}{2}$, the Luo-Rudnick-Sarnak bound (Theorem 0.2) implies that $\sum_{m \geq 1} c_m(S) m^{-s}$ converges in $\{\Re(s) \geq 1\}$.

One has a similar statement for $\log(L_S(\bar{\pi}' \otimes \pi, s))$, $\log(L_S(\bar{\pi}' \otimes \pi, s))$, and $\log(L_S(\bar{\pi}' \otimes \pi', s))$. So we get the following

Lemma 2.5 *Let S be as in Lemma 2.3. As s goes to 1 from the right on the real line, we have*

$$\log Z_S(s) \;=\; o\left(\log \frac{1}{s-1}\right). \tag{2.6}$$

Now, since $\pi_v \simeq \pi_v'$ for all but a finite number of places of X outside S, (2.6) holds with S replaced by X, which contradicts the following consequence of (2.2-a):

$$\log Z_X(s) \;=\; (\mu + \mu') \log \frac{1}{s-1} + o\left(\log \frac{1}{s-1}\right), \tag{2.7}$$

since $\mu = \mu' \geq 1$.

Thus $L_S(\bar{\pi}' \otimes \pi, s)$ must have a pole at $s = 1$, which implies that $\pi_i \simeq \pi'_j$ for *some* (i, j). If π or π' is cuspidal, then both will need to be cuspidal with $\pi = \pi_i \simeq \pi'_j = \pi'$, an so we are done in this case. We may assume that π, π' are non-cuspidal. Consider then the isobaric automorphic representations Π, Π' such that

$$\pi = \Pi \boxplus \pi_i, \quad \pi' = \Pi' \boxplus \pi'_j.$$

The Π, Π' satisfy the hypotheses of Proposition 2.1, and we may find as before cuspidal isobaric summands π_k of Π and π'_m of Π' which are isomorphic. Continuing thus, by infinite decent, we arrive finally at the situation when one of the isobaric forms is cuspidal, which we have already taken care of. This proves Proposition 2.1.

Done.

3. A brief digression on the tempered case

The general Ramanujan conjecture predicts that for any cuspidal π on $\mathrm{GL}(n)/K$, π_v is tempered at all v. This is far from being known. Let us preserve the notations of the previous section.

At any finite place v where π is unramified,

(3.1)

$$\pi_v \text{ tempered} \quad \Longrightarrow \quad |\alpha_{j,v}| \leq 1, \ \forall\, j \leq n,$$

where $\{\alpha_{1,v}, \ldots, \alpha_{n,v}\}$ is the Langlands class attached to π_v.

Lemma 3.2 *Suppose π, π' are cuspidal, tempered automorphic representations of $GL_n(\mathbb{A}_K)$ such that $\pi_v \simeq \pi'_v$, for all finite places outside a set S of lower Dirichlet density $\delta < 1/2n^2$. Then $\pi \simeq \pi'$.*

Proof. Using (3.1) in (2.4-b), we get, for any set Y of finite places of K,

$$|c_m(Y)| \leq n^2 \sum_{M=\{(v,r)\in Y\times\mathbb{N}\,:\,m=Nv^r\}} \frac{1}{r},$$

which implies, by (2.4-a), that for any set Y of finite places,

$$\log L_Y(s) = \log L_{Y^1}(s) + o\left(\log \frac{1}{s-1}\right), \tag{3.3}$$

where Y^1 is the subset of Y consisting of degree 1 primes.

Next put

$$D_Y(s) = L_Y(\bar{\pi} \times \pi, s) L_Y(\bar{\pi}' \times \pi', s) L_Y(\bar{\pi} \times \pi', s) L_Y(\bar{\pi}' \times \pi, s), \tag{3.4}$$

which converges absolutely in $\mathrm{Re}(s) > 1$ and is of positive type, i.e., $\log(D_S(s))$ is a Dirichlet series with non-negative real coefficients (cf. [3]). Since

$$Z_Y(s) = L_Y(\overline{\pi} \times \pi, s)^2 L_Y(\overline{\pi'} \times \pi', s)^2 / D_Y(s),$$

we apply (3.1) and obtain the following for all real $s = 1 + \epsilon$ with ϵ small and positive:

$$\log Z_X(s) = \log Z_S(s) \,\leq\, 4n^2 \delta \log \frac{1}{s-1} + o\left(\log \frac{1}{s-1}\right), \tag{3.5}$$

where we have used the fact that $\sum_{v \in S^1} Nv^{-s}$ is $\delta \log \frac{1}{s-1} + o\left(\log \frac{1}{s-1}\right)$.

Finally, since by hypothesis, $\delta < 1/2n^2$, (3.5) contradicts the fact that $Z_X(s)$ has a pole of order 2 at $s = 1$. Hence π must be isomorphic to π'. Done!

Remark 3.6 By considering the partial sums $\sum\limits_{v\,:\,Nv \leq x} |a_v|^2$, one can show in this tempered case that Lemma 3.2 even holds for δ the *natural density* of S, not just for its Dirichlet density. This will take us far afield to pursue it here, and besides, this approach does not work well in the non-tempered situation.

4. The central character

Suppose π, $\pi' \in \mathcal{A}_u(n, K)$ are of (respective) central characters ω, ω', such that $\pi_v \simeq \pi'_v$ for all but a finite number of primes v of K of degree 1 over F. Then ω and ω' agree at all (but a finite number of) the degree one places v, which forces the global identity

$$\omega \,=\, \omega'. \tag{4.1}$$

In fact, by Hecke, this conclusion will result as soon as ω and ω' agree at a set of primes of density $> 1/2$.

Of course, if $\pi \simeq \boxplus_{i=1}^{m} \pi_i$ and $\pi' \simeq \boxplus_{j=1}^{m'} \pi'_j$, with π_i, π'_j cuspidal with (respective) central characters ω_i, ω'_j, we can only conclude that $\prod_i \omega_i \,=\, \prod_j \omega'_j$.

5. On the number of self-twists

We need the following, which is likely known to experts:

Lemma B *Fix $n \geq 1$, a number field K, and consider any cuspidal element π in $\mathcal{A}(n, K)$. Denote by $\Gamma(\pi)$ the group of self-twists of π, i.e., the group of characters λ of C_K such that $\pi \otimes \lambda \simeq \pi$. Then the order of Γ is $\leq n^2$.*

It follows that any $\pi \in \mathcal{A}_u(n, K)$ admits only a finite number of self-twists.

Before giving a proof of Lemma B, let us note that this is as it should be: If π were attached to a Galois representation ρ of dimension n, the existence of a self-twist by some λ implies that this character embeds in $\rho \otimes \rho^\vee$, thus restricting the number of such λ to be $\leq n^2$. In the automorphic case one if far from being able to prove that $\pi \boxtimes \pi^\vee$ is in $\mathcal{A}(n^2, K)$, let alone obtain such a λ as an isobaric summand, except for $n = 2$ ([13]). But luckily, one is able to avoid knowing it.

Proof. This is obvious for $n = 1$ for any K, so let $n > 1$ and assume by induction that the Lemma holds for all cusp forms η in $\mathcal{A}(m, E)$ for all $m < n$ and for all number fields E. If $\Gamma(\pi)$ has order 1, there is nothing to prove, so assume that there is a non-trivial character λ of K such that $\pi \otimes \lambda \simeq \pi$. By looking at the central characters, it is immediate that the order of λ is a divisor of n. Replacing λ by a suitable power, if necessary, we may assume that λ has prime order p. Write $n = mp$ and denote by E the cyclic p-extension of K cut out by λ with $\mathrm{Gal}(E/K)$ generated by θ. Then by [1], π is the automorphic induction $I_E^K(\eta)$ by a cuspidal automorphic representation η of $\mathrm{GL}_m(\mathbb{A}_E)$, necessarily with $\eta^\theta \not\simeq \eta$. Then the base change π_E is an isobaric sum of $\eta \circ \theta^j$, as j runs from 0 to $p - 1$. By induction, there are at most $m^2 = n^2/p^2$ elements in $\Gamma(\eta)$. Suppose μ is another element of $\Gamma(\pi)$. Then, since $\pi_E \otimes \mu_E \simeq \pi_E$, $\eta \otimes \mu$ must be isomorphic to some $\eta \circ \theta^j$. By the pigeon hole principle, μ_E has at most $p|\Gamma(\eta)| \leq n^2/p$ choices. Moreover, if $\mu' \in \Gamma(\pi)$ is such that $\mu'_E = \mu_E$, then $\mu' = \mu\lambda^i$, for some $0 \leq i \leq p$. So, all together, there are at most $p(n^2/p)^2$ elements.
Done.

6. Trivialization of certain torsion classes in Galois cohomology

The following Lemma will play a key role for us, and it is needed to avoid assuming that F contains sufficiently many roots of unity.

Lemma 6.1 *Let K/F be a finite Galois extension of number fields, with p a prime divisor of the order of Gal(K/F). Let $\mathcal{I}$ be the set of intermediate fields K' in the extension K/F such that $[K : K'] = p$. For each $K' \in \mathcal{I}$, fix a class $\beta(K')$ in $H^2(K', \mathbb{Z}/p)$ whose restriction to K is trivial (in $H^2(K, \mathbb{Z}/p)$). Fix also an auxiliary finite set T_0 of finite places of F which are prime to p and unramified in K, and let T be the places of K above T_0.Then there exists a cyclic extension E/F such that*

(a) *E/F is linearly disjoint from K/F, and is of degree p^r or $2p^r$, with r being independent of T;*
(b) *Every place in T splits completely in KE;*
(c) *For every $K' \in \mathcal{I}$, and for every subfield M' of KE containing K' with $[KE : M'] = p$, the restriction of $\beta(K')$ in $H^2(M', \mathbb{Z}/p)$ is trivial.*

Proof. For $0 \leq i \leq 3$, consider the map

$$\alpha_i = (\alpha_{i,u}) : H^i(K', \mathbb{Z}/p) \to \prod_u H^i(K'_u, \mathbb{Z}/p),$$

where u runs over all the places of K', K'_u denotes the local completion of K' at u, and α_u the restriction at u. It is known (see [9], chapter 1) that for any $\beta \in H^2(K', \mathbb{Z}/p)$, there is a finite set $X(K')$ of places of K' such that $\alpha_{i,u}(\beta)$ is zero at every place u outside $X(K')$, which is seen by noting that β must be in the image of $H^2(\mathrm{Gal}(L/K'), \mathbb{Z}/p)$ for a finite

Galois extension L/K'. Moreover, the kernel $Ш^i(K', \mathbb{Z}/p)$ of α_i is, by Tate, in duality with $Ш^{3-i}(K', \mu_p)$ ([9]), where μ_p denotes the Galois module of p-th roots of unity; $\mathbb{Z}/p$ is as usual the trivial Galois module. By Artin-Tate [2], $Ш^1(K', \mu_m)$, and hence $\mathrm{Sha}^2(K', \mathbb{Z}/m)$, is either trivial or of order 2. In fact, for $m = p$, this kernel is trivial.

Sublemma 6.2 *For every $u \in X(K')$, the restriction of $\beta_u := \alpha_{2,u}(\beta(K'))$ becomes trivial over the unique unramified p-extension $L_1(u)$, say, of K'_u.*

Proof. We may (and we will) assume from here on that β_u is non-trivial on K'_u for any $u \in X(K')$. Indeed, if we put $k = K'_u[\mu_p]$, then the restriction $\beta_{u,k}$ of β_u to k lies in $H^2(k, \mu_p) = Br_k[p]$, the p-torsion subgroup ($\simeq \mathbb{Z}/p$) of the Brauer group of k. It is well known (cf. [15]) that any class in $Br_k[p]$ becomes trivial over the unramified p-extension k' of k. Let δ denote the unramified character of ${K'_u}^*$ of order p. The pull-back by norm of δ to k is evidently non-trivial and cuts out the extension k'/k. Let $\beta_{u,1}$ be the restriction of β_u to $L_1(u)$. Then its restriction to k' is trivial, and since the composition of restriction followed by norm is multiplication by the degree, we see that $[k' : L_1(u)]\beta_{u,1} = 0$. Since$[k' : L_1(u)] = [k : K'_u]$ divides $p - 1$, and since $\beta_{u,1}$ is killed by p, we must have $\beta_{u,1} = 0$, proving the assertion of the Sublemma.

Proof of Lemma 6.1 (*contd.*) Now let X_0 denote the finite set of places of F above which lie all the places of $X(K')$ for all $K' \in \mathcal{I}$. Since β_u is non-trivial on K'_u for any $u \in X(K')$, u cannot split in K (as β becomes trivial over K). Let v be the unique place over u; K_v/K'_u may or may not be ramified. In any case choose, for each $u_0 \in X_0$, a finite, cyclic, unramified extension $E(u_0)$ of F_{u_0}, say of degree p^r, for large enough $r > 0$, such that for every u above u_0 lying in some $X(K')$, the compositum $K_vE(u_0)$ contains $L_2(u)$, the unramified (cyclic) p^2-extension of K'_u. (For each u_0, there is an $r > 0$ which works, and since there are only a finite number of u_0's in X_0, we may choose an r which works for all of them.) In particular, the restriction of β_u to the unramified extension $K'_uE(u_0)$ is trivial ($\forall K' \in \mathcal{I}$). Now, by appealing to the Grunewald-Wang theorem ([2]), we may choose a global cyclic extension E/F of degree tp^r, with $t \in \{1, 2\}$, such that

(i) the local extension of E/F at any divisor of u_0 is $E(u_0)/F_{u_0}$,
(ii) E/F and K/F are linearly disjoint from each other, and
(iii) every place in T_0 splits completely in E.

Then KE is Galois over F, and hence over K' for every $K' \in iI$. Also, KE is abelian over K' with Galois group $(\mathbb{Z}/p) \times (\mathbb{Z}/tp^r)^2$. By construction, $u \in X(K')$ is unramified in $K'E$, splitting into a product of places $u_1, \ldots, u_m$ (of places in $K'E$) such that the restriction of β_u to each $(K'E)_{u_j}$ is trivial, implying that the global class β restricts to 0 in $H^2(K'E, \mathbb{Z}/p)$, and hence in $H^2(K'E, \mathbb{Z}/p)$. We have to prove furthermore that β has trivial restriction to any intermediate field M' in KE/K' with $[KE : M'] = p$, not just to $K'E$. Fix such an M', which will be normal over K' since KE/K' is abelian. Consider any place $\tilde{u}$ of M' above u in $X(K')$. As noted above, since β becomes trivial upon restriction to K, and since β_u is non-trivial on K'_u, u cannot split in K, and we write v for the unique place of K above u, and $\tilde{v}$ a place of KE above v such that $\tilde{u}$ lies below it in M'. By construction, $(KE)_{\tilde{v}}$

contains $L_2(u)$. It follows that, since $[KE : M'] = p$, the local field $M'_{\tilde{u}}$ must at least contain $L_1(u)$, over which the local class β_u becomes trivial (by Sublemma 6.2). So β_u will become trivial over $M'_{\tilde{u}}$. This holds for every $u \in X(K')$, and so the global class β restricts to zero over M', as asserted. Done.

7. The width

For any finite Galois extension K/F of number fields, and positive integer j, put

$$S^j_{K/F} := \{v \in S_K \mid \deg_F(v) = j\},$$

where $\deg_F(v)$ denotes the degree v over F.

Next, for such K/F, and for any pair π, π' in $\mathcal{A}(n, K)$ such that $\pi_v \simeq \pi'_v$ for all but finitely many places v in $S^1_{K/F}$, put

$$\Sigma := \Sigma(\pi, \pi'; K/F) \;=\; \bigcup_{2 \leq i \leq \frac{n^2+1}{2}} \{v \in S^i(\pi, \pi'; K/F) \mid \pi_v \not\simeq \pi'_v\}, \tag{7.1}$$

where $S^i(\pi, \pi'; K/F)$ is just $S^i_{K/F}$ if there are infinitely many v of degree i (over F) with $\pi_v \not\simeq \pi'_v$, but equals $\emptyset$ otherwise.

The degrees of the places in Σ are powers of rational primes lying in a finite set, and can be ordered. Put

$$h(\Sigma) \;=\; \max\{j \leq [\frac{n^2+1}{2}] + 1 \quad : \quad S^i(\pi, \pi'; K/F) = \emptyset, \;\; \forall\, i < j\}. \tag{7.2}$$

The *width* of Σ is defined as follows:

$$w(\Sigma) = w(\Sigma(\pi, \pi'; K/F)) \;=\; [\frac{n^2+1}{2}] + 1 - h(\Sigma). \tag{7.3}$$

Note that if $w(\Sigma) = 0$, then $\pi_v \simeq \pi'_v$ for all but a finite number of v, and thus π, π' are isomorphic by the strong multiplicity one theorem.

When $w(\Sigma) > 0$, we will show in the next section how to transfer the problem to a bigger Galois extension L/F with L/K solvable, such that $w(\Sigma(\pi_L, \pi'_L; L/F))$ is strictly smaller than $w(\Sigma)$, where π_L, π'_L denote the respective base changes of π, π' respectively to GL(n)/L, which exist by [1] for L/K solvable and normal. In fact, we have a lot of freedom in the choice of L, and the situation gets better over each such L. The idea is to repeat the process to shrink the weight once again, and proceed *ad infinitum* till we bring the weight down to zero. The base changes of π, π' will be shown to be isomorphic over a family of suitably chosen Galois extensions R/F with R/K solvable, and where an auxiliary finite set of places of K split completely. Then we will descend in steps the isomorphisms to one over $M = KE$, and then over K.

8. Proof of the main result

Fix any integer $n \geq 1$. As Theorem A evidently holds for $n = 1$, so we may take $n > 1$ and assume by induction that the assertion is correct for all $m < n$.

Let $\Sigma = \Sigma(\pi, \pi'; K/F)$ be as above, with width $w(\Sigma)$, which is non-negative and $\leq [(n^2+1)/2]$. Clearly, if $w(\Sigma) = 0$, then $\pi \simeq \pi'$ as we saw above. So we may assume that the width is positive, and moreover, that the assertion holds for smaller widths.

Let v_0 be a place in Σ with degree equal to $h(\Sigma)$, which is $\leq (n^2+1)/2$. Write $\deg(v_0) = h(\Sigma) = p^m$, where p is a rational prime.

Let $\mathcal{I}$ denote (as before) the finite collection of intermediate fields K' of K/F such that $[K : K'] = p$. For each such $\mathcal{I}$, denote by $\varphi(K')$ the character of order p of the absolute Galois group $\Gamma_{K'}$ of K' cutting out the cyclic p-extension K/K'. The surjective p-power map $z \mapsto z^p$ on $\mathbb{C}^*$ gives a short exact sequence of *trivial* $\Gamma_{K'}$-modules

$$0 \to \mathbb{Z}/p \to \mathbb{C}^* \to \mathbb{C}^* \to 1.$$

(We are looking at the trivial Galois action since we are interested in $\mathrm{Hom}(\Gamma_{K'}, \mathbb{C}^*)$.) The associated long exact sequence in Galois cohomology yields

$$H^1(\Gamma_{K'}, \mathbb{C}^*) \to H^1(\Gamma_{K'}, \mathbb{C}^*) \to H^2(K', \mathbb{Z}/p),$$

which shows that the obstruction to $\varphi = \varphi(K') \in \mathrm{Hom}(\Gamma_{K'}, \mathbb{C}^*)$ being a p-th power of another character of $\Gamma_{K'}$ is the class $\partial(\varphi) \in H^2(K', \mathbb{Z}/p)$, where ∂ is the connecting morphism from $H^1(\Gamma_{K'}, \mathbb{C}^*)$ into $H^2(K', \mathbb{Z}/p)$. Put

$$\beta = \beta(K') := \partial(\varphi(K')), \ \ \forall\ K' \in \mathcal{I}.$$

Note that since the restriction map $\mathrm{res}_{K/K'} : H^i(K', -) \to H^i(K, -)$ commutes with ∂, and since $\varphi(K')$ restricts to the trivial character on Γ_K, the restriction of $\beta(K')$ is trivial in $H^2(K, \mathbb{Z}/p)$. Hence the collection $\{\beta(K') \,|\, K' \in \mathcal{I}\}$ satisfies the hypothesis of Lemma 6.1. Consequently, given any auxiliary finite set T_0 of finite places of F which are prime to p and unramified in K, with T denoting the set of places of K above T_0, we can find a cyclic extensions extension E of F of degree a power of p, such that the conclusions (a), (b) and (c) of Lemma 6.1 hold. In particular, for every $K' \in \mathcal{I}$, and for every subfield M' of

$$M := KE, \ \text{with}\ M' \supset K',\ [M : M'] = p,$$

we have

$$\mathrm{res}_{M'/K'}(\beta(K')) \,=\, 0.$$

It is important to note that by construction (cf. Lemma 6.1), the same E works for all K' in $\mathcal{I}$. Since by definition $\beta(K') = \partial(\varphi)$, and as the restriction map commutes with ∂, we get

$$\partial(\varphi|_{\Gamma_{M'}}) \,=\, 0.$$

Hence $\varphi(K')_{|M'} = \psi^p$ for some $\psi \in \mathrm{Hom}(\Gamma_{M'}, \mathbb{C}^*)$, necessarily of order p^2 since $\varphi(K')_{|M'}$is still of order p, cutting out M over M'. Put

$$L := M'(\psi) \,\supset\, M = K'(\varphi)E = M'(\varphi),$$

where $M'(\nu)$ denotes, for any character ν of $\Gamma_{M'}$, the cyclic extension of M' cut out by ν. Then L is cyclic of degree p^2 over M' and of degree p over M. This way we get a collection $\mathcal{J}$ of cyclic p-extensions L of M, one for each M' as above, as K' varies over $\mathcal{I}$.

For every τ in Γ_F, $K^\tau = K$ since K/F is Galois, while L^τ need not be L, though still cyclic of degree p over $M = KE$. Put

$$\tilde{L} := \prod_{\tau \in \Gamma_F} L^\tau,$$

and

$$R := \prod_{L \in \mathcal{J}} \tilde{L}.$$

Then R/F is a finite Galois extension, and since R is a compositum of cyclic p-extensions of K, R/K is solvable, even abelian. So the base changes π_R, π'_R are defined in $\mathcal{A}(n, R)$.

Lemma 8.1 *Let $w(\Sigma) = w(\Sigma(\pi, \pi'; K/F))$ be positive. Then for R as above,*

$$w(\Sigma(\pi_R, \pi'_R; R/F)) \; < \; w(\Sigma).$$

Proof. Let p^m be the smallest prime power for which there exist infinitely many places v of K of degree p^m over F such that $\pi_v \not\simeq \pi'_v$, so that $w(\Sigma) = [\frac{n^2+1}{2}] + 1 - p^m$. First note that if w is any place of R having degree less than p^m over F, then the place v, say, of K below w will have degree less than p^m, and thus, outside a finite set of such places, $\pi_v \simeq \pi'_v$, implying that $\pi_{R,w} \simeq \pi'_{R,w}$ (by the basic properties of base change). It follows that

$$w(\Sigma(\pi_R, \pi'_R; R/F)) \; \leq \; w(\Sigma).$$

To show that this inequality is strict, it suffices to show that for all but a finite number of places v of K which are of degree p^m over F, if w is a place of R above v, then $\deg_{R/F}(w) \geq p^{m+1}$. (The finite number of places v which are ignoring are the ones above T.) Pick any such v, with v_1 denoting the place of F below it. Since v has degree divisible by p, its decomposition group (over F) contains a cyclic subgroup H of order p in $\mathrm{Gal}(K/F)$. Let $K' \in \mathcal{I}$ correspond to H. Let v_L be the place of L below w. It suffices to show that v_L has degree $\geq p^{m+1}$ over F. If $v_{M'}$, resp. v', is the place of M', resp. K', below v_L, then the degree of $v_{M'}$ over F is at least as big as the degree of v' over F, which is p^{m-1}, since v divides v' and is of degree p over K'. Putting these together, we see that it suffices to check that v_L has degree p^2 over M'. By construction, M'/L is cyclic of degree p^2 and moreover, $v_{M'}$ is inert in KE, which is cyclic of degree p over M'. Then $v_{M'}$ must be inert all the way in L. Indeed, if it were false, v_L would have degree p over M' and its decomposition group in $\mathrm{Gal}(L/M')$ would be the unique cyclic subgroup C, say. Then KE would necessarily be the fixed field of C, in which case $v_{M'}$ would split in KE, which contradicts the fact that $v_{M'}$ is inert in KE. Thus v_L has degree p^2 over M', and this phenomenon recurs at every possible M' as K' varies over $\mathcal{I}$. The Lemma is now proved.

Thanks to this Lemma, we have, albeit at the cost of replacing K by R, reduced to the case where the width is smaller and where the assertion holds. Thus

$$\pi_R \simeq \pi'_R,$$

where $R = \prod_{L\in\mathcal{J}} \tilde{L}$ as above.

Lemma 8.2 *We have*

$$\pi_M \simeq \pi'_M,$$

where $M = KE$, for each choice of the auxiliary cyclic extension E/F.

Proof. Fix E/F as above. Recall that for each M', the extension L of M' containing KE depends on a choice of a p-root ψ of the character $\varphi(K')_{|M'}$. We can always multiply ψ by a character λ of $\Gamma_{M'}$ of order p, which we may choose to be non-trivial when restricted to Γ_{KE}. The effect of modifying ψ is to replace $L = M'(\psi)$ by $L_1 = M'(\psi\lambda)$, which is another cyclic p-extension of KE. Now let Y be a finite set of primes in $M = KE$ away from the ramification loci (of R/F and p. Then we may, by appealing to the Grunwald-Wang theorem ([2]), choose λ in such a way that $\lambda_y = \psi_{M,y}^{-1}$ for every y in Y, so that the primes in Y split in L_1. In fact we can do this so that for all τ in Γ_{KE}, λ^τ is the inverse of ψ^τ at any y in Y, and so Y will split in $\tilde{L}_1 = \prod_\tau L_1^\tau$. We can do this at every M', and finally we get a Galois extension R_1/F, solvable over EK of degree p^k (with k independent of Y), such that Y splits completely in R_1. Using this we can conclude, using the isomorphism over R, that

$$\pi_{M,y} \simeq \pi'_{M,y}, \ \forall\, y \in Y.$$

Since Y is arbitrary outside the fixed finite set of places where M, π or π' may be ramified, we conclude that π_M and π'_M are isomorphic by the strong multiplicity one theorem.

The final step is to descend the isomorphism over $M = KE$ to one over K. For this recall that E depends on a finite set T of auxiliary places of K at which K, π or π' is not ramified, and which split completely in E. Let us write $E(T)$ to indicate this dependence, and appeal to Lemma 8.2 to get an isomorphism over $M(T) := KE(T)$. By construction, as places in T splits completely in $M(T)$, we obtain $\pi_v \simeq \pi'_v$ for all v in T. By varying T, we get the desired isomorphism $\pi \simeq \pi'$ over K.

This finishes the proof of Theorem A. And it is clear why Corollary B follows.

References

[1] James Arthur and Laurent Clozel, *Simple algebras, base change, and the advanced theory of the trace formula*, Annals of Mathematics Studies, vol. 120, Princeton University Press, Princeton, NJ, 1989. MR1007299 (90m:22041)

[2] Emil Artin and John Tate, *Class field theory*, AMS Chelsea Publishing, Providence, RI, 2009. Reprinted with corrections from the 1967 original. MR2467155 (2009k:11001)

[3] Jeffrey Hoffstein and Dinakar Ramakrishnan, *Siegel zeros and cusp forms*, Internat. Math. Res. Notices **6** (1995), 279–308, DOI 10.1155/S1073792895000225. MR1344349 (96h:11040)

[4] H. Jacquet, I. I. Piatetskii-Shapiro, and J. A. Shalika, *Rankin-Selberg convolutions*, Amer. J. Math. **105** (1983), no. 2, 367–464, DOI 10.2307/2374264. MR701565 (85g:11044)

[5] H. Jacquet and J. A. Shalika, *On Euler products and the classification of automorphic forms. II*, Amer. J. Math. **103** (1981), no. 4, 777–815, DOI 10.2307/2374050. MR623137 (82m:10050b)

[6] Henry H. Kim, *Functoriality for the exterior square of* GL_4 *and the symmetric fourth of* GL_2, J. Amer. Math. Soc. **16** (2003), no. 1, 139–183 (electronic), DOI 10.1090/S0894-0347-02-00410-1. With appendix 1 by Dinakar Ramakrishnan and appendix 2 by Kim and Peter Sarnak. MR1937203 (2003k:11083)

[7] R. P. Langlands, *Automorphic representations, Shimura varieties, and motives. Ein Märchen*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 205–246. MR546619 (83f:12010)

[8] Wenzhi Luo, Zeév Rudnick, and Peter Sarnak, *On the generalized Ramanujan conjecture for* GL(n), Automorphic forms, automorphic representations, and arithmetic (Fort Worth, TX, 1996), Proc. Sympos. Pure Math., vol. 66, Amer. Math. Soc., Providence, RI, 1999, pp. 301–310. MR1703764 (2000e:11072)

[9] J. S. Milne, *Arithmetic duality theorems*, 2nd ed., BookSurge, LLC, Charleston, SC, 2006. MR2261462 (2007e:14029)

[10] C. S. Rajan, *On strong multiplicity one for l-adic representations*, Internat. Math. Res. Notices **3** (1998), 161–172, DOI 10.1155/S1073792898000142. MR1606395 (99c:11064)

[11] ______, *On strong multiplicity one for automorphic representations*, J. Number Theory **102** (2003), no. 1, 183–190, DOI 10.1016/S0022-314X(03)00066-0. MR1994478 (2004f:11050)

[12] Dinakar Ramakrishnan, *A refinement of the strong multiplicity one theorem for* GL(2)*. Appendix to: "l-adic representations associated to modular forms over imaginary quadratic fields. II" [Invent. Math.* **116** *(1994), no. 1-3, 619–643; MR1253207 (95h:11050a)] by R. Taylor*, Invent. Math. **116** (1994), no. 1-3, 645–649, DOI 10.1007/BF01231576. MR1253208 (95h:11050b)

[13] ______, *Modularity of the Rankin-Selberg L-series, and multiplicity one for* SL(2), Ann. of Math. (2) **152** (2000), no. 1, 45–111, DOI 10.2307/2661379. MR1792292 (2001g:11077)

[14] ______, *A mild Tchebotarev theorem for* GL(n), Preprint **arXiv:1003.4498** (2010), 1–15.

[15] Jean-Pierre Serre, *Local fields*, Graduate Texts in Mathematics, vol. 67, Springer-Verlag, New York, 1979. Translated from the French by Marvin Jay Greenberg. MR554237 (82e:12016)

[16] Freydoon Shahidi, *On the Ramanujan conjecture and finiteness of poles for certain L-functions*, Ann. of Math. (2) **127** (1988), no. 3, 547–584, DOI 10.2307/2007005. MR942520 (89h:11021)

[17] Nahid Walji, *A further refinement of strong multiplicty one for* GL(2), Preprint (2011), 1–17.

[18] Wei Zhang, *Fourier transform and the global Gan-Gross-Prasad conjecture for unitary groups*, Preprint (2012), 1–73.

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