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Siegel Zeros and Cusp Forms

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Introduction

Given a Dirichlet series $L(s)$ with Euler product in $\{\Re(s) > 1\}$, admitting a meromorphic continuation to the whole s-plane and a functional equation relating s to $1-s$, a fundamental problem is to know if $L(s)$ has a *Siegel zero*, i.e., a real zero in $(1-\delta, 1)$ for a small δ. (See §2 for a precise definition.)

This question has been extensively studied for the class of $L(s)$ attached to Dirichlet characters χ, the only relevant ones being quadratic characters. Existence of a Siegel zero for $L(s,\chi)$ exerts an influence on the distribution of zeros of $L(s,\chi')$ for characters χ' different from χ, and it also affects the distribution of primes in arithmetic progressions. On the other hand, when $L(s,\chi)$ does not have a Siegel zero, one gets a good lower bound for $L(1,\chi)$ leading to information on the growth of class numbers of imaginary quadratic fields. The only nonabelian results so far are the recent theorems of [HL] and [GHL] establishing the nonexistence of Siegel zeros for the symmetric square L-functions of nondihedral cusp forms g on GL(2). These led to a sharp lower bound for the residue of the Rankin-Selberg L-function $L(s, g \times g)$ at $s = 1$.

This paper was inspired by the beautiful lecture of D. Goldfeld at the workshop on automorphic forms at Mathematical Sciences Research Institute in October 1994, where he stressed the importance of establishing the conjecture that the standard (degree-n) L-series of cusp forms on GL(n) should not, for $n > 1$, admit Siegel zeros. (This should also be the case for $n = 1$, but it remains a very deep unsolved problem.)

In this article we prove the following (over any number field F): (1) The conjecture is true for cusp forms π on GL(2); it is also true on GL(3), unless the form in question is

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self-dual with nontrivial (quadratic) central character, in which case Hypothesis 6.1 must be assumed. (2) If we consider the class $\mathcal{A}_0$ of all cusp forms on GL(n)/F for *all* $n \geq 1$, such that the conductor is bounded from above by a fixed constant, then for every appropriate range to the left of $s = 1$, there is *at most one* element π_0 of $\mathcal{A}_0$ (up to equivalence) whose L-series has a Siegel zero in this range (see Theorem A for a precise statement). (3) If we admit the conjecture of Langlands, that given any pair of cusp forms π, π' on GL(n) (GL(n′), respectively), there should exist an automorphic representation $\pi \boxtimes \pi'$ of GL(nn′), whose L-function is the Rankin product L-function of (π, π'), then no cusp form on GL(n), $n > 1$, can have a Siegel zero.

The proof of (1) uses certain analytic properties of the symmetric cube L-functions of GL(2), as established in [BGH] and [Sh3], and of the symmetric square L-functions of GL(3), as in [BG]. In particular, a crucial result for us is the analyticity [BGH] of the symmetric cube L-function, with the ramified Euler factors removed, of a cuspidal automorphic representation π of GL(2)/F in the real interval (3/4, 1).

As one would expect, the good zero-free regions of (1) for L-series of cusp forms π on GL(2) or GL(3) lead to sharp lower bounds for $L(1, \pi)$ (see Theorems C and D). As a potential application, it may be worthwhile to note that, for holomorphic eigenforms of odd weight k, if the relevant Bloch-Kato conjectures are established at the edge of absolute convergence, which corresponds to $s = (k + 1)/2$ in the classical normalization, then our result will give some information on the growth of appropriate higher analogs of the class group; similarly for the symmetric squares of even-weight forms.

Note that (2) implies in particular that no *non-self-dual* cusp form on GL(n)/F admits a Siegel zero for any n. This extends, to $n \geq 3$, a result of [M] for $n = 2$, and generalizes the classical theorem that $L(s, \chi)$, for a complex character χ, has no Siegel zero. See [M] also for a very thorough discussion of some of the analytic details that are omitted here. In addition, there are some related results in [Gr], and both [M] and [Gr] describe prime number theorem analogs that are consequences of zero-free regions for cusp forms on GL(2).

Though (3) is conditional, it shows the power of functoriality, and moreover, the relevant hypothesis can be verified for π attached to idele class characters of cyclic extensions [AC]. If one also admits the general principle that the L-function of any global Galois representation σ arising in number theory or arithmetical algebraic geometry is identifiable with an automorphic L-function, then our result implies that the only time $L(s, \sigma)$ could possibly have a Siegel zero is when the base field is $\mathbb{Q}$ and σ is a quadratic character. When σ is of Artin type, i.e., it has finite image, this has already been noticed in a different way, as a consequence of Artin's conjecture, by Stark in [St], where he proves, in addition, without any hypothesis, that for any *Galois* number field F, Siegel zeros of the Dedekind zeta function of F, if any, are inherited from quadratic subextensions $K/\mathbb{Q}$ of $F/\mathbb{Q}$.

The astute reader will observe that, modulo some technicalities, the real workhorse behind our result is a general L-function identity giving, for any $\pi \in \mathcal{A}_0$, a factorization of a suitable Dirichlet series $D(s)$ with positive coefficients, divisible by $L(s, \pi)$ to a degree larger than the order of pole of $D(s)$ at $s = 1$. For $n = 2$, the relevant identity (see (5.3)) was found independently by the two authors. The rest resulted from collaboration.

1 Preliminaries

Let F be a number field with adele ring $\mathbb{A}_F$. For every integer $n \geq 1$, denote by $\mathcal{A}_0(n, F)$ the set of unitary cuspidal automorphic representations of $GL(n, \mathbb{A}_F)$ up to equivalence. For each $\pi \simeq \otimes_v \pi_v \in \mathcal{A}_0(n, F)$, let $\mathfrak{f}(\pi)$ denote its conductor, and $\pi^\vee$ the contragredient, which identifies with the complex conjugate representation $\overline{\pi}$ by unitarity.

For every finite place v not dividing $\mathfrak{f}(\pi)$, there is an associated diagonal matrix $A_v(\pi) = [\alpha_{1,v}(\pi), \ldots, \alpha_{n,v}(\pi)]$ in $GL(n, \mathbb{C})$, unique up to permutation. For every finite-dimensional representation r of $GL(n, \mathbb{C})$, and for every finite set S of places containing those dividing $\mathfrak{f}(\pi)$ and the archimedean places, one has an incomplete Euler product (absolutely convergent in some right half-plane)

$$L^S(s, \pi, r) = \prod_{v \notin S} L(s, \pi_v, r),$$

where (for all $v \notin S$)

$$L(s, \pi_v, r) = \det(1 - r(A_v(\pi))(Nv)^{-s})^{-1}.$$

For $v \in S$, if v is archimedean or if $n \leq 3$, one knows how to associate to π_v an n-dimensional representation σ_v of the (extended) Weil group W_{F_v} of F_v, and one sets

$$L(s, \pi_v, r) = L(s, r \circ \sigma_v).$$

Since the nature of the archimedean factors are crucial to our work, we will briefly recall their definition. It is convenient to set

$$\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s) \quad \text{and} \quad \Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2).$$

We may decompose $r(\sigma_v)$ as a direct sum of irreducibles τ_j, $1 \leq j \leq k$, each occurring with multiplicity e_j, and set

$$L(s, \pi_v, r) = \prod_{j=1}^{k} L(s, \tau_j)^{e_j}. \tag{1.1}$$

It remains to define the local factors of irreducibles. If v is a complex place, every irreducible τ of $W_{\mathbb{C}} = \mathbb{C}^*$ is one-dimensional of the form $z \to (z/|z|)^m |z|^t$, and the corresponding local factor is

$$L(s,\tau) = \Gamma_{\mathbb{C}}\left(s + \frac{m}{2} + t\right).$$

If v is a complex place, $W_{\mathbb{R}}$ is the unique nontrivial extension of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ by $\mathbb{C}^*$, and every irreducible τ is one- or two-dimensional. In the former case, $\tau(x)$ is of the form $\mathrm{sgn}(x)^m|x|^t$ with $m \in \{0, 1\}$, and

$$L(s,\tau) = \Gamma_{\mathbb{R}}(s + m + t).$$

In the latter case, τ is induced by a character χ of $\mathbb{C}^* = W_{\mathbb{C}}$, and one puts

$$L(s,\tau) = L(s,\chi).$$

For example, consider the case $n = 2$ and $F = \mathbb{Q}$, with π of trivial central character. If π is defined by a holomorphic eigenform of weight $k \geq 2$, σ_∞ is induced by the character $z \to (z/|z|)^{k-1}$, and so (in our unitary normalization)

$$L(s,\pi_\infty) = \Gamma_{\mathbb{C}}(s + (k-1)/2) = 2(2\pi)^{-s-(k-1)/2}\Gamma\left(\frac{s + (k-1)/2}{2}\right)\Gamma\left(\frac{s + (k+1)/2}{2}\right).$$

If π corresponds to a Maass wave form of weight zero, σ_∞ is of the form $x \to (|x|^w \oplus |x|^{-w})$, and we have

$$L(s,\pi_\infty) = \Gamma_{\mathbb{R}}(s + w)\Gamma_{\mathbb{R}}(s - w) = \pi^{-s}\Gamma\left(\frac{s+w}{2}\right)\Gamma\left(\frac{s-w}{2}\right).$$

We will say that the complete L-function

$$L(s,\pi,r) = \prod_{\text{all } v} L(s,\pi_v,r)$$

is *nice* if (i) all the factors are defined with the product absolutely convergent in $\Re(s) > 1$, (ii) it admits a meromorphic continuation to the whole s-plane, bounded in vertical strips, with no poles outside $\{0, 1\}$, and (iii) it satisfies a functional equation of the form

$$L(s,\pi,r) = \epsilon(s,\pi,r)L(1-s,\overline{\pi},r),$$

with

$$\epsilon(s,\pi,r) = \left(D_F^{\dim(r)}N(\pi,r)\right)^{1/2-s} W(\pi,r),\ N(\pi,r) = \mathcal{N}\mathfrak{f}(\pi,r),$$

for some constant $N(\pi,r)$ and root number $W(\pi,r)$. Here D_F denotes the absolute value of the discriminant of F.

When r is the standard representation on $\mathbb{C}^n$, we will write $N(\pi)$ for $N(\pi,r)$, $\epsilon(\pi)$ for $\epsilon(\pi,r)$, and $L(s,\pi)$ for $L(s,\pi,r)$, which is nice with $N(\pi,r) = \mathcal{N}(\mathfrak{f}(\pi))$. (Here $\mathcal{N}$ denotes the absolute norm.) Even better, it is entire unless $n = 1$ and π is trivial, in which case it is the Dedekind zeta function of F.

Now let r be the symmetric square representation on $\mathbb{C}^{n(n+1)/2}$. When $n = 2$, one knows by the work of Gelbart and Jacquet [GJ], that for any $\pi = \pi_\infty \otimes \pi_f \in \mathcal{A}_0(2, F)$, there is an automorphic representation $S^2(\pi)$ of $GL(3, \mathbb{A}_F)$ such that $L(s, \pi, S^2) = L(s, S^2(\pi))$, and when π is not asssociated to a grossencharacter of a quadratic extension of F, $S^2(\pi)$ is cuspidal. For $n = 3$, one knows by the work of Bump and Ginzburg [BG] that $L^S(s, \pi_f, S^2)$ is entire for any finite set S containing the ramified and archimedean places.

We also need the Rankin-Selberg L-function attached to the pairs $\pi \in \mathcal{A}_0(n, F)$ and $\pi' \in \mathcal{A}_0(n', F)$, defined by an Euler product

$$L(s, \pi \times \pi') = \prod_v L(s, \pi_v \times \pi'_v),$$

where, for all finite v not dividing $f(\pi)f(\pi')$, we have

$$L(s, \pi_v \times \pi'_v) = \det(1 - (A_v(\pi) \otimes A_v(\pi'_v))(Nv)^{-s})^{-1}.$$

Remark 1.2. This L-series is nice (cf. [JS], [JPSS], [Sh1], [Sh2], [MW]), with no pole at $s = 1$, unless $n = n'$ and $\pi \simeq \overline{\pi}'$, in which case the order of the pole is 1. Moreover, the functional equation it satisfies is of the form

$$L(s, \pi \times \pi') = \epsilon(s, \pi \times \pi')L(1 - s, \overline{\pi} \times \overline{\pi}'),$$

with

$$\epsilon(s, \pi \times \pi') = (D_F^{nn'} N(\pi \times \pi'))^{1/2-s} W(\pi \times \pi'),$$

for some constant $N(\pi \times \pi')$ and a complex number $W(\pi \times \pi')$ of absolute value 1.

Set

$$\mathcal{A}_0(F) = \bigcup_{n \geq 1} \mathcal{A}_0(n, F).$$

Given $\pi_j \in \mathcal{A}_0(n_j, F)$, $1 \leq j \leq k$, Langlands's theory of Eisenstein series gives the construction of an automorphic representation $\boxplus_{j=1}^k \pi_j$ of $GL(\sum_{j=1}^k n_j, \mathbb{A}_F)$ such that

$$L(s, \boxplus_{j=1}^k \pi_j) = \prod_{j=1}^k L(s, \pi_j).$$

Moreover, by a uniqueness result of Jacquet and Shalika [JS], one knows that if $\boxplus_{j=1}^k \pi_j \simeq \boxplus_{j=1}^\ell \pi'_j$ with $\pi_j \in \mathcal{A}_0(n_j, F)$ and $\pi'_j \in \mathcal{A}_0(m_j, F)$, then, after renumbering if necessary, one has $k = \ell$, $n_j = m_j$ and $\pi_j \simeq \pi'_j$ for all j.

An automorphic representation π of $GL(n, \mathbb{A}_F)$ is called *isobaric* if there is a partition $n = \sum_{j=1}^k n_j$, and $\pi_j \in \mathcal{A}_0(n_j, F)$ (for all j) such that $\pi \simeq \boxplus_{j=1}^k \pi_j$. By a theorem of

Langlands [La], every automorphic representation π' of $GL(n, \mathbb{A}_F)$ is nearly equivalent to an isobaric automorphic representation π; i.e., for almost all places v, π'_v is isomorphic to π_v.

Given isobaric representations $\pi = \boxplus_{i=1}^k m_i \pi_i$ and $\pi' = \boxplus_{j=1}^{\ell} m'_j \pi'_j$ with π_i (resp. π'_j) cuspidal and not isomorphic to π_a (resp. π'_b) for $i \neq a$ (resp. $j \neq b$), the associated Rankin-Selberg L-function is

$$L(s, \pi \times \pi') = \prod_{i,j} L(s, \pi_i \times \pi_j)^{m_i m'_j},$$

which is nice. Moreover, its order of pole at $s = 1$ is given by the formula

$$-\operatorname{ord}_{s=1} L(s, \pi \times \pi') = |\{(i, j) \mid \overline{\pi}_i \simeq \pi'_j\}|. \tag{1.3}$$

Definition 1.4. Let π be an isobaric automorphic representation of $GL(n, \mathbb{A}_F)$ such that $L(s, \pi, r)$ is nice for some r, with the order of the pole at $s = 1$ being $m \geq 0$. Put $d = [F : \mathbb{Q}]$ and

$$L(s, \pi_\infty, r) = A(B)^{-m \dim(r) s/2} \prod_{j=1}^{m \dim(r)} \Gamma((s + b_j(\pi, r))/2),$$

for suitable $A, B > 0$ and $b_j(\pi, r) \in \mathbb{C}$. Put

$$M = M(\pi, r) = D_F^{\dim(r)} \mathcal{N} f(\pi, r) \left(1 + \sum_{j=1}^{\dim(r)} |b_j(\pi, r)| \right).$$

If $c > 0$, then we say that $L(s, \pi, r)$ has a *Siegel zero* relative to c if there exists a real point β in $(1 - c/\log M, 1)$ such that

$$L(\beta, \pi, r) = 0.$$

For $R > M^C$, C an absolute constant, it will sometimes be convenient to refer to a real zero β of $L(s, \pi, r)$ in the interval $(1 - c/\log R, 1)$ as a Siegel zero relative to c and R.

Definition 1.5. Given isobaric automorphic representations π and π' of $GL(n, \mathbb{A}_F)$ and $GL(n', \mathbb{A}_F)$ respectively, we set

$$M(\pi \times \pi') = D_F^{nn'} N(\pi \times \pi') \left(1 + \sum_{i=1}^{dn} \sum_{j=1}^{dn'} |b_i(\pi) + b_j(\pi')| \right).$$

We can use this to define Siegel zeros of $L(s, \pi \times \pi')$ (as above) for all such isobaric pairs (π, π').

2 Three basic lemmas

The purpose of this section is to establish Lemmas a, b, and c below concerning the L-functions of $GL(n) \times GL(n')$. We will not be surprised if they are known to experts. Since we could not find any published reference dealing with the first two lemmas (except for small n, n'), we have, at the suggestion of the referee, included complete proofs. Finally, given Lemma a, the truth of Lemma c is an immediate consequence of the basic lemma of [GHL].

Lemma a. For any unitary, isobaric automorphic representation $\pi = \pi_\infty \otimes \pi_f$, $L(s, \pi_f \times \overline{\pi}_f)$ defines a Dirichlet series with nonnegative coefficients. □

Proof. It suffices to check, at every finite place v, that the power series expansion in Nv^{-s} of $\log(L(\pi_v \times \overline{\pi}_v, s))$ has nonnegative coefficients. Note that π_v is unitary and generic. We can write it as an isobaric sum $\boxplus_{j=1}^r \pi_j$, with each π_j an irreducible (generic) square-integrable representation of $GL(n_j, F_v)$, with $\sum_j n_j = n$. One has the consequent factorization (cf. [JPSS, Theorem 9.5]):

$$L(s, \pi_v \times \overline{\pi}_v) = \prod_{i,j=1}^{r} L(s, \pi_i \times \overline{\pi}_j). \tag{2.1}$$

By the Bernstein-Zelevinsky classification of discrete series representations [BZ], we can associate to each π_j a positive divisor d_j of n_j, with $m_j = n_j/d_j$, and a supercuspidal representation β_j of $GL(d_j, F_v)$, and write π_j as the unique quotient, which we denote by $sp(\beta_j, n_j)$, of the induced representation

$$\mathrm{Ind}(GL(n_j, F_v), P; (\beta_j \otimes |\cdot|^{(m_j+1)/2-1}, \ldots, \beta_j \otimes |\cdot|^{(m_j+1)/2-m_j}) \otimes 1).$$

Here $P = MU$ is the parabolic subgroup associated to the partition $(d_j, d_j, \ldots, d_j)$, $M = GL(d_j, F_v) \times GL(d_j, F_v) \times \cdots \times GL(d_j, F_v)$, and the inducing representation of P is trivial on U.

If $n_i \le n_j$, then by [JPSS, Theorem 8.2], we have

$$L(s, \pi_i \times \overline{\pi}_j) = \prod_{k=1}^{m_i} L\left(s, \beta_i \times \overline{\beta}_j \otimes |\cdot|^{\frac{m_i+m_j}{2}-k}\right). \tag{2.2}$$

Moreover, applying Proposition 8.1 of [JPSS], we see that

$$L(s, \beta_i \times \overline{\beta}_j \otimes |\cdot|^{(m_i+m_j)/2-k}) = 1 \quad \text{if } d_i \ne d_j,$$

and that, if $d_i = d_j$,

$$L(s, \beta_i \times \overline{\beta}_j \otimes |\cdot|^{(m_i+m_j)/2-k}) = \prod_{\chi} L(s, \chi \otimes |\cdot|^{(m_i+m_j)/2-k}), \tag{2.3}$$

where the product on the right runs over the set of *unramified* (quasi) characters χ such that $\beta_i \simeq \beta_j \otimes \chi$. Note that if χ, χ' are two such characters, then β_j admits a self-twist by $\chi^{-1}\chi'$.

(In the statement of Proposition 8.1 of [JPSS], there is a typographical error, which should be corrected as follows: in the last line, $\pi \otimes \alpha^{-u} \simeq \sigma$ should read $\pi \otimes \alpha^{-u} \simeq \sigma^\vee$, where $\sigma^\vee$ denotes the contragredient.)

Note that (2.3) allows us to establish the positivity separately for each twist-equivalence class of $\{\beta_j\}$. In other words, the lemma is a consequence of the following.

Sublemma 2.4. Let K be any finite extension of F_v with residue field $\mathbb{F}_{q^f}$. Let β be an irreducible, unitary supercuspidal representation of $GL(d, K)$, $\chi_1, \chi_2, \dots, \chi_\ell$ (quasi) characters of K^*, and $m_1, m_2, \dots, m_\ell$ positive integers. Then the isobaric representation

$$\Pi = \boxplus_{j=1}^{\ell} sp(\beta \otimes \chi_j, dm_j)$$

is of positive type, i.e., the power series expansion in q^{-fs} of $\log L(s, \Pi \times \overline{\Pi})$ has nonnegative coefficients. □

Proof of Sublemma. Let $\mathcal{G} = \mathcal{G}(\beta)$ denote the set of unramified (quasi) characters ν of K^* such that $\beta \simeq \beta \otimes \nu$, and let $N \geq 1$ be the cardinality of $\mathcal{G}$. We may, after rearranging, assume that $m_1 \leq m_2 \leq \cdots \leq m_\ell$.

First we will prove the sublemma when $N = 1$, i.e., when β has no nontrivial unramified self-twist. Then by applying (2.3), we get (for all $i, j \leq \ell$)

$$L(s, sp(\beta \otimes \chi_i, dm_i) \times sp(\overline{\beta} \otimes \chi_j^{-1}, dm_j)) = L(s, sp(\chi_i, m_i) \times sp(\chi_j^{-1}, m_j)), \tag{2.5}$$

which identifies (by (2.2)) with

$$\prod_{1 \leq i < j \leq \ell} \prod_{k=1}^{m_i} L(s, \chi_i \overline{\chi}_j \cdot |\cdot|^{(m_i+m_j)/2-k}). \tag{2.6}$$

So, to prove the sublemma for $N = 1$, we may (and we will) assume that $d = 1$ and $\beta = 1$.

Let $m = m(\Pi)$ denote the maximum of $\{m_j \mid 1 \leq j \leq \ell\}$, which is simply m_ℓ by our ordering, and fix a uniformizer ϖ_K of K. We will prove the sublemma by induction on m. If $m = 1$, then

$$\Pi = \boxplus_{j=1}^{\ell} \chi_j.$$

Clearly, since all the χ_j are unramified, the coefficient of q^{-nfs} in the expansion of $\log L(\Pi \times \overline{\Pi}, s)$ equals, for every $n \geq 1$, the nonnegative number

$$\left(\sum_{i=1}^{\ell} \chi_i(\varpi_K^n)\right)\left(\sum_{i=1}^{\ell} \overline{\chi}_i(\varpi_K^n)\right).$$

Now assume $m > 1$ and that the sublemma has been proved for all $m' < m$ (still with $N = 1$). Consider first the case when some $m_j < m$, and denote by $r < \ell$ the largest positive integer with $m_r < m$, and put $m' = m_r$. Consider the (isobaric) representations

$$\Pi' = (\boxplus_{j=1}^{r} sp(\chi_j, m_j)) \boxplus (\boxplus_{j=r+1}^{\ell} sp(\chi_j|\cdot|^{(m-m')/2}, m')), \tag{2.7}$$

and

$$\eta = \boxplus_{j=r+1}^{\ell} sp(\chi_j, m - m').$$

Then we see, using (2.6), that

$$L(s, \Pi \times \overline{\Pi}) = L(s, \Pi' \times \overline{\Pi}')L(s, \eta \times \overline{\eta}). \tag{2.8}$$

Since m' and $m - m'$ are both (strictly) smaller than m, we have by the induction assumption that both $L(s, \Pi' \times \overline{\Pi}')$ and $L(s, \eta \times \overline{\eta})$ are of positive type. Thus the same is true of $L(s, \Pi \times \overline{\Pi})$.

It remains to consider the case when $m_j = m$ for all $j \leq \ell$. Here we have a factorization (by appealing to (2.6))

$$L(s, \Pi \times \overline{\Pi}) = \prod_{k=1}^{m} L(s, \Pi_k \times \overline{\Pi}_k),$$

where

$$\Pi_k = \boxplus_{j=1}^{\ell} \chi_j|\cdot|^{(m-k)/2}.$$

The positivity now follows as each $m(\Pi_k)$ is 1. This also completes the proof of the sublemma for $N = 1$.

Now suppose $N = |\mathcal{G}| > 1$. First we claim that $\mathcal{G}$ is a finite cyclic group of order dividing d. Indeed, given $\mu_1, \mu_2 \in \mathcal{G}$, we have $\beta \otimes \mu_1\mu_2^{-1} \simeq \beta \otimes \mu_1 \simeq \beta$, and so $\mathcal{G}$ is a group. Comparing central characters, we see that every element of $\mathcal{G}$ has order dividing d. Finally, the compositum E of the cyclic extensions $K(\mu)$ cut out by all the μ (in the finite group $\mathcal{G}$) is unramified over K and is hence cyclic. Clearly, we can find a ν in $\mathcal{G}$ such that $E = K(\nu)$ as claimed. Put $r = [E : K]$. Set

$$\Pi' = \boxplus_{j=1}^{\ell} sp(\chi_j, m_j).$$

Then, by using (2.2) and (2.3), we get

$$L(s, \Pi \times \overline{\Pi}) = \prod_{j=0}^{r-1} L(s, \Pi' \times \overline{\Pi}' \otimes \nu^j). \tag{2.9}$$

But on the other hand, the product on the right is, by [AC], the same as

$$L(s, \Pi'_E \times \overline{\Pi}'_E),$$

where Π'_E is the base change of Π' to $GL(d)/E$. If we write d' for d/r, then, again by [AC], there exists a supercuspidal representation β' of $GL(d', E)$ such that β is (automorphically) induced from β'. If τ denotes a generator of $Gal(E/K)$, then the only supercuspidals giving rise to the discrete series occurring in the (isobaric) decomposition of Π_E are of the form $\beta' \circ \tau^j$, for $j = 0, 1, \ldots, r$. Moreover, $N(\beta' \circ \tau^j) = 1$ for all j; for otherwise β', and hence β, would be induced from a larger unramified extension E'/K, contradicting the maximality of E. Thus we see, by our proof above for $N = 1$, that $L(s, \Pi'_E \times \overline{\Pi}'_E)$ is of positive type. Then the same holds for $L(s, \Pi \times \overline{\Pi})$ by virtue of (2.9). ■

This finishes the proof of Lemma a. ■

The next lemma bounds the "level" $M(\pi \times \pi')$ (see Definition 1.5) in terms of the levels of π and π'. The lower bound is obtained by a simple variant of the Stark-Odlyzko method for discriminants for number fields. Our proof of the upper bound uses Henniart's proof of the "numerical local Langlands conjecture" and the cyclic base change theorem of Arthur and Clozel, in conjunction with Jordan's theorem, which gives constraints on the possible finite subgroups of $GL(n, \mathbb{C})$.

Lemma b. Let $n, n' \geq 1$. Then there exist positive integers m, m', depending only on n, n', and F, such that

$$M(\pi)^{-m'} M(\pi')^{-m} \leq M(\pi \times \pi') \leq M(\pi)^{m'} M(\pi')^{m}. \qquad \square$$

Proof. First we verify the assertion for the archimedean part, and call it $M_\infty(\pi \times \pi')$, of $M(\pi \times \pi')$. Let w be an archimedean place of F. Write $\pi_w = \boxplus_{i=1}^k \eta_i$ and $\pi'_w = \boxplus_{j=1}^r \eta'_j$, with η_i, η'_j in the discrete series. By combining the results of Shahidi [Sh1], [Sh2] with Vogan's classification of unitary respresentations of $GL(n, F_w)$ as in [BR, §2], we can factor $L(s, \pi_w \times \pi'_w)$ as a product $\prod_{i,j} L(s, \tau_i \otimes \tau'_j)$, where τ_i and τ'_j are irreducibles of the Weil group W_{F_w}. When $F_w = \mathbb{C}$, each of the τ_i, τ'_j is one-dimensional, and it follows easily that

$$M_w(\pi)^{-n'} M_w(\pi')^{-n} \leq M_w(\pi \times \pi') \leq M_w(\pi)^{n'} M_w(\pi')^{n}, \tag{2.10}$$

where $M_w(-)$ has the obvious meaning. When $F_w = \mathbb{R}$, irreducibles of $W_\mathbb{R}$ are of dimension one or two. Suppose τ is irreducible of dimension two, so that we may write $\tau = \mathrm{Ind}(W_\mathbb{R}, \mathbb{C}^*; \chi)$. Then $\tau \otimes \tau'$ decomposes as $\mathrm{Ind}(W_\mathbb{R}, \mathbb{C}^*; \chi \otimes \tau')$. If τ' is also two-dimensional, given as $\tau' = \mathrm{Ind}(W_\mathbb{R}, \mathbb{C}^*; \chi')$, then $\tau \otimes \tau' \simeq \mathrm{Ind}(W_\mathbb{R}, \mathbb{C}^*; \chi\chi') \oplus \mathrm{Ind}(W_\mathbb{R}, \mathbb{C}^*; \overline{\chi}\chi')$. In either case, using the fact that $L(s, \mathrm{Ind}(W_{\mathbf{R}}, \mathbf{C}^*; \nu)) = L(s, \nu)$ for any ν, we see that that (2.10) holds at any real place w as well.

To establish the lower bound for $N(\pi \times \pi')$, we need the following lemma provided by the Stark-Odlyzko method.

Lemma 2.11. Let $\phi(s)$ be a Dirichlet series which is absolutely convergent for $\Re(s) > 1$. Suppose for real $s > c$, c a fixed constant, we have $\phi'(s)/\phi(s) \ll 1$, with the implied constant depending only on c. Suppose also that $\phi(s)$ has an Euler product, so $\phi(s) \neq 0$ in $\Re(s) > 1$. Suppose finally that $\phi(s)$ satisfies a functional equation of the form

$$\Lambda(s) = s^m(1-s)^m N^{s/2} G(s)\phi(s) = W(\phi)\overline{\Lambda(1-\bar{s})},$$

with $\Lambda(s)$ entire of order one, N the level of ϕ, $W(\phi)$ a constant, and $G(s)$ a product of gamma factors of the form $\prod_{i=1}^{k} \Gamma_{\mathbf{R}}(s + c_i)$. Let $R = 2 + \sum_{i=1}^{k} |c_i|$. Then there exists an effective constant A, depending only on c and m, such that $N > R^{-A}$. □

Proof. Write

$$\Lambda(s) = e^{A+Bs} \prod \left(1 - \frac{s}{\rho}\right) e^{s/\rho},$$

where the product runs over the set of nontrivial zeros of $\Lambda(s)$. Taking the real part of the logarithmic derivative, and applying the functional equation to the right-hand side, we get

$$\Re(B) + \Re \sum \frac{1}{s-\rho} - \Re \sum \frac{1}{\rho} = -\Re(\overline{B}) + \Re \sum \frac{1}{s-(1-\overline{\rho})} + \Re \sum \frac{1}{\overline{\rho}}.$$

After noting that the two sums involving s are equal, as they run over the same set of zeros, it follows that

$$\Re(B) = \Re \sum \frac{1}{\rho}.$$

Solving for the level, we thus have, for s real,

$$\log N = -\frac{m}{s} - \frac{m}{s-1} - \Re\left(\frac{G'(s)}{G(s)}\right) - \Re\left(\frac{\phi'(s)}{\phi(s)}\right) + \Re \sum \frac{1}{s-\rho}.$$

If s is chosen larger than c, then $\Re(\phi'(s)/\phi(s))$ will be bounded above by an absolute constant. Also, $\Re(G'(s)/G(s))$ will be bounded by an absolute constant times $k \log R$. The two terms involving m contribute at most a constant times m. Finally, for each $\rho = \beta + i\rho$ in the sum over zeros, we have

$$\Re \sum \frac{1}{s-\rho} = \frac{s-\beta}{(s-\beta)^2 + \gamma^2)} > 0,$$

as $s > 1$. We thus have $\log N \gg -k \log R$, from which the Lemma 2.11 follows. ■

We apply this lemma to $\phi(s) = L(s, \pi\times\pi')$. We claim that we may choose c to be $2+\epsilon$, for any $\epsilon > 0$. Indeed, if we denote the set of ramified primes for (π, π') by S, then one knows by [JS] (or [Sh1]) that at any finite place v not in S, the coefficients $a_{v^m}(\pi)$ and $a_{v^m}(\pi')$ are bounded by $nN(v)^{m/2}$ and $n'N(v)^{m/2}$ respectively. This shows that the logarithmic derivative of the incomplete L-function $L^S(s, \pi\times\pi')$ is bounded (in absolute value) in $\Re(s) > 2+\epsilon$ by nn' times (the corresponding incomplete part of) $\zeta_K'(s-1)/\zeta_K(s-1)$, which is bounded in this range. Moreover, since the Euler product converges in $\Re(s) > 1$, we also know that the bad Euler factors $L(s, \pi_v\times\pi_v')$ have no singularities in in $\Re(s) > 1$, and so their logarithmic derivative can be bounded by an absolute constant in $\Re(s) > 2\}$. Since the R given by Lemma 2.11 is essentially $M_\infty(\pi\times\pi')$ in this case, and since the order of pole of $L(s, \pi\times\pi')$ at $s = 1$ is bounded by nn', we get the lower bound of Lemma b by appealing to (2.10).

As a supplement, note that our proof shows that the lower bound depends only on n, n' and the archimedean factors of π, π', and not on the components at finite places. In particular, the lower bound is the same for $L(s, \pi\times\pi'\otimes\chi)$, for any finite order character χ which is trivial at infinity.

For any number field K, $r \geq 1$, and $\Pi = \otimes_v\Pi_v$ an isobaric automorphic representation $GL(r, \mathbb{A}_K)$, we will say that Π is good (or almost semistable) if every local component Π_v is an isobaric sum of (quasi) characters and special representations, i.e., if we have $\Pi_v = \boxplus_{j=1}^k sp(\beta_j, n_j)$, with each β_j a (quasi) character. (Note that when $n_j = 1$, $sp(\beta_j, n_j) = \beta_j$. Also, one says that Π is semistable if each β_j is moreover unramified.) ■

We will now prove that Lemma b holds for (π, π') if π and π' are both good. For this we first observe that, as the epsilon factor is a product of local factors coming from the local functional equations, we can factor $N(\pi\times\pi')$ as $\prod_v N_v(\pi\times\pi')$, where v runs over the finite places of F. This is a finite product; in fact, we have, for every v unramified for both π and π', $N_v(\pi\times\pi') = N_v(\pi) = N_v(\pi') = 1$. So we may assume that v belongs to the (finite) set S of divisors of $\mathfrak{f}(\pi)\mathfrak{f}(\pi')$. Write $\pi_v = \boxplus_i sp(\beta_i, n_i)$ and $\pi_v' = \boxplus_j sp(\beta_j', n_j')$. Then one has a factorization of $\epsilon(s, sp(\beta_i, n_i) \times \epsilon(sp(\beta_j', n_j')$ analogous to (2.2) (see [JPSS, §8]), i.e., as a product of abelian ϵ-factors as β_i, β_j' are (quasi) characters. The claim follows easily; in fact, we have (in this case)

$$1 \leq N_v(\pi\times\pi') \leq N_v(\pi)^{n'}N_v(\pi')^n. \tag{2.11}$$

The general case is more subtle, and Henniart has pointed out to us that it is an open problem to know that $N_v(\pi\times\pi_v')$ is ≥ 1. Luckily we do not need it.

Our basic idea is to use the fact that, after a finite solvable base change K/F, π and π' become good. We control this base change carefully so we can use induction in cyclic layers.

First we need some preliminaries. Fix any finite place v of F. For every $k \geq 1$, let $\mathfrak{A}(k, F_v)$ (resp. $\mathfrak{R}(k, F_v)$) denote the set of irreducible admissible (resp. semisimple k-dimensional) representations, up to equivalence, of $GL(k, F_v)$ (resp. W'_{F_v}). Then one knows by a theorem of Henniart ("the numerical local Langlands conjecture") [He1], that there is a bijection

$$\sigma_k : \mathfrak{A}(k, F_v) \to \mathfrak{R}(k, F_v), \tag{2.12}$$

which preserves conductors and is functorial for (i) twisting by characters, and (ii) base change in cyclic extensions [AC]. Moreover, it maps the subset $\mathfrak{A}_0(k, F_v)$ consisting of supercuspidal representations of $GL(k, F_v)$ onto the set $\mathfrak{R}_0(k, F_v)$ of irreducible representations of the Weil-Deligne group of F_v in $\mathfrak{R}(k, F_v)$. A consequence of this work of Henniart is that there exists, for each $\eta \in \mathfrak{A}_0(n, F_v)$, a finite Galois extension E/F_v such that the base change η_E of η to $GL(k)/E$ is an isobaric sum $\mu \boxplus \mu \boxplus \cdots \boxplus \mu$, for an unramified (quasi) character μ of E^*. Indeed, the irreducibility of $\sigma_k(\eta)$ implies that it is of the form $\tau \otimes \mu$ with μ an unramified character and τ an irreducible representation of $Gal(\overline{F}_v/F_v)$, necessarily factoring through a finite Galois extension E/F_v (corresponding to the kernel of τ). Thus $\sigma_k(\eta)_E \simeq k\mu$, which implies the asserted consequence by the functorility of σ_k relative to (i) and (ii) and by the solvability of E/F_v. Note that the extension E/F_v can be arbitrarily large, but viewing it as a (finite) subgroup of $GL(k, \mathbb{C})$, we are saved by Jordan's theorem (see [R]), which says that there exists an abelian normal subgroup A of $Gal(E/F_v)$ such that $[Gal(E/F_v) : A] \leq C'_k$, for an absolute constant C'_k depending only on k. Then the restriction of τ to A will decompose as $\oplus_i \mu_i$, with each μ_i one-dimensional, and if K denotes the fixed field of A, η_K will be isomorphic to $\boxplus \mu_i$.

Now suppose π is an arbitrary irreducible admissible representation of $GL(n, F_v)$. Writing it as an isobaric sum of discrete series representations, we then conclude by the above that there is a finite Galois extension K/F such that the base change π_K is good, and moreover

$$[K : F_v] \leq C_n, \tag{2.13}$$

for an absolute constant C_n depending only on n. (C_n is bounded by $\prod_k C'_k$.) Note also that this bound is independent of the base field F_v.

Now we begin the proof of the upper bound of Lemma b. It suffices to find absloute constants m, m' such that the conductor $N(\pi \times \pi')$ is bounded above by $M(\pi)^{m'} M(\pi')^{m}$. Let X denote the set of (rational) primes $\ell \leq C_n C_{n'}$. Let E be the abelian extension of F obtained by adjoining the ℓth roots of unity, for all $\ell \in X$. Then $[E : F]$ is bounded by $\prod_{\ell \in X}(\ell - 1)$.

We first claim that, for any pair (π, π'), if we have an upper bound of the sort we seek over E, then we have one over F. Since this is trivial if E is F, we assume that $E \neq F$ and argue by induction on $[E : F]$. Choose a a cyclic subextension L/F of E/F of degree a prime p, and denote by η the idele class character of F corresponding to L/F given by class field theory. Then one has, by Arthur-Clozel [AC, Prop. 6.9], the identities of epsilon factors

$$\epsilon(s, \pi_L \times \pi'_L) = \epsilon(s, \mathrm{Ind}(W_F, W_L; 1))^{-nn'} \prod_{i=0}^{p-1} \epsilon(s, \pi \times \pi' \otimes \eta^i), \tag{2.14}$$

and

$$\epsilon(s, \beta_L) = \epsilon(s, \mathrm{Ind}(W_F, W_L; 1))^{-r} \prod_{i=0}^{p-1} \epsilon(s, \beta \otimes \eta_j^i), \tag{2.15}$$

where β denotes π or π', and r is n or n' accordingly. By the inductive hypothesis, we have (universal) constants $m(L)$ and $m'(L)$ depending only on L, n and n', such that

$$N(\pi_L \times \pi'_L) \leq M(\pi_L)^{m'(L)} M(\pi'_L)^{m(L)}. \tag{2.16}$$

One knows (cf. [He2]) that we have the bound $1 \leq N(\beta \otimes \omega) \leq N(\beta) N(\omega)^r$, for any character ω of F^*. Using this along with the inductive assumption, the expression of $N(\mathrm{Ind}(W_F, W_L; 1))$ in terms of the discriminant of L, and also the lower bound for $M(\pi \times \pi' \otimes \eta^i)$ (already established, independently of η^i), we see that (2.14) and (2.15) imply the following inequalities for conductors:

$$N(\pi \times \pi') \leq D_L^{nn'} (M(\pi)^{m'} M(\pi')^{m})^{p-1} M(\pi_L)^{m'(L)} M(\pi'_L)^{m(L)} \tag{2.17}$$

and

$$N(\pi_L)^{m'(L)} N(\pi_L)^{m(L)} \leq D_L^{-nm'(L) - n'm(L)} N(\pi)^{pm'(L)} N(\pi')^{pm(L)}.$$

Moreover, one sees without much trouble that $M_\infty(\pi_L) \leq M_\infty(\pi)^{3np/2}$ and that $M_\infty(\pi'_L) \leq M_\infty(\pi')^{3n'p/2}$. The claim follows by taking $m(F) = (p-1)m + (3np)/2m(L)$ and $m'(F) = (p-1)m' + (3n'p)/2m'(L)$. (To avoid confusion, note that m, m' were given by the lower bound, and to get also the upper bound, we have to increase them to $m(F), m'(F)$.)

Thanks to this claim, we may enlarge F and assume that it contains ℓth roots of unity for all $\ell \in X$. (This process is absolute and does not depend on (π, π').) Let S denote (as before) the set of bad places for (π, π'). Choose, for each $v \in S$, a finite Galois extension K^v/F_v of degree bounded by $C_n C_{n'}$ such that π_v and π'_v are both good over K^v. Put

$r = r(F) = \sum_{v \in S}[K^v : F_v]$. If $r = 1$, we are done by (2.12). So assume that $r > 1$ and argue by induction on r. Pick any $v_0 \in S$ and a cyclic subextension L^{v_0}/F_{v_0} of K^{v_0}/F_{v_0} of degree a prime ℓ. Denote by χ_{v_0} the character of F_{v_0} corresponding to L^v. Fix also unramified characters χ_v at all the other places v in S, each of order ℓ.

We now need the following lemma. We are indebted to D. Rohrlich for his help with the proof.

Lemma 2.18. Let ℓ be a prime and k a number field containing all the ℓth roots of unity. Fix a finite set S of finite places, and at each $v \in S$, fix a character of k_v^* of order ℓ. Then there exists a global (ray class) character χ of k, with components χ_v at each $v \in S$, such that

$$N(\chi) \leq \prod_{v \in S}(A + N(\chi_v))^B,$$

for some universal constants A, B. □

One hopes that this result can be freed from the hypothesis that k contains ℓth roots of unity.

Proof. At each $v \in S$ where χ_v is nontrivial, we can find an element $\alpha_v \in k_v^*$ by Kummer theory such that the extension defined by χ_v is obtained by adjoining an ℓth root of α_v. By the Chinese remainder theorem, we can find an $\alpha \in k^*$ such that at each such v, α defines the same class mod $(k_v^*)^\ell$ as α_v, and such that α is zero, modulo v, at each $v \in S$ where χ_v is trivial. This gives a specific congruence condition on α, and we can choose it to be of norm bounded by the norm of an ideal determined by $\{N(\chi_v)\}$. We take χ to be associated to the extension defined by the ℓth root of α. The bound on the norm of α also gives the bound of the desired form on $N(\chi)$. ■

Applying this to our situation (i.e., with $k = F$), we get a global character χ of conductor satisfying the bound above. Let L denote the cyclic extension of F degree ℓ corresponding to χ. By construction, v_0 is inert in L with the local extension at v_0 being L^{v_0}/F_{v_0}. Then $r(L) < r$, and so by the inductive hypothesis, we have the requisite upper bound over L relative to absolute constants $m(L)$, $m'(L)$. Note also that the relative discriminant of L over F is bounded by a power of $N(\pi)N(\pi')$ (by Lemma 2.18). Appealing to (2.14) through (2.17) (with $p = \ell$, $\eta = \chi$), we obtain (as before) absolute constants $m(F)$, $m'(F)$, and the desired upper bound. ■

Remark. Needless to say, we have been quite wasteful above in estimating $N(\pi \times \pi')$, partly caused by our use of Jordan's theorem. We believe that the proposition should hold with $m = n$ and $m' = n'$, which will be very nice to establish. Clearly, it suffices to prove it locally for pairs of supercuspidal representations. Our weak result is, however, sufficient for our purposes.

The following is an easy consequence of the crucial basic lemma from [GHL] giving a bound on the number of possible real zeros near $s = 1$ of a nice positive Dirichlet series, when combined with Lemma a.

Lemma c. Fix a number field F, and let π be an isobaric automorphic representation of $GL(n, \mathbb{A}_F)$ with $L(s, \pi_f \times \overline{\pi}_f)$ having a pole of order $m \geq 1$ at $s = 1$. Then there is an effective constant $c > 0$, depending only on n and m, such that $L(s, \pi \times \overline{\pi})$ has *at most* m *real zeros* in the interval

$$1 - c/\log(M(\pi \times \overline{\pi})) < \Re(s) < 1. \tag{2.19}$$

Furthermore, suppose that there is a factorization

$$L(s, \pi_f \times \overline{\pi}_f) = L_1(s)L_2(s), \tag{2.20}$$

with $L_2(s)$ holomorphic in $\{s \in \mathbb{C} \mid \Re(s) \in (t, 1)\}$, for some fixed real $t < 1$ depending only on n. Then there is an effective constant $c' > 0$, depending only on n, t, and m, such that $L_1(s)$ has *at most* m *real zeros* in the interval (2.20) with c replaced by c'. □

Proof. We now know (by Lemma a) that $L(s, \pi_f \times \overline{\pi}_f)$ defines a Dirichlet series with non-negative coefficients. It has a convergent Euler product in $\{\Re(s) > 1\}$ and admits a meromorphic continuation to the whole s-plane, having no pole outside $s = 0, 1$ (see Remark 1.2). Set $\Lambda(s) = s^m(1 - s)^m N(\pi \times \overline{\pi})^{-s/2} L(s, \pi \times \overline{\pi})$. Then $\Lambda(s)$ is entire of order 1 and satisfies

$$\Lambda(s) = W(\pi \times \overline{\pi})\Lambda(1 - s).$$

So we may apply the lemma of [GHL, page 178] and conclude that $L(s, \pi_f \times \overline{\pi}_f)$ has at most m real zeros in the interval (2.20) for an effective constant $c > 0$ depending only on n and m. (To be precise, the statement of the lemma in [GHL] assumes that the constant in the functional equation of $\Lambda(s)$ is 1, but the argument there does not make use of this. It is easy to see that $W(\pi \times \overline{\pi}) = \pm 1$, but it is an open problem to prove that it is $+1$, already for π on GL(2), except when π is dihedral or holomorphic.)

This then implies the same for $L_1(s)$ for s in the interval $\max(t, 1 - c/\log(M(\pi \times \overline{\pi}))) < \Re(s) < 1$, since $L_2(s)$ has no singularities in $(t, 1)$. The assertion follows for an appropriate $c' > 0$, depending only on c and t. ■

3 The rarity of Siegel zeros

Theorem A. Fix an integer $R > 0$, and consider the subset $\mathcal{A}_0(F, R)$ of $\mathcal{A}_0(F)$ consisting of cusp forms π on $GL(n)/F$ such that $D_F^{n^2} M(\pi \times \pi) \leq R$. Then there exists an absolute effective constant $c > 0$ such that there is *at most one* member of $\mathcal{A}_0(F, R)$ which has a Siegel zero relative to c and R. □

Proof. Let $\pi = \pi_\infty \otimes \pi_f$, $\pi' = \pi'_\infty \otimes \pi'_f$ be arbitrary, nonisomorphic, nontrivial members of $\mathcal{A}_0(F, R)$. Put

$$D(s) = L(s, (1 \boxplus \pi_f \boxplus \pi'_f) \times (1 \boxplus \overline{\pi}_f \boxplus \overline{\pi}'_f)),$$

which is a Dirichlet series with nonnegative coefficients. (See Lemma a of §2.) Moreover, applying (1.3), we see that

$$-\mathrm{ord}_{s=1} D(s) = 3.$$

On the other hand, we have the factorization

$$D(s) = L_1(s)L_2(s), \tag{3.1}$$

where

$$L_1(s) = L(s, \pi_f)^2 L(s, \pi'_f)^2$$

and

$$L_2(s) = \zeta_F(s) L(s, \pi_f \times \overline{\pi}_f) L(s, \pi'_f \times \overline{\pi}'_f) L(s, \pi_f \times \pi'_f)^2.$$

By Lemma b of §2, we see that the conductor of D(s) is bounded between R^{-C} and R^C, with C an absolute constant. Suppose both $L(s, \pi)$ and $L(s, \pi')$ have Siegel zeros in $(1 - c/\log R)$. Then, since $L_2(s)$ is nice and in particular does not have poles in $\{\Re(s) \in (1/2, 1)\}$, Lemma c says that there is an effective constant $c > 0$ such that $L_1(s)$ can have at most three Siegel zeros relative to c and R. This clearly precludes both $L(s, \pi_f)$ and $L(s, \pi'_f)$ having Siegel zeros simultaneously. ■

Corollary 3.2. Let π be a non-self-dual cusp form on $GL(n)/F$ for any $n \geq 1$. Then $L(s, \pi)$ never admits a Siegel zero. □

Proof. Suppose $L(s, \pi)$ has a Siegel zero $s = \beta$. Then, since β is real, $L(s, \overline{\pi})$ will also have a Siegel zero at $s = \beta$. Since the conductors of π and $\overline{\pi}$ have the same norm, we get a contradiction to the theorem above, since π is not isomorphic to $\overline{\pi}$. ■

Remark 3.3. As mentioned in the introduction, this extends, to arbirary n, the classical result for GL(1), i.e., that the L-series of complex characters have no Siegel zeros. For $n = 2$, this has been found by C. Moreno [M].

4 A consequence of functoriality

Let $\pi \in \mathcal{A}_0(n, F)$. In this section (and *nowhere else*), we will make the following hypothesis impied by the principle of functoriality of Langlands.

Hypothesis H(π). For all $m \leq n^2$ and $\eta \in \mathcal{A}_0(m, F)$, there exists an isobaric automorphic representation $\pi \boxtimes \eta$ of $GL(nm, \mathbb{A}_F)$ such that, at every finite place v where π and η are unramified, we have

$$A_v(\pi \boxtimes \eta) = A_v(\pi) \otimes A_v(\eta) \in GL(nm, \mathbb{C}). \tag{4.1}$$

It may be useful to note that this defines the hypothetical representations $\pi \boxtimes \eta$ uniquely. Indeed, we have the following.

Lemma 4.2. Suppose there are two isobaric automorphic representations, say β and β', of $GL(nm, \mathbb{A}_F)$ satisfying (4.1) at every unramified v. Then $\beta \simeq \beta'$. □

Proof. Write $\beta = \boxplus m_j \pi_j$, with $m_j > 0$ and $\pi_j \in \mathcal{A}_0(n_j, \mathbb{A}_F)$, such that π_j is not isomorphic to π_k if $j \neq k$. Then, for every j, $L(s, \beta \times \bar{\pi}_j)$ has a pole of order m_j at $s = 1$ (see (1.3)). Since $A_v(\beta) \otimes A_v(\bar{\pi}_j)$ is by (4.1) the same as $A_v(\beta') \otimes A_v(\bar{\pi}_j)$, and since the finite number of bad and archimedean factors is invertible at $s = 1$ (see [JS], and also [BR, §2]), we conclude that $L(s, \beta' \times \bar{\pi}_j)$ also has a pole of order m_j at $s = 1$. Then π_j occurs in the $\boxplus$ decomposition of β' with multiplicity m_j. Since $\sum_j m_j n_j = n^2$, β' must be isomorphic to β. ■

Remark 4.3. When $n = 1$, π is simply an idele class character of F, and $\pi \boxtimes \eta = \pi \otimes \eta$. See §5 for a discussion of the case $n = 2$. For any n, if π is associated to an idele class character of a cyclic extension [AC], this hypothesis can be verified.

Theorem B. Let π be a cusp form on $GL(n)/F$, $n > 1$, for which the hypothesis $H(\pi)$ holds. Then $L(s, \pi)$ has *no Siegel zero*. □

Proof. In view of Corollary 3.2, we may, and we will, assume that π is self-dual. Put $\Pi = \pi \boxtimes \pi$. First we need the following.

Lemma 4.4. There exists a $\tau \in \mathcal{A}_0(F)$ such that

(1) τ is not isomorphic to 1 or π; and

(2) τ occurs in the $\boxplus$ decomposition of Π, i.e., $L(s, \Pi \times \bar{\tau})$ has a pole at $s = 1$. □

Proof. Write $\Pi = \boxplus_j m_j \eta_j$, with η_j cuspidal and $\eta_j \neq \eta_k$ if $j \neq k$. Since π is cuspidal and self-dual, $L(s, \pi \times \pi)$ has a pole of order one at $s = 1$. Thus, for some j, we have $\eta_j = 1$ and $m_j = 1$. Suppose the lemma is false. Then we must have $\Pi = 1 \boxplus m\pi$, for some $m \geq 1$. This implies that $n^2 = 1 + mn$, which is impossible unless $n = 1$ and $m = 0$. We are done, as we assume that $n > 1$. ■

Consider the Dirichlet series

$$D(s) = L(s, (1 \boxplus \tau_f \boxplus \pi_f) \times (1 \boxplus \overline{\tau}_f \boxplus \pi_f)),$$

which has nonnegative coefficients by Lemma a of §2. Applying (1.3), we see that

$$-\operatorname{ord}_{s=1} D(s) = 3.$$

Also, using Lemma b and the fact that τ occurs in $\pi \boxtimes \pi$, we see that the level of $D(s)$ is bounded above by $M(\pi)^C$, for an absolute constant C, where M is as in Definition 2.5. On the other hand, we have the factorization

$$D(s) = \zeta_F(s)L(s, \Pi_f)L(s, \tau_f \times \overline{\tau}_f)L(s, \tau_f)L(s, \overline{\tau}_f)L(s, \pi_f)^2 L(s, \pi_f \times \overline{\tau}_f)L(s, \pi_f \times \tau_f).$$

Claim 4.5. There exist nice Dirichlet series $D_1(s)$ and $D_2(s)$ such that

$$L(s, \pi_f \times \overline{\tau}_f) = L(s, \pi_f)D_1(s) \tag{a}$$

and

$$L(s, \pi_f \times \tau_f) = L(s, \pi_f)D_2(s). \tag{b}$$

□

Indeed, since τ occurs in the isobaric sum decomposition of Π, $L(s, \Pi \times \overline{\tau})$ has a pole at $s = 1$. But

$$L(s, \Pi \times \overline{\tau}) = L(s, \pi \times (\pi \boxtimes \overline{\tau})),$$

which shows that π occurs in the decomposition of $\pi \boxtimes \overline{\tau}$. (Note that since 1 occurs in Π, $\tau \in \mathcal{A}_0(k, F)$ for some $k \leq n^2 - 1$, and so $\pi \boxtimes \overline{\tau}$ exists by the hypothesis $H(\pi)$.) This gives (a). Furthermore, since π is self-dual, the occurrence of τ in Π implies also that $\overline{\tau}$ occurs in Π. Then π occurs in $\pi \boxtimes \tau$, furnishing (b). We have proved the claim.

Thanks to the claim, we can write

$$D(s) = L_1(s)L_2(s),$$

where

$$L_1(s) = L(s, \pi_f)^4$$

and

$$L_2(s) = \zeta_F(s)L(s, \Pi_f)L(s, \tau_f \times \overline{\tau}_f)L(s, \tau_f)L(s, \overline{\tau}_f)D_1(s)D_2(s).$$

Now suppose $L(s, \pi)$ has a Siegel zero. Then $L_1(s)$ has a Siegel zero of order 4. But, on the other hand, since the Dirichlet series $L_2(s)$ is nice and has no pole in $\{\Re(s) \in (0, 1)\}$ (see

Remark 1.2), and since D(s) has a pole of order 3 at $s = 1$, Lemma c implies that $L_1(s)$ can have at most three zeros in the relevant range to the left of $s = 1$. This gives the desired contradiction. ∎

Remark 4.7. In the proof of Theorem B, we did not need the full force of the hypothesis $H(\pi)$, only that $\pi \boxtimes \pi$ exists, and that $\pi \boxtimes \tau$ exists for every τ occurring in Π.

Remark 4.8. Suppose π itself occurs in the $\boxplus$ decomposition of $\Pi = \pi \boxtimes \pi$; i.e., the triple product L-function $L(s, \pi \times \pi \times \pi)$ has a pole at $s = 1$. This happens, for example, when π is a self-dual cusp form on GL(3)/F with trivial central character (cf. §6). Then the proof of Theorem B can be simplified by considering, instead of D(s), the Dirichlet series $L(s, (1 \boxplus \pi_f) \times (1 \boxplus \pi_f))$, which has positive coefficients and is divisible by $\zeta_F(s)^2 L(s, \pi_f)^3$.

5 Forms on GL(2)

The object of the section is to prove *unconditionally* the absence of Siegel zeros for the standard L-series of a self-conjugate (Hecke eigen-)cusp form π on GL(2) over a number field F.

We will continue to use the adelic language, and the classically minded reader may consult [Ge] to learn how to go back and forth between the two ways of describing automorphic forms. See also the passage following (1.1), where we described the infinity type of cuspidal automorphic representations π of $GL(2, \mathbb{A}_\mathbb{Q})$ associated to holomorphic (Hecke eigen-)cusp forms on the upper half-plane of weight $k \geq 2$ relative to a congruence subgroup.

Because of Remark 4.3, we may assume that π is not associated to a grossencharacter of a quadratic extension. Then by [GJ] we know that its symmetric square $S^2(\pi)$ is cuspidal.

Before beginning the proof, it may be worthwhile to see what happens if we try to verify the hypothesis $H(\pi)$ (of §4). Since we know by [GJ] that the symmetric square of π is cuspidal, we *can* construct $\pi \boxtimes \pi$ as $S^2(\pi) \boxplus 1$. But the problem is that, if we want to appeal to Theorem B, we also need (see Remark 4.7) the automorphy of $\pi \boxtimes S^2(\pi)$, which is essentially the same as asking for the automorphy of the symmetric cube $S^3(\pi)$. This is definitely beyond the reach of current technology. We get around this problem below by showing that it suffices to know the location of poles of $L(s, \pi, S^3)$ in some real interval $(t, 1)$ for an absolute $t < 1$. Doing this, however, involves changing the base field to $F[\sqrt{-3}]$, as the desired facts are known only when F contains the cube roots of unity.

Theorem C. Let π be a cusp form on $GL(2)/F$. Then we have:

(1) There is an effective absolute constant $c > 0$ such that $L(s,\pi)$ has no zero in the interval $(1 - c/\log M, 1)$, where M is as in Definition 2.5.

(2) If $F = \mathbb{Q}$ we have, for any $\epsilon > 0$,

$$L(1,\pi) \gg M^{-\epsilon},$$

where the implied constant depends on ϵ and is effective.

(3) There is an absolute effective constant $c_2 > 0$ such that $L(s,\pi)$ has no zero in the region

$$\left\{ s = \sigma + it \mid \sigma \geq 1 - \frac{c_2}{\log(\max\{1,|t|\} + M)} \right\}. \qquad \square$$

Remark. Recently, W. Luo found a different, and more direct, proof of the lower bound in (2), once given the nonexistence of the Siegel zero. When π is tempered, i.e., when π satisfies the Ramanujan conjecture, and $F = \mathbb{Q}$, his method gives, for any $\epsilon > 0$,

$$L(1,\pi) \geq \frac{c_1(\epsilon)}{(\log M)^{2+\epsilon}}$$

where $c_1(\epsilon)$ is an effective positive constant depending on ϵ. Luo makes use of the full zero-free region of (3) and takes the square root of $L(s,\pi)$ inside this region. He then takes advantage of the fact that the product of this series with $\zeta(s)$ must have positive coefficients.

Proof. We may assume (as above) that $\pi = \pi_\infty \otimes \pi_f$ is self-conjugate and nondihedral. Consider the Dirichlet series

$$D(s) = L(s, (1 \boxplus S^2(\pi_f) \boxplus \pi_f) \times (1 \boxplus S^2(\pi_f) \boxplus \pi_f)). \tag{5.1}$$

This has nonnegative coefficients by Lemma a of §2, and we have, by (1.3),

$$-\operatorname{ord}_{s=1} D(s) = 3. \tag{5.2}$$

Moreover, by Lemma b of §2 it has a conductor bounded between M^{-C} and M^C, for an absolute constant C, where M is as in Definition 2.5. On the other hand, noting that

$$L(s, S^2(\pi_f) \times \pi_f) = L(s, \pi_f, S^3)L(s, \pi_f),$$

we obtain the factorization

$$D(s) = \zeta_F(s)L(s, S^2(\pi_f) \times S^2(\pi_f))L(s, \pi_f \times \pi_f)L(s, S^2(\pi_f))^2 L(s,\pi_f)^4 L(s, \pi_f, S^3)^2 \tag{5.3}$$

where S^j denotes, for any $j \geq 1$, the symmetric jth power representation of $GL(2, \mathbb{C})$. One sees, by the Rankin-Selberg theory and the cuspidality of $S^2(\pi)$, that $\zeta_F(s)L(s, S^2(\pi_f) \times S^2(\pi_f))L(s, \pi \times \pi)L(s, S^2(\pi_f))^2$ has no pole in $(1/2, 1)$. (In fact, it is analytic everywhere outside $s = 1$ where it has a pole of order 3.) When coupled with (5.2) and (5.3), this will rule out, by Lemma c of §2, any Siegel zero for $L(s, \pi)$ if we know that $L(s, \pi_f, S^3)$ also has no pole in an interval $(t, 1)$ for an absolute $t < 1$.

Many of the basic analytic properties of $L(s, \pi_f, S^3)$ have been established by Shahidi [Sh3] and by Bump, Ginzburg, and the first author [BGH]. In particular, we have the following.

Theorem [BGH]. If F contains the cube roots of unity, and the central character of π is trivial at infinity, then $L^S(s, \pi_f, S^3)$ does not have any pole in $(3/4, 1)$, where S is any finite set of places containing those where π_f is ramified. □

The bad Euler factors do not introduce any poles near $s = 1$ either. More precisely, we have the following.

Lemma 5.4. Let F be any number field, v a finite place of F, and π a unitary cuspidal automorphic representation of $GL(2, \mathbb{A}_F)$. Then $L(s, \pi_v, S^3)$ is holomorphic in $\{\Re(s) > 3/4\}$. □

Proof. Suppose π_v is a principal series representation $\mu_1 \boxplus \mu_2$. Write each μ_j as $\nu_j|\cdot|^{s_j}$, with ν_j a possibly ramified (finite-order) character, and s_j a complex number. If the (quasi) characters μ_1, μ_2 are both unramified, i.e., if $\nu_1 = \nu_2 = 1$, then one knows by [GJ] that $\Re(s_j) < 1/4$ for $j = 1, 2$. (Though we do not need it, we note that for π of trivial central character, we have the stronger bound $\Re(s_j) < 1/5$ due to Shahidi. If, moreover, the base field is $\mathbb{Q}$, then one has the bound $\Re(s_j) < 5/28$ by the work of D. Bump, W. Duke, H. Iwaniec, and the first author.) It is easy to see that

$$L(s, \pi_v, S^3) = L(s, |\cdot|^{3s_1})L(s, |\cdot|^{2s_1+s_2})L(s, |\cdot|^{s_1+2s_2})L(s, |\cdot|^{3s_2}).$$

Since for any w, $L(s, |\cdot|^w) = L(s + w, 1)$, the assertion of the lemma follows in this case.

If ν_1 or ν_2 is not trivial, we base change π to an abelian extension K of F with $K_u \simeq F_v(\nu_1, \nu_2)$, for some place u above v. Then $\pi_{K,u}$ is unramified and the resulting estimate of $|\cdot|^{s_j} \circ \mathcal{N}_{K_u/F_v}$ gives the desired result for $L(s, \pi_v, S^3)$.

Now suppose π_v is a special representation $sp(\beta, 2)$ (see §2 for the notation), with β a (necessarily) unitary charater of F_v^*. Then by the local theory, the associated representation σ_v of the modified Weil group $\tilde{W}_{F_v} = W_{F_v} \times SL(2, \mathbb{C})$ is given by $\mu \otimes st$, where st denotes the standard (two-dimensional) representation of $SL(2, \mathbb{C})$. Then the symmetric cube of σ_v is simply associated to $\mu^3 \otimes S^3(st)$ of $\tilde{W}_{F_v}$, which in turn defines the special representation $sp(\beta^3, 4)$ of $GL(4, F_v)$. The assertion of the lemma then follows by [JPSS, equation (6), page 445].

The remaining case to consider is when π_v is supercuspidal. While it is true that $L(s, \pi_v)$ is then 1, it may (and does) happen that $L(s, \pi_v, S^3) \neq 1$. By Kutzko's proof of the local Langlands conjecture for GL(2) [Ku], we can find an irreducible two-dimensional representation σ_v of the Weil group W_{F_v} such that $L(s, \pi_v, r) = L(s, r(\sigma_v))$, for any polynomial representation r of $GL(2, \mathbb{C})$. We will apply this to $r = S^3$. Moreover, it is well known (see [He2] for example) that we can find an irreducible representation τ_v of $Gal(\overline{F}_v/F_v)$ and an unramified (quasi) character χ of $F_v^* \simeq W_{F_v}^{ab}$ such that $\sigma_v \simeq \tau_v \otimes \chi$. Clearly, σ_v factors through a finite Galois extension E of F_v, and the unitarity of π_v then forces χ to be a unitary character. Then we see that

$$L(s, \pi_v, S^3) = \prod_{\mu} L(s, \mu\chi^3),$$

where the product on the right runs over one-dimensional (unramified) subrepresentations μ of $S^3(\tau_v)$, which we identify (by class field theory) with (quasi) characters of F_v^*. On the other hand, since τ_v is in effect a representation of the finite group $Gal(E/F_v)$, we see that each μ must be of finite order. Combined with the unitarity of χ, we get the holomorphy of $L(s, \pi_v, S^3)$ in $\Re(s) \in (0, 1)$. This completes the proof of Lemma 5.4. ■

Recall that we may assume that π is self-conjugate and not induced by an idele-class character, of a quadratic extension. Then it has trivial central character, and so we may apply the theorem above (from [BGH]) if $\sqrt{-3} \in F$. Thus (1) is proved in this case.

For general F, we set $K = F[\sqrt{-3}]$ and consider the base change π_K of π to K. Since π is nondihedral, π_K will still be cuspidal on GL(2)/K. Then $L(s, \pi_K)$ has no Siegel zero. The assertion on $L(s, \pi)$ is then deduced by appealing to the factorization

$$L(s, \pi_K) = L(s, \pi)L(s, \pi \otimes \chi),$$

where χ is the quadratic character of F attached to K. Indeed, since $L(s, \pi \otimes \chi)$ is entire, any zero of $L(s, \pi)$ is also a zero of $L(s, \pi_K)$. Hence the assertion (1).

Part (3) of the theorem follows from the result in [M]. This gave essentially the same zero-free region, except for the possibility of a Siegel zero, which has now been eliminated.

It remains to prove part (2). First we claim that

$$L(s, S^2(\pi_f) \times S^2(\pi_f)) = \zeta_F(s)L(s, \pi_f, S^4)L(s, \pi_f, S^2). \tag{5.5}$$

Indeed, as the left-hand side is a product of $L(s, S^2(\pi_f), S^2)$ and $L(s, S^2(\pi_f), \Lambda^2)$, and since $L(s, S^2(\pi_f), S^2)$ factors as $\zeta_F(s)L(s, \pi_f, S^4)$, it suffices to check that

$$L(s, S^2(\pi_f), \Lambda^2) = L(s, S^2(\pi_f)),$$

i.e., that the central character of $S^2(\pi_f)$ is trivial. But this follows from the fact that π_f has trivial central character. (It is not sufficient to know that $S^2(\pi_f)$ is self-dual, as this only shows that its central character is at most quadratic.)

Combining (5.3) and (5.5), and using the elementary factorization $L(s,\pi_f\times\pi_f) = \zeta_F(s)L(s,S^2(\pi_f))$, we get the identity

$$D(s) = \zeta_F(s)^3 L(s,S^2(\pi_f))^4 L(s,\pi_f)^4 L(s,\pi_f,S^3)^2 L(s,\pi_f,S^4). \tag{5.6}$$

Recall that there is a positive number $M = M(\pi)$ associated to $L(s,\pi)$ (cf. Definition 1.5).

We now need the following.

Lemma 5.7. Let $K = \mathbb{Q}[\sqrt{-3}]$, and let π_K denote the base change of π to $GL(2)/K$. Fix $\epsilon > 0$. Let $L(s)$ denote $L'(s,\pi_{K,f})$ or $L(s,\pi_{K,f},S^j)$, for some $j \le 4$. Then for every real c with $0 \le |c| \le \epsilon$, we have

$$L(1+c) \ll_\epsilon M^\epsilon,$$

where the implied constant is effective and depends only on ϵ. □

Proof. First consider the case when $L(s)$ is $L(s,\pi_{K,f},S^j)$, for some $j \le 4$. Then, if $\sum_{n\ge1} a_n n^{-s}$ is the Dirichlet series representing $L(s)$ in $\{\Re(s) > 1\}$, one has

$$L(1+\epsilon) \ll_\epsilon \left(\sum_{n\ge1}\frac{|a_n|^4}{n^{1+\epsilon}}\right)^2 \ll_\epsilon M^\epsilon.$$

Indeed, the second inequality follows from Lemma 2.1 of [HL], while the first follows by using the factorization $L(s) = L(s,\pi_f,S^j)L(s,\pi_f\otimes\chi,S^j)$, and by bounding each factor at $s = 1+\epsilon$ from above by $\sum |a_n|^4 n^{-1-\epsilon}$. The result for $L(1+c)$ then follows, for $j \le 2$, from the functional equation satisfied by $L(s,\pi_{K,\infty},S^j)L(s,\pi_{K,f},S^j)$, and the Phragmen-Lindelöf principle. This is also the case for $j = 3$, but one must take into account two factors. The first is that, by Shahidi's work [Sh3], we know the functional equation and meromorphic continuation of the complete Langlands L-function. One sees that $L(s,\pi_K,S^3)$ is bounded in vertical strips in a right half-plane, and so by Shahidi's functional equation, also in a left half-plane. Since $L_\infty(s,\pi_K,S^3)$ is a finite product of standard gamma factors, its product with a suitable polynomial (which cancels its poles) is holomorphic and bounded in a vertical strip $\{1-\sigma_0 < \Re(s) < \sigma_0\}$, for large σ_0. Finally, by [Sh3], one also knows that the global L-function $L(s,\pi_K,S^3)$ has only a finite number of poles, and modifying it by multiplication by a an appropriate polynomial (invariant under $s \to 1-s$), we can conclude polynomial growth for this modified function, which allows us to apply Phragmen-Lindelöf. The second factor is that the possibility of poles of $L(s,\pi_{K,f},S^3)$ in $\{\Re(s) \in [1/4,3/4]\}$ can contribute at most another factor of M^ϵ. For $j = 4$, we apply Phragmen-Lindelöf to the function $\zeta_K(s)L(s, \pi_{K,f}, S^4)$, which equals $L(s, S^2(\pi_f), S^2)$; it has at most a simple pole at $s = 1$ by the result of [BG], which establishes it with (the finite set of) bad Euler factors removed, together with Lemma 6.5 below. The result for $L'(1+c, \pi_{K,f})$ follows from that for $L(1+c, \pi_{K,f})$. ■

Now we begin the proof of (2). First consider the case when π is self-dual, and set

$$D(s) = \zeta_K(s)^3 L(s, \pi_{K,f})^4 L(s, \pi_{K,f}, S^2)^4 L(s, \pi_{K,f}, S^3)^2 L(s, \pi_{K,f}, S^4),$$

which defines, as seen earlier, a Dirichlet series $\sum_{n\geq 1} b_n n^{-s}$ with $b_n \geq 0$ and $b_1 = 1$. Pick $\epsilon > 0$. By the lemma above, we have the upper bound

$$L(1+c, \pi_{K,f}, S^2)^4 L(1+c, \pi_{K,f}, S^3)^2 L(1+c, \pi_{K,f}, S^4) \ll_\epsilon M^{7\epsilon},$$

holding for $0 \leq |c| \leq \epsilon$. Choose $\beta < 1$ such that $1-\beta < \epsilon$, and suppose that

$$L(1, \pi_{K,f}) < 1 - \beta.$$

Put $\alpha = 1 - \beta$. Note first that $D(1+\alpha) \geq 1$. Now

$$L(1+\alpha, \pi_{K,f}) = L(1, \pi_{K,f}) + \alpha L'(c, \pi_{K,f}),$$

for some $1 \leq c \leq \alpha$. By the lemma above, we then have

$$L(1+\alpha, \pi_{K,f}) \ll (1-\beta)(1+M^\epsilon) \ll (1-\beta)M^\epsilon,$$

from which it follows that

$$\zeta_K(1+\alpha)^3 L(1+\alpha, \pi_{K,f})^4 = L(1+\alpha, \pi_{K,f}) \left(\zeta_K(1+\alpha)L(1+\alpha, \pi_{K,f})\right)^3 \ll L(1+\alpha, \pi_{K,f})M^{3\epsilon}.$$

Thus we have

$$L(1+\alpha, \pi_{K,f})M^{10\epsilon} \gg_\epsilon D(1+\alpha) \geq 1.$$

But

$$L(1+\alpha, \pi_{K,f})M^{10\epsilon} = L(1, \pi_{K,f})M^{10\epsilon} + O_\epsilon(\alpha M^{11\epsilon}).$$

If $\alpha \ll M^{-11\epsilon}$ with a sufficiently large constant, then the last term is < 1 and we have

$$L(1, \pi_{K,f}) \gg M^{-10\epsilon}.$$

Thus we have a contradiction, as we assumed that $L(1, \pi_{K,f})$ was less than α. It follows then that

$$L(1, \pi_{K,f}) \gg M^{-11\epsilon}.$$

Since $L(1, \pi_{K,f}) = L(1, \pi_f)L(1, \pi_f \otimes \chi)$, and since we have the upper bound M^ϵ for $L(1, \pi_f \otimes \chi)$, part (2) of the theorem follows, after replacing ϵ by $\epsilon/12$.

If π is not self-dual, we consider (as in the proof of Theorem A)

$$D(s) = L(s, (1 \boxplus \pi_f \boxplus \overline{\pi}_f) \times (1 \boxplus \pi_f \boxplus \overline{\pi}_f)),$$

which is, by Remark 2.4, a Dirichlet series with nonnegative coefficents and factors as

$$\zeta(s)^3 L(s, \pi_f, S^2 \otimes \omega^{-1})^2 L(s, \pi_f \times \pi_f)^2 L(s, \overline{\pi}_f \times \overline{\pi}_f)^2.$$

The proof then goes identically, and is in fact easier as the symmetric cube, and fourth power L-functions are not involved. ∎

6 Forms on GL(3)

Let π be a self-dual cusp form on GL(3)/F. The self-duality forces the *square* of the central character ω of π to be trivial. For any quadratic character ν, the central character of $\pi \otimes \nu$ is $\omega\nu$, and this shows that there exist self-dual π with nontrivial central character. In either case, the general method of this note requires us to find a cuspidal automorphic representation $\tau \neq 1$ of GL(m)/F, $m \geq 1$, such that $L(s, \tau)$ divides $L(s, \pi \times \pi)$. It turns out (see below) that there is a canonical choice for τ, namely $\pi \otimes \omega$, with the quotient of $L(s, \pi \times \pi)$ by $L(s, \tau)$ being identifiable with $L(s, \pi, \mathrm{sym}^2)$.

The theorem below generalizes the result of [GHL] on the symmetric square L-series of forms on GL(2). In that paper, a key ingredient was the result of Bump and Ginzburg [BG] on the analyticity of the symmetric square L-series (with the bad Euler factors removed) of an automorphic form on GL(3) at points $s \neq 0, 1$. As will be indicated below, if ω is trivial, the argument of [GHL] essentially goes through for any self-dual π on GL(3), not just a symmetric square lift, to prove that $L(s, \pi)$ does not have a Siegel zero. If ω is quadratic, a bit more than the result of [BG] is needed. In particular we require the following.

Hypothesis 6.1. Let π be a cusp form on GL(3)/F, ω a quadratic character, and S the (finite) set of places containing the archimedean and ramified places. Then $L^S(s, \pi_f, S^2 \otimes \omega)$ has no pole in $\{\Re(s) \in (t, 1)\}$, for some absolute positive $t < 1$. □

In fact, the local theory necessary to prove Hypothesis 6.1 was worked out by W. Banks in his dissertation [B]. The global theory, however, has not yet been completed. Our result is the following.

Theorem D. Let π be a cusp form on GL(3)/F. If π is not self-dual with quadratic central character, then $L(s, \pi)$ has no Siegel zeros. If π is self-dual with quadratic central character, the result is true upon the assumption of Hypothesis 6.1. □

Remark. One can also establish, as in the case of GL(2), effective lower bounds for $L(1,\pi)$ and a zero-free region for $L(s,\pi)$.

Proof. By Corollary 3.2, we may assume that π is self-dual, in which case, as mentioned above, the central character ω must be trivial or quadratic. Suppose ω is nontrivial. Consider the Dirichlet series

$$D(s) = L(s, (1 \boxplus \pi_f \boxplus (\pi_f \otimes \omega)) \times (1 \boxplus \pi_f \boxplus (\pi_f \otimes \omega))),$$

which is nice by Remark 1.4, has nonnegative coefficients by Lemma a, and (by Lemma b) has its level bounded between $M(\pi)^{-C}$ and $M(\pi)^{C}$, for an absolute constant C. Clearly,

$$-\operatorname{ord}_{s=1} D(s) = 3. \tag{6.2}$$

Let v be a finite place where π is unramified. Then the self-duality and unitarity of π implies that

$$A_v(\pi, \Lambda^2) \otimes \omega = A_v(\pi),$$

where Λ^2 denotes the exterior square representation of $GL(3,\mathbb{C})$. So, if $S = S_f \cup S_\infty$ denotes the finite set of ramified and archimedean places, we have

$$L(s,\pi,\Lambda^2 \otimes \omega) = Z_S(s) L(s,\pi), \tag{6.3}$$

where

$$Z_S(s) = \prod_{v \in S} \frac{L(s,\pi_v,\Lambda^2\omega)}{L(s,\pi_v)}.$$

Using (6.3) and expanding the right-hand side of (6.2), incorporating the decomposition of the tensor square of the standard representation into a direct sum of the symmetric and exterior square representations, we get the following identity:

$$D(s) = \zeta_F(s) L(s, \pi_f \times \pi_f)^2 L(s, \pi_f \otimes \omega)^2 L(s, \pi_f, S^2 \otimes \omega) Z_{S_f}(s)^2 L(s,\pi_f)^4. \tag{6.4}$$

Since π is generic, each of the local factors at $v \in S_f$ is invertible in $\{\Re(s) > 1/2\}$. At this point we appeal to Hypothesis 6.1 to eliminate the possibility that $L^S(s,\pi_f,S^2 \otimes \omega)$ has a pole in $\{\Re(s) \in (t,1)\}$. It turns out the bad Euler factors do not contribute any pole either near $s = 1$. More precisely, we have the following.

Lemma 6.5. For any finite place v of F, the local factor $L(s,\pi_v,S^2 \otimes \omega)$ is holomorphic in $\{1 - 2/(3[F:\mathbb{Q}]+1) + \epsilon < \Re(s) < 1\}$, for any $\epsilon > 0$. □

Proof. Replacing π by $\pi \otimes \omega$, we may assume that π has trivial central character ω. Suppose π_v is a principal series representation. Then the unitarity, together with self-duality, of π implies that the three (quasi) characters defining π_v must be of the form $\mu|\cdot|^t, 1, \mu|\cdot|^{-t}$, for a character μ of order 1 or 2 and some real number $t \geq 0$. Then the vth coefficient of the (positive) Dirichlet series defined by $L(s, \pi_f \times \pi_f)$ is bounded, by a version of Landau's lemma (cf. [CN]), by $Nv^{(1-2/(3[F:\mathbb{Q}]+1)+\epsilon)}$, for any $\epsilon > 0$. Since $L(s, \pi_f \times \pi_f) = L(s, \pi_f, S^2)L(s, \pi_f)$, we see that $2t$ must be smaller than $1 - 2/(3[F : \mathbb{Q}] + 1)) + \epsilon$, whence the lemma (in this case).

Suppose π_v is a special representation $sp(\mu, 3)$. Then μ is unitary, and moreover, the associated representation of the extended Weil group $\tilde{W}_{F_v} = W_{F_v} \times SL(2, \mathbb{C})$ is given by $\mu \otimes S^2(st)$. It is easy to see that $L(s, \pi_v, S^2) = L(s, \mu^2 \otimes S^4(st))L(s, 1)$, which has no pole in $\Re(s) \in (0, 1)$.

Finally, suppose π_v is a supercuspidal representation. It is then attached by the local Langlands correspondence for GL(3) established by Henniart [He3], to an irreducible representation σ_v of W_{F_v}. The conclusion now follows by arguing as in the proof of Lemma 5.4. In fact, one sees that $L(s, \pi_v, S^2)$ is holomorphic in $\{\Re(s) \in (0, 1)\}$ in this case. ∎

Consequently, if $L(s, \pi)$ has a Siegel zero, then $D(s)$ will have a zero of order 4 near 1. This leads to the desired contradiction in view of Lemma c and (6.2).

If $\omega = 1$, one uses the Dirichlet series

$$D(s) = L(s, (1 \boxplus \pi_f) \times (1 \boxplus \pi_f)).$$

The proof is easier in this case. In fact, once one observes that $A_v(\pi) = A_v(\pi, \Lambda^2)$ at unramified v, and one takes note of Lemma 6.5, it is identical to the argument given in [GHL] for the symmetric square of a GL(2) cusp form. ∎

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