ON THE GLOBAL ROOT NUMBERS OF GL(n)$\times$ GL(m)

DIPENDRA PRASAD AND DINAKAR RAMAKRISHNAN

To Professor G. Shimura

1. Introduction

Let F be a number field, $n, m \geq 1$, and $\pi = \otimes_v' \pi_v$, $\pi' = \otimes_v' \pi_v'$ cuspidal, unitary automorphic representations of $\mathrm{GL}(n, \mathbb{A}_F)$, $\mathrm{GL}(m, \mathbb{A}_F)$ respectively. To this data is associated an Euler product

$$L(s, \pi \times \pi') \, = \, \prod_v L(s, \pi_v \times \pi_v'),$$

which converges absolutely in $\Re(s) > 1$. One knows by Jacquet, Piatetski-Shapiro and Shalika ([JPSS]), and Shahidi ([Sh1]), that this L-function extends to a meromorphic function on all of $\mathbb{C}$ and admits a functional equation

$$L(s, \pi \times \pi') \, = \, W(\pi \times \pi')(N(\pi \times \pi')d_F^n)^{\frac{1}{2}-s} L(1-s, \pi^\vee \times \pi'^\vee).$$

Here $\pi^\vee$ (resp. $\pi'^\vee$) denotes the contragredient of π (resp. π'), d_F the discriminant of F, and $N(\pi \times \pi')$ a positive rational number ("the conductor"). The arithmetically important **global root number** $W(\pi \times \pi')$ is non-zero and satisfies $W(\pi \times \pi')W(\pi^\vee \times \pi'^\vee) \, = \, 1$. We will be particularly interested in the **self-dual** situation, when

$$W(\pi \times \pi') \, = \, \pm 1.$$

It is of importance to determine the sign. Our object here is to make some modest progress on this question.

We will say that a cuspidal automorphic representation η of $\mathrm{GL}(r, \mathbb{A}_F)$ is of **symplectic type** if, for a finite set S of places, the (incomplete) exterior square L-function $L^S(s, \eta, \Lambda^2)$ (see section 3 below for a definition) admits a pole at $s = 1$. One knows that this cannot happen if r is odd ([JS1]).

Conjecture I. *Suppose n, m are even, and π, π' are both cusp forms of symplectic type. Then $W(\pi \times \pi') \, = \, 1$.*

As positive evidence, we will prove (in section 5) the following

Theorem A *Let π, π' be as in the conjecture with $n, m \leq 4$. If n or m is 4, assume that the conductor of the representation is prime to 2. Then*

$$W(\pi \times \pi') \, = \, 1.$$

In fact, our proof will show a bit more. If n or m is 4, it suffices to assume that, at every place v dividing 2, the base change of the local component at v to some quadratic extension of F_v is not supercuspidal.

If π (resp. π') corresponds to a Galois representation σ (resp. σ') of dimension $2n$ (resp. $2m$) of Artin type, then π (resp. π') being symplectic is equivalent to the image of σ (resp. σ') being contained in $\mathrm{Sp}(2n, \mathbb{C})$ (resp. $\mathrm{Sp}(2m, \mathbb{C})$). So $\sigma \otimes \sigma'$ would then be orthogonal, and one can deduce the fact that $W(\sigma \otimes \sigma') = 1$ by appealing to a well known result due to Fröhlich-Queyrut [Fr-Q] and Deligne [De1]. The problem in attempting such an argument in the automorphic setup is that there is **no global group $\mathfrak{L}_F$** available (as of yet) whose n-dimensional representations parametrize automorphic representations of $\mathrm{GL}(n, \mathbb{A}_F)$. In (the motivational) section 2, we indicate how to get the result on $W(\sigma \otimes \sigma')$ by using only a local result of Deligne relating root numbers to Stiefel-Whitney classes. (Recently, we have come to know that this local argument has already been found by D. Rohrlich in [Ro], section 1, Prop. 2.) Putting this in a general framework leads to Conjecture II.

After some preliminaries in section 3, we discuss the status of the local Langlands conjecture for GL(4) in section 4. In particular we show that the local correspondence works (see Prop. 4.1) for the class of representations of GL(4) satisfying the ramification condition at primes above 2 which was alluded to above. (In the odd residual characteristic case, the correspondence for GL(4) is a consequence of the recent works of M. Harris [Ha] and Jeff Chen [Ch].)

The heart of this paper is in section 5, where we study of $W(\pi \times \pi')$ by some global arguments for automorphic forms on $\mathrm{GL}(4)/F$ and lifting to $\mathrm{GSp}(4)/F$. The key problem (in proving Theorem A) becomes one of showing that, locally at the ramified places, the representations σ_v of the Weil-Deligne group defined by a global π of symplectic type have images in the symplectic group (see Theorem 5.1 and Propositions 5.1 - 5.3). Another probem is to know that at such a place, $W(\pi_v \times \pi'_v) = W(\sigma_v \otimes \sigma'_v)$, which is addressed earlier in section 4 (cf. Proposition 4.2).

When $n = 2$, it is easy to see that π is of symplectic type iff its central character is trivial. Thus Theorem A proves in particular the triviality of the root number $W(\pi \times \pi')$ associated to a pair of cuspidal automorphic representations of π, π' $\mathrm{PGL}(2, \mathbb{A}_F)$. However, the difficult part of the Theorem deals with GL(4)×GL(2) and GL(4)×GL(4).

Given any cuspidal automorphic representation π of $\mathrm{GL}(2, \mathbb{A}_F)$, it is natural to look at the symmetric power L-functions $\{L(s, \pi, \mathrm{Sym}^r) | r \geq 1\}$ (see section 4). These were first defined by Langlands ([La1]) at almost all places. Thanks to the work of Shahidi ([Sh 1,2], one knows, for each $r \leq 5$, how to extend $L(s, \pi, \mathrm{Sym}^{\mathrm{r}})$ to a meromorphic function satisfying a functional equation, with a good definition of bad factors as well. For $r = 2$, one knows much more by the work of Gelbart and Jacquet ([GJ]), namely that $L(s, \pi, \mathrm{Sym}^2)$ is the standard L-function of an automorphic form $\mathrm{Sym}^2(\pi)$ on $\mathrm{GL}(3)/F$. For $r = 3$, one also knows, by the work of Bump, Ginzburg and Hoffstein, that the (symmetric cube) L-function is holomorphic in $\{\Re(s) \geq 3/4\}$; but we will not have occasion to use this.

Using [Sh1, 2], we can therefore define the global root number $W(\pi, \mathrm{Sym}^r)$, for each $r \leq 5$. In section 6, we will prove

Theorem B *Let π be a cuspidal automorphic representation of $GL(2, \mathbb{A}_F)$ of trivial central character. Then*

$$W(\pi, \mathrm{Sym}^2) \,=\, W(\pi, \mathrm{Sym}^4) \,=\, 1.$$

It should be noted that if π corresponds to a compatible system $\{\sigma_\ell\}$ of ℓ-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$, then $\mathrm{Sym}^{2k}(\sigma_\ell)$ will be an orthogonal similitude representation of even (motivic) weight, and the assertion that the global root number is 1 already follows from the work of T. Saito ([Sa]), who generalized the results of Fröhlich and Deligne ([De1]) on Galois representations of Artin type, and also the result of Coates and Schmidt ([Co-S]) on the symmetric square of a modular elliptic curve. But our result appears to be new for Maass forms over $\mathbb{Q}$ and for forms of arithmetic type over number fields with a complex place.

Professor Shimura established a number of important special value results for L-series of pairs (π, π') of arithmetic type. He also was the first one to prove the holomorphy of the symmetric square L-function associated to elliptic modular newforms. Here we study problems of a different sort, but for the same types of L-series, and we are honoured to dedicate this paper to him.

We would like to thank F. Shahidi for helpful discussions concerning his results on L-functions, and for his help with Lemma 5 of section 5. We would also like to thank M. Harris, H. Jacquet, D. Rohrlich and J.-P. Serre for useful conversations/ correspondence. The first author would like to thank Caltech for an invitation to visit during February 97, when this work was begun. The second author would like to thank the NSF for support through the grant DMS-9501151.

2. Motivation and generalization

Suppose σ (resp. σ') is a continuous, irreducible $\mathbb{C}$-representation of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ of dimension n (resp. m). Then the analogue of $W(\pi \times \pi')$ is the **global Artin root number** $W(\sigma \otimes \sigma')$. One says that σ is symplectic if the image is in fact contained in $\mathrm{Sp}(n, \mathbb{C})$, or in other words, if the exterior square of σ contains the trivial representation. This is known to be equivalent to the existence of a pole at $s = 1$ of the Artin L-function $L(s, \Lambda^2(\sigma))$. Indeed, this is so because $L(s, \tau)$ is invertible at $s = 1$ for any non-trivial irreducible τ. (This can be seen, for example, by Prop. 3.4 of [Ta] together with the functional equation.) Clearly, every symplectic σ is self-dual and of even dimension. Similarly for σ'.

Though we will stick to Galois representations of Artin type below, much of what we say will also be valid for compatible systems. However, it should be noted that, for an irreducible ℓ-adic representation σ_ℓ, the equivalence between being symplectic and having the exterior square L-function admit a pole is not known, though predicted by the Tate conjectures.

One knows by a theorem of Langlands (see [De2] for an elegant global proof) that there is a factorization

$$W(\sigma \otimes \sigma') \,=\, \prod_v W_v(\sigma \otimes \sigma'),$$

where each local constant $W_v(\sigma \otimes \sigma')$ depends only on the restriction $\sigma_v \otimes \sigma'_v$ to the decomposition group D_v at v.

Now fix a place v, and suppose τ is a virtual **orthogonal** representation of $\mathrm{Gal}(\overline{F}_v/F_v)$ of determinant 1 and dimension 0. Then, by a fundamental theorem of Deligne ([De1]), one has the equality

$$W_v(\tau) = e^{\pi i w_2(\tau)}, \tag{2.1}$$

where $w_2(\tau) \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ is the image of the second Stiefel-Whitney class of τ under the isomorphism $H^2(F_v, \{\pm 1\}) = {}_2\mathrm{Br}(F_v) \simeq \frac{1}{2}\mathbb{Z}/\mathbb{Z}$.

The motivation for the conjecture made in the introduction arises from the following

Proposition 2.1. *Let σ, σ' be both symplectic of respective dimensions $2n, 2m$. Then*

$$W(\sigma \otimes \sigma') = 1.$$

When σ, σ' are symplectic, their tensor product is clearly orthogonal, and so this Proposition is an immediate consequence of a theorem of Fröhlich-Queyrut ([Fr-Q]) and Deligne ([De1]). But in the automorphic context, there is (as of yet) no analog of the global Galois group, and a starting point for this paper is supplied by the following

Alternate Proof. Let v be any place. Since σ_v and σ'_v are both symplectic, their tensor product must be orthogonal and of determinant 1. Put

$$\tau = (\sigma_v \otimes \sigma'_v) \ominus 4nm[1],$$

where [1] denotes the trivial representation. Then τ is of dimension 0, determinat 1, and of orthogonal type, and so by Deligne, $W_v(\tau) = e^{\pi i w_2(\tau)}$. Note that $W_v([1])^4 = 1 = e^{\pi i w_2(4[1])}$. For any $N \geq 1$, let $\mathrm{Spin}(N, \mathbb{C})$ denote the two-fold covering group of $\mathrm{SO}(N, \mathbb{C})$. Clearly, $\sigma_v \otimes \sigma'_v$ lies in $\mathrm{SO}(4nm, \mathbb{C})$. It follows then, by the definition and properties of Stiefel-Whitney classes (cf. [De1]), that $W_v(\sigma \otimes \sigma')$ can be -1 iff $\sigma_v \otimes \sigma'_v$ cannot be lifted to a representation into $\mathrm{Spin}(4nm, \mathbb{C})$. But this representation factors through $\mathrm{Sp}(2n, \mathbb{C}) \times \mathrm{Sp}(2m, \mathbb{C})$, which is *simply connected* as an algebraic group. Now we appeal to the following basic Lemma, whose proof is left to the reader.

Lemma 1. *Let $\phi : H \to H'$ be a morphism of semisimple algebraic groups over $\mathbb{C}$, and let H be connected and simply connected as an algebraic group. Let $\tilde{H}'$ denote the universal cover of H' (in the sense of algebraic groups), with covering map $p : \tilde{H}' \to H'$. Then there exists a lifting $\tilde{\phi} : H \to \tilde{H}'$ such that $\phi = p \circ \tilde{\phi}$.*

We apply this with $H = \mathrm{Sp}(2n, \mathbb{C}) \times \mathrm{Sp}(2m, \mathbb{C})$ and $H' = \mathrm{SO}(4nm, \mathbb{C})$, and note that $\tilde{H}'$ is then none other than $\mathrm{Spin}(4nm, \mathbb{C})$. Consequently, $w_2(\sigma_v \otimes \sigma'_v)$ is 1. So is $W_v(\sigma_v \otimes \sigma'_v)$ by (2.1). Now we are done as *every* local factor of $W(\sigma \otimes \sigma')$ is 1. □

Remark: We have recently learnt from D. Rohrlich that this argument above has already been found by him, see [Ro], section 1, Prop. 2 and the following remark.

More generally, let σ be a Galois representation with values in the $\mathbb{C}$-points of a simply connected semisimple group $\hat{G}$, and let

$$r : \hat{G} \to \mathrm{GL}(N, \mathbb{C}),$$

be an algebraic representation whose image is contained in SO($N,\mathbb{C}$). Then the argument proves that $W_v(r(\sigma)) = 1$, for every v.

This suggests a *general conjecture* about the root numbers of automorphic L-functions associated to a split semisimple group G over F when the (Langlands) dual group $\hat{G}$ is simply connected. The reason is this. The automorphic representations of $G(\mathbb{A})$ are expected, modulo some anomalous ones, to be parametrized by the conjugacy classes of homomorphisms ϕ into $\hat{G}$ from a conjectural, pro-reductive group $\mathfrak{L}_F$, which should be an extension of $\mathrm{Gal}(\overline{F}/F)$ by its connected component. (See [La3] for a discussion of anomaly.) The cuspidal ones among them should correspond to those (classes of) ϕ with $\mathrm{im}(\phi)$ not contained in any Levi subgroup.

Conjecture II. *Let G be a split semisimple group over F with a simply connected (Langlands) dual group $\hat{G}$, and let π be a non-anomalous, cuspidal automorphic representation of $G(\mathbb{A}_F)$. Suppose r is an algebraic representation of $\hat{G}$ which is orthogonal. Then one should have $W_v(\pi, r) = 1$ for every v.*

Whether or not $\hat{G}$ is simply connected, the global root number of π relative to an orthogonal representation r of $\hat{G}$ is expected to be 1; but it will be very subtle to establish, requiring a product formula, as the local root numbers need not be 1.

When $G = \mathrm{PGL}(n, \mathbb{A}_F)$, $\hat{G}$ is $\mathrm{SL}(n,\mathbb{C})$, and every cuspidal automorphic representation of $G(\mathbb{A}_F)$ is non-anomalous, called *isobaric* in this context. When $G = \mathrm{SO}(2n+1)\times \mathrm{SO}(2m+1)$, $\hat{G}$ is $\mathrm{Sp}(2n,\mathbb{C})\times \mathrm{Sp}(2m,\mathbb{C})$, and if r is the tensor product representation of $\hat{G}$ into $\mathrm{GL}(4nm,\mathbb{C})$, then $\mathrm{im}(r)$ lies in $\mathrm{SO}(4nm,\mathbb{C})$, and the Conjecture II applies, giving a strengthening of Conjecture I stated in the introduction. The connection is understood via the functoriality principle, which predicts that isobaric automorphic forms of symplectic type on $\mathrm{GL}(2n)/F$ correspond to packets of non-anomalous ones on $\mathrm{SO}(2n+1)/F$.

3. Preliminaries

Given any unitary, cuspidal automorphic representation π of $\mathrm{GL}(n,\mathbb{A}_F)$, one has, at every finite place v where π_v is unramified, a (Langlands) conjugacy class $A_v(\pi)$ in $\mathrm{GL}(n,\mathbb{C})$, represented by a diagonal matrix $[\alpha_{1,v},\dots,\alpha_{n,v}]$. Let S be any finite set of places containing the archimedean and ramified places (for π). If r is any algebraic representation of $\mathrm{GL}(n,\mathbb{C})$, we put

$$L^S(s,\pi,r) = \prod_{v\notin S} L(s,\pi,r),$$

where

$$L(s,\pi,r) = \det(I-(Nv)^{-s}r(A_v(\pi)))^{-1}.$$

Of particular interest is when r is the exterior square Λ^2, or the symmetric kth power Sym^k, for some $k\geq 1$, of the standard represenation.

One knows (cf. [JS1], [BF], [Sh1]) that $L^S(s,\pi,\Lambda^2)$ converges absolutely in $\Re(s)>1$ and admits a meromorphic continuation with a functional equation of the usual type relating s and $1-s$.

Definition. *π is of symplectic type iff $L^S(s,\pi,\Lambda^2)$ has a pole at $s=1$.*

Here is a simple Lemma which will be needed later.

Lemma 2. *If π is of symplectic type, then it is self-dual.*

Proof. We have the factorization

$$L^S(s, \pi \times \pi) \,=\, L^S(s, \pi, \Lambda^2) L^S(s, \pi, \mathrm{Sym}^2).$$

One knows by Shahidi ([Sh1]) that $L^S(s, \pi, \mathrm{Sym}^2)$ has no zero at $s = 1$. Thus, if π is of symplectic type, the pole at $s = 1$ of $L^S(s, \pi, \Lambda^2)$ will introduce a pole of $L^S(s, \pi \times \pi)$ at $s = 1$. This implies, by a theorem of Jacquet and Shalika ([JS2]), that π is equivalent to $\pi^\vee$.

□

Thus, for every place v, the local component π_v of a cusp form π on $\mathrm{GL}(n)/F$ of symplectic type is self-dual. It is not clear, however, that the conjecturally associated representation (by the local Langlands conjecture)

$$\sigma_v : W_{F_v'} \to \mathrm{GL}(n, \mathbb{C})$$

should be symplectic. (One could equally well use the modified Weil group $\tilde{W}_{F_v} = W_{F_v} \times \mathrm{SL}(2, \mathbb{C})$; in this setting, the indecomposable module $\mathrm{sp}(n)$ corresponds to $1 \otimes \mathrm{Sym}^{n-1}(st)$, where st denotes the standard representation of $\mathrm{SL}(2, \mathbb{C})$.) This leads to the following

Conjecture III. *Let $n \geq 1$, and let π be a cuspidal automorphic representation of $GL(2n, \mathbb{A}_F)$ of symplectic type. Let v be a place such that π_v is functorially associated to a $2n$-dimensional representation σ_v of W'_{F_v}. Then σ_v is symplectic, i.e., the image of σ_v lies in $Sp(2n, \mathbb{C})$.*

If $n = 2$, it is easy to see that this conjecture is true. Indeed, as remarked earlier, a cusp form on $\mathrm{GL}(2)/F$ is of symplectic type iff its central chacter ω is trivial. Now if v is a place, then the determinant of σ_v is equal to ω_v, which is trivial. Since σ_v is two-dimensional, it must then be symplectic. In the next section, on the way to proving Theorem A, we will prove this conjecture for cusp forms (of symplectic type) on $\mathrm{GL}(4)/F$ satisfying a condition above 2.

We end this section discussing **base change and automorphic induction** for $\mathrm{GL}(n)/F$. We will rely on the results of Arthur and Clozel in [AC]. We will use the modern terminology of *isobaric representations* together with the "sum operation" $\boxplus$ ([La3], [JS2]). These representations are suitable subquotients of parabolically induced representations from essentially unitary cuspidal (resp. essentially square-integrable) representations when F is global (resp. local). For every $n \geq 1$, denote by $\mathrm{Isob}(n, F)$ (resp. $\mathrm{Isob}(n, F_v)$) the set of irreducible, isobaric automorphic (resp. admissible) representations of $\mathrm{GL}(n, \mathbb{A}_F)$ (resp. $\mathrm{GL}(n, F_v)$). Then in [AC], chapter 3 (sec. 3 - 6), one finds a construction, for any cyclic extension K/F with $[K : F]$ a prime ℓ, of maps

$$b_{K/F} : \mathrm{Isob}(n, F) \to \mathrm{Isob}(n, K),\, \pi \to \pi_K \quad (\textit{base change})$$

and

$$I_{K/F} : \mathrm{Isob}(n, K) \to \mathrm{Isob}(n\ell, F),\, \pi \to I(\pi) \quad (\textit{automorphic induction}),$$

such that at every place v of F which is finite and unramified for the representations and K/F (or archimedean) and a place w of K above v, we have (respectively)

$$\mathrm{res}^{F_v}_{K_w}(\sigma(\pi_v)) \,\simeq\, \sigma((\pi_K)_w), \tag{3.1}$$

and
$$\sigma(I(\pi)_v) \simeq \mathrm{ind}_{K_w}^{F_v}(\pi_w).$$
Given any local field E, if β an unramified representation of GL(r, E) (or if E is archimedean), we write $\sigma(\beta)$ to signify the associated r-dimensional representation of W'_E (or W_E).

There are also local analogs of these maps, customarily called "local base change" and "local automorphic induction". The former is constructed and discussed in great detail in chapter 1 of [AC]. The later is discussed briefly in [C] and the relevant assertions are consequences of the results of [AC]. An alternate construction of the local automorphic induction for essentially tempered representations, which works also in characteristic p, is given in the paper [HH] of Henniart and Herb. The properties we will need are summarized below:

Proposition 3.1. *Let K/F be a cyclic extension of number fields or local fields of degree ℓ, a prime. Let θ be a generator of Gal(K/F), and let χ denote the character of the idele class group (resp. multiplicative group) of F in the global (resp. local) case associated to K. Then*

1. *The image of $b_{K/F}$ consists precisely of those $\beta \in$Isob(n, K) such that $\beta \simeq \beta \circ \theta$.*
2. *The image of $I_{K/F}$ consists precisely of those $\pi \in$Isob$(n\ell, F)$ such that $\pi \simeq \pi \otimes \chi$.*
3. *For every $\pi \in$Isob(m, F) and $\beta \in$Isob(n, K), we have the* **adjointness** *property:*
$$L(s, \pi \times I(\beta)) = L(s, \pi_K \times \beta),$$
and
$$\varepsilon(s, \pi \times I(\beta))\varepsilon(s, 1_K)^{nm} = \varepsilon(s, \pi_K \times \beta) \prod_{j=0}^{\ell-1} \varepsilon(s, \chi^j)^{nm}.$$
4. *Suppose β is cuspidal (resp. supercuspidal) in Isob(n, K), for K global (resp. local). Then*
$$I(\beta)_K \simeq \boxplus_{j=0}^{\ell-1} \beta \circ \theta^j.$$
Moreover, $I(\beta)$ is cuspidal (resp. supercuspidal) iff β is **not** *isomorphic to $\beta \circ \theta$.*
5. *Suppose π is cuspidal (resp. supercuspidal) in Isob(n, F), for F global (resp. local). Then*
$$I(\pi_K) \simeq \boxplus_{j=0}^{\ell-1} \pi \otimes \chi^j.$$
Moreover, π_K is cuspidal (resp. supercuspidal) iff π is not isomorphic to $\pi \otimes \chi$.

For a proof of 1., 2. and the second half of 5., see [AC], [C] and [HH]. There one also finds formulae for the L- and ε-factors of pairs (π_K, π'_K) and $(I(\beta'), I(\beta))$, which do not quite imply the identities of 3. In any case, one gets the adjointness formulae, as it is well known to experts, by a similar global argument, which we will briefly indicate here for completeness. First let us consider the global case. By using the identities (3.1), one sees easily that for almost all v (including the unramified and archimedean ones), the local factors at v of the functions $L(s, \pi \times I(\beta))$ and $L(s, \pi_K \times \beta)$, both viewed as Euler products over F, coincide. Such a relationship holds with π replaced by $\pi \otimes \mu$, for any character μ. Now fix a ramified place v_0. We may choose a finite order character μ which is 1 at v_0 and is sufficiently ramified at all the other ramified places u to ensure that the L-factors on both sides (and their contragredient factors) are 1 at each u. (This is easily done by using the formula in [JS2] for the L-factors for $\mathrm{GL}(m)\times\mathrm{GL}(n\ell)$.) Consequently, one gets, by comparing the functional equations of both global L-functions, the desired identity of the local factors at v_0. The L-function identity of 3. follows in the global case. To get this locally, one reduces the problem, by inductivity, to the case when the representations are supercuspidal, which can then be realized as the local components of cuspidal automorphic representations by using the trace formula. The local identity then follows from the global one. (This argument is very similar to the proof of Proposition 6.9 in chapter 1 of [AC].) For epsilon factors, which are non-trivial only at a finite set S of places, we exploit the following trick used in [He2]. (We may assume that S consists only of finite places as the archimedean comparison poses no problem.) By a theorem of Jacquet and Shalika ([JS3]), if μ is sufficiently ramified at a place v, then $\epsilon(s, \pi_v \times \pi'_v \otimes \mu_v)$ has a simple expression, depending only on the central characters of π, π' and the twisting character. Now pick any place v_0 in S, and pick μ to be 1 at v_0, but sufficiently ramified at every other place u in S. Then, by comparing the functional equations of the μ-twisted L-functions in question, we can isolate the epsilon factors at v_0 and deduce the assertion. (It should be noted that the extraneous looking factors appear for the epsilon identity as these factors are inductive only in degree zero.)

Now we prove part 4. In the global case, once again, the L-functions of both sides of the identity agree at all the unramified (and archimedean) places. Twisting by a suitable character and comparing functional equations, we get the assertion at every place. (Locally, we employ the same trick as above.) Suppose first that β is not isomorphic to $\beta \circ \theta$. Let π be a cuspidal element of $\mathrm{Isob}(m, F)$ which occurs in the isobaric sum decomposition ([La2]) of $I(\beta)$. Then $L(s, \pi^\vee \times I(\beta))$, and hence $L(s, \pi_K^\vee \times \beta)$ by the adjointness formula, admits a pole at $s = 1$ by [JS2]. (In the local case, one exploits the pole at $s = 0$.) This means that π_K contains β in its isobaric sum decomposition. But π_K is invariant under θ by part 1. So it must contain $\beta \circ \theta^j$ for all j, and so $I(\beta)_K$. Since π occurs in $I(\beta)$, we must have $\pi \simeq I(\beta)$, and so $I(\beta)$ is cuspidal. Conversely, suppose $I(\beta)$ is cuspidal. If we had $\beta \simeq \beta \circ \theta$, then by part 1., we can find some $\eta \in\mathrm{Isob}(n, F)$ such that $\beta \simeq \eta_K$. Now we interject and observe that the identity of part 5. can be established by the same idea used for the identity of part 4. Consequently,

$$I(\beta) \simeq I(\eta_K) \simeq \boxplus_{j=1}^{\ell-1} \eta \otimes \chi^j,$$

which contradicts the cuspidality of $I(\beta)$. The local argument is the same.

Thus the assertions of the Proposition hold.

4. On the local correspondence for GL(4)

We begin with a technical definition restricting the amount of ramification, to be used at places above 2. This is necessitated by the lack of knowledge of the local Langlands conjecture for GL(4).

Definition *Let E be a non-archimedean local field. An irreducible, admissible representation η of $GL(4, E)$ is* **allowable** *iff the following condition is satisfied:*

($R(E)$) *There exists a quadratic extension K of E such that the base change η_K of η to K is not supercuspidal.*

Proposition 4.1. *Let E be a non-archimedean local field of characteristic zero. Then the local Langlands conjecture holds for $GL(4)$ if the residual characetristic is odd. In the even residual characteristic case, it holds for the class of irreducible admissible representations β of $GL(4, E)$ satisfying the condition ($R(E)$). To be explicit, this class is in bijection, preserving local factors, with the class of representations τ of W'_E such that the restriction of τ to W'_K, for some quadratic extension K of E, is not irreducible.*

Proof. Let p denote the residual characteristic of E. By using inductivity and the proof of the local Langlands conjecture for GL(2) and GL(3) due respectively to Kutzko ([Ku]) and Henniart ([He1]), we may reduce to proving the bijection between supercuspidals of $\mathrm{GL}(4, E)$ and irreducibles of W_E of dimension 4, satisfying the hypothesis for $p = 2$. A recent theorem of Michael Harris (Ha]) furnishes, for any $n \geq 1$, a family of bijections $\phi_m : \beta \to \tau$ between supercuspidals β of $\mathrm{GL}(m, E)$ and irreducibles τ of W_E of dimension m, for *every* $m \leq n$, compatible with taking contragredients, such that, *if* m, n *are prime to* p, the following identity of epsilon factors holds, for all pairs (β, β') of supercuspidals of $\mathrm{GL}(n, E)$ and $\mathrm{GL}(m, E)$ respectively:

$$\varepsilon(s, \beta \times \beta') = \varepsilon(s, \tau \otimes \tau'). \tag{4.1}$$

From this, using a criterion of Henniart ([He3]), he concludes the local Langlands conjecture for $n < p$, which of course gives the Proposition for GL(4) if $p \geq 5$. Another recent result, due to Jeff Chen ([Ch]), gives a finer criterion than that of Henniart for $n = 4$, and says that it suffices, in order to verify that a given bijection is the right one, to check the equality of epsilon factors of pairs for $(n, m) = (4, 1)$ and $(4, 2)$. So for any odd p, we have the needed identity thanks to Harris (namely (4.1)), and so the local Langlands conjecture follows for GL(4) for *any* odd p.

Now we begin the proof for $p = 2$. Since β satisfies ($R(E)$), there exists a quadratic extension K/E such that the base change β_K is not supercuspidal. Appealing to properties of base change ([AC]), more precisely Proposition 3.1 of this article, we then see that β must be a local automorphic induction $I_{K/E}(\lambda)$, for a supercuspidal λ of $\mathrm{GL}(2, K)$. Let μ be the associated irreducible, two-dimensional representation of W_K, given by [Ku]. Since β is cuspidal, λ must be inequivalent to $\lambda \circ \rho$, where ρ denotes the non-trivial automorphism of K over E (see part 4. of the same Prop. 3.1). This implies that μ is not isomorphic to the representation $\mu^{[\rho]}$, defined by $w \to \mu(\rho w \rho^{-1})$ ($\forall w \in W_K$). Let τ be the representation of W_E induced by μ. Then it must be irreducible by Mackey theory. Having defined $\beta \to \tau$, we must check that it is well defined, i.e., independent of K, and has the right functorial properties. For this, it suffices, by Chen ([Ch]), to show that

$$\varepsilon(s, \beta \times \beta') = \varepsilon(s, \tau \otimes \tau'), \tag{$*$}$$

for all supercuspidal representations β' of $\mathrm{GL}(k, E)$, $k \leq 2$, with corresponding irreducibles τ' of W_E of dimension k. But $\tau \otimes \tau'$ is isomorphic to $\mathrm{Ind}_K^E(\mu \otimes \tau'_K)$, where Ind_K^E denotes the induction from W_K to W_E and τ'_K the restriction of τ' to W_K. The epsilon factor is additive, but inductive only in dimension zero; so we get

$$\varepsilon(s, \tau \otimes \tau')\varepsilon(s, \mathrm{Ind}_K^E(1_K))^{-2k} = \varepsilon(s, \mu \otimes \tau'_K)\varepsilon(s, 1_K)^{-2k},$$

where 1_K denotes the trivial representation of W_K. By the base change theory (see Prop. 3.1), we know that a similar formula holds with $\tau\otimes\tau'$ (resp. $\mu\otimes\tau'_K$) replaced by $\pi\times\pi'$ (resp. $\lambda\times\beta'_K$).

Then the desired identity (*), and hence Proposition 4.1, is a consequence of knowing that the factor $\varepsilon(s,\lambda\times\beta'_K)$ is the same as $\varepsilon(s,\mu\otimes\beta'_K)$. If $k=1$, or if $k=2$ with β'_K reducible, this is clear. The case when β_K is irreducible is settled by appealing to the $n=m=2$ case of the following

Proposition 4.2. *Let E be a non-archimedean local field, $n,m\in\{2,4\}$, and λ (resp. λ') a supercuspidal representation of $GL(n,E)$ (resp. $GL(m,E)$), functorially associated to an irreducible μ (resp. μ') of W_E. IF n or m is 4 and E has residual characteristic 2, assume that the supercuspidal representation in question is allowable. Then we have*

$$\varepsilon(s,\lambda\times\lambda')\;=\;\varepsilon(s,\mu\otimes\mu').$$

Proof. If the residual characteristic p is odd, this is the content of Harris's identity (4.1). So assume that $p=2$. If μ and μ' are local automorphic inductions of characters of cyclic extensions, the assertion is a special case of a general result of Henniart (see part (iii) of Theorem on page 146 of [He2]). In the general case, the proof is still similar, the key point being the existence of global (automorphic) representations Π, Π' with local components λ, λ' respectively. We give the complete argument.

To begin, we may assume that $n\geq m$, and after twisting by suitable unramified characters, that μ and μ' are continuous (irreducible) representations of $\mathrm{Gal}(\overline{E}/E)$, where $\overline{E}$ denotes an algebraic closure of E. Let L (resp. L') denote the Galois extension of E cut out by the kernel of τ (resp. τ'), and let $\tilde{L}$ denote the compositum of L and L'.

The following simple, but useful, lemma was shown to the second author some time ago by J.-P. Serre.

Lemma 3. *Let E be a non-archimedean local field of characteristic zero and of residual characteristic p. Fix a finite set S of rational primes other than p, but possibly including ∞. Let E'/E be a finite Galois extension. Then there exists a finite Galois extension k'/k of number fields, and a place v of k extending to a place v' of k', such that (i) $k_v=E$, (ii) $k'_{v'}=E'$, and (iii) the decomposition group of v' in Gal(k'/k) is the whole group. Moreover, all the primes in S split completely in k'.*

This lemma is a consequence of Krasner's lemma and its proof will be left to the reader. A slightly weaker version can be found in [De2], page 544, as Lemme 4.13.

Applying this lemma to our setup, we get a Galois extension $\tilde{M}/k$ of number fields with local extension $\tilde{L}/E$, such that $\mathrm{Gal}(\tilde{M}/k)=\mathrm{Gal}(\tilde{L}/E)$. Then $\mathrm{Gal}(\tilde{E}/L)$ and $\mathrm{Gal}(\tilde{E}/L')$ are subgroups of $\mathrm{Gal}(\tilde{M}/k)$, and let us denote their fixed fields in $\tilde{M}$ by M' and M respectively. It is easy to see that L (resp. L') is a local completion of M (resp. M') at a place u (resp. u') with $\mathrm{Gal}(L/E)=\mathrm{Gal}(M/k)$ (resp. $\mathrm{Gal}(L'/E)=\mathrm{Gal}(M'/k)$. Thus we get continuous, irreducible representations β and β' of $\mathrm{Gal}(\overline{\mathbb{Q}}/k)$, acting via the respective quotients $\mathrm{Gal}(L/k)$ and $\mathrm{Gal}(L'/k)$, such that $\mu=\beta_u$ and $\mu'=\beta'_{u'}$.

Note that the image of β (resp. β') in $\mathrm{GL}_n(\mathbb{C})$ (resp. $\mathrm{GL}_m(\mathbb{C})$) is solvable.

Now suppose $n = m = 2$. Then, by the theorems of Langlands ([La2]) and Tunnell ([Tu]) on tetrahedral and octahedral Galois representations, there exist cuspidal automorphic representations Π and Π' of $\mathrm{GL}(n, \mathbb{A}_k)$ and $\mathrm{GL}(m, \mathbb{A}_k)$ respectively, such that we have the global L-function identities $L(s, \Pi \otimes \chi) = L(s, \beta \otimes \chi)$ and $L(s, \Pi' \otimes \chi) = L(s, \beta' \otimes \chi)$, for all idele class characters χ of E.

It then follows ([La2]) that, for all quasi-characters ν of E^*,

$$\varepsilon(s, \Pi_u \otimes \nu) = \varepsilon(s, \mu \otimes \nu),$$

and

$$\varepsilon(s, \Pi'_u \otimes \nu) = \varepsilon(s, \mu' \otimes \nu).$$

Then, by uniqueness, Π_u must be isomorphic to λ, and $\Pi'_{u'}$ to λ'. Now applying Theorem 4.1 of [He3], we get

$$\varepsilon(s, \lambda \times \lambda') \frac{L(1-s, \lambda^\vee \times \lambda'^\vee)}{L(s, \lambda \times \lambda')} = \varepsilon(s, \mu \otimes \mu') \frac{L(1-s, \mu^\vee \otimes \mu'^\vee)}{L(s, \mu \otimes \mu')}.$$

Suppose μ' is not of the form $\mu \otimes \nu$, for a quasi-character ν. Then $L(s, \mu \otimes \mu')$ and $L(1-s, \mu^\vee \otimes \mu'^\vee)$ are both 1. In this case, λ' cannot be a character twist of λ either, as μ (resp. μ') is functorially associated to λ (resp. λ'). Then one sees by Jacquet-Shalika ([JS2]) that the corresponding $L-$factors are also 1. So, to prove Proposition 4.2, we may assume that $\mu' \simeq \mu \otimes \nu$ and $\lambda' \simeq \lambda \otimes \nu$, for some ν. Then we have the factorizations

$$L(s, \mu \otimes \mu') = L(s, \mathrm{Sym}^2(\mu) \otimes \nu) L(s, \omega\nu)$$

and

$$L(s, \lambda \times \lambda') = L(s, \mathrm{Sym}^2(\lambda) \otimes \nu) L(s, \omega\nu),$$

where $\mathrm{Sym}^2(\lambda)$ is the representation of $\mathrm{GL}(3, K)$ associated to λ by the symmetric square lifting (cf. [GJ]), and ω the central character of λ. Since μ is associated to λ, it follows that $\mathrm{Sym}^2(\mu)$ is associated to $\mathrm{Sym}^2(\lambda)$, and so we get the equality of $L(s, \mu \otimes \mu') = L(s, \lambda \times \lambda')$. Similarly for their contragredients. This proves Proposition 4.2 in this ($n = m = 2$) case, and hence also Proposition 4.1.

Next suppose $(n, m) = (4, 2)$. Then, since λ satisfies $(R(E))$, $\mu = \beta_u$ must be, by an earlier argument, induced by an irreducible θ of $\mathrm{Gal}(\overline{E}/K)$, for some quadratic extension K of E. Clearly, K must be subfield of L, and so must correspond, by construction, to a quadratic extension N of k contained in M; in other words, there is a place w of N extending u, and lying below u', such that $N_w = K$. Then β must be induced by an irreducible δ of $\mathrm{Gal}(\overline{k}/N)$, with $\delta_w = \theta$. Since δ is two-dimensional and has solvable image, we can apply Langlands and Tunnell once again to get a cuspidal automorphic representation α of $\mathrm{GL}(2, \mathbb{A}_N)$ functorially associated to δ. Then we can conclude that α_w is a supercuspidal representation of $\mathrm{GL}(2, K)$ whose induction to E is isomorphic to λ. Then λ is the component of Π at u. Since $m = 2$, we already knew that there exists cuspidal Π' with $\Pi'_u = \lambda'$. Applying what was proved above for the $(2, 2)$ case, we get

$$\varepsilon(s, \alpha_w \times \lambda'_K) = \varepsilon(s, \theta \otimes \mu'_K).$$

The Proposition then follows (in this case) by using the way the epsilon factors change under induction, since $\mathrm{Ind}_K^E(\theta) = \mu$ and $\mathrm{Ind}_K^E(\alpha_w) = \lambda$).

Finally, when $n = m = 4$, we get the same identity of epsilon factors over K, but with μ'_K four-dimensional. But then we can apply the $(4, 2)$ case (just proved) and deduce what we want.

□

5. Proof of Theorem A

In this section we will prove the following strengthening of Theorem A:

Theorem A' *Let $m, n \leq 4$, and π, π' unitary, cuspidal automorphic representations of $GL(n, \mathbb{A}_F)$, $GL(m, \mathbb{A}_F)$ respectively, which are of symplectic type. If n or m is 4, assume that, for every place v above 2, the local component at v of the representation in question is allowable. Then*

$$W(\pi \times \pi') = 1.$$

Our proof of theorem A' depends crucially on the following

Theorem 5.1. *Let π be a cuspidal automorphic representation of $GL(4, \mathbb{A}_F)$ of symplectic type, and at each v, let σ_v be the W'_{F_v}-module associated to π_v by Prop. 4.1. Assume that for every v above 2, π_v is allowable. Then σ_v is* ***symplectic*** *at* ***every*** *v.*

Theorem 5.1 $\Longrightarrow$ Theorem A'. If π and π' are as in the Theorem A', we can, by Proposition 4.1, functorially associate at every v, representations σ_v, σ'_v of W'_{F_v} to the local components π_v, π'_v respectively. These are symplectic by Theorem 5.1. Setting $s = 1/2$ in the epsilon factor identity of Prop. 4.2, we get $(\forall v)$

$$W(\pi_v \times \pi'_v) = W(\sigma_v \otimes \sigma'_v). \tag{5.1}$$

Since we have the product formula $W(\pi \times \pi') = \prod_v W(\pi_v \times \pi'_v)$, it suffices to show that $W(\pi_v \times \pi'_v) = 1$ at each v. So we get what we want by appealing to the reasoning in section 2 (see the proof of Prop. 2.1). Note that, though we considered only the representations of W_{F_v} in section 2, the crucial result (2.1) of Deligne applies to those of W'_{F_v} as well, as seen by the discussion in section 5.4 of [De1].

□

Proof of Theorem 5.1. First we need the following

Lemma 4. *Let E be a non-archimedean local field of odd residual characteristic, β a self-dual supercuspidal representation of $GL(4, E)$ of trivial central character, and τ the irreducible 4-dimensional representation of W_E associated to β by the local Langlands correspondence (cf. Prop. 4.1). Suppose τ is not symplectic. Then β is allowable.*

Proof. Since the local correspondence is compatible with taking contragredients, τ is self-dual. Moreover, since the central character of β is trivial, the determinant of τ must be trivial. Thus the image of τ lands in either $\mathrm{Sp}(4, \mathbb{C})$ or $\mathrm{SO}(4, \mathbb{C})$. By hypothesis, we are not in the former case. We may view τ as a representation of $\Gamma_E = \mathrm{Gal}(\overline{E}/E)$.

Consider $M_2(\mathbb{C})$ as a four dimensional quadratic space under the non-degenerate symmetric bilinear form $B(X, Y) = \mathrm{tr}({}^tXY)$. Then $\mathrm{GO}(4, \mathbb{C})$ identifies with the group of similitudes of B. Let $\mu(g)$ denote the similitude factor of g in $\mathrm{GO}(4, \mathbb{C})$. Define $\mathrm{GSO}(4, \mathbb{C})$ to be the kernel of the map $g \to \mu(g)^{-2}\det(g)$. (Some authors write SGO(4) instead of GSO(4).) Then there is a well known short exact sequence of $\mathbb{C}$-algebraic groups

$$1 \to \mathbb{C}^* \to \mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(2,\mathbb{C}) \to \mathrm{GSO}(4,\mathbb{C}) \to 1,$$

where the map on $\mathbb{C}^*$ sends t to $(tI, t^{-1}I)$, and the one on $\mathrm{GL}(2,\mathbb{C})\times\mathrm{GL}(2,\mathbb{C})$ sends (g_1, g_2) to the similitude $(X \to {}^t g_1 X g_2)$. Viewing this as an exact sequence of trivial Γ_E-modules, we get the following (part of the) associated long exact sequence in cohomology:

$$\mathrm{Hom}(\Gamma_E, \mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(2,\mathbb{C})) \to \mathrm{Hom}(\Gamma_E, \mathrm{GSO}(4,\mathbb{C})) \to H^2(\Gamma_E, \mathbb{C}^*),$$

where the left arrow identifies with the tensor product map. But $H^2(\Gamma_E, \mathbb{C}^*)$ is trivial by a theorem of Tate (see [Se] for a proof). Note that $\mathrm{SO}(4,\mathbb{C})$ is a subgroup of $\mathrm{GSO}(4,\mathbb{C})$; it is the image of the subgroup of $\mathrm{GL}(2,\mathbb{C})\times\mathrm{GL}(2,\mathbb{C})$ consisting of (g_1, g_2) such that $\det(g_1)^2\det(g_2)^2 = 1$. So we may view τ as an element of $\mathrm{Hom}(\Gamma_E, \mathrm{GSO}(4,\mathbb{C}))$. Then, by Tate's theorem, we can find 2-dimensional representations τ_j, $j = 1, 2$, of Γ_E such that $\tau \simeq \tau_1 \otimes \tau_2$. (The choice of (τ_1, τ_2) is not unique, as we can replace it by $(\tau_1 \otimes \nu, \tau_2 \otimes \nu^{-1})$, for a character ν, but this will not matter to us.) Since τ is irreducible, τ_1 and τ_2 must be irreducible.

Since the residual characteristic is odd, we know that τ_1 must be induced by a (linear) character χ of Γ_K, for a quadratic extension K of E. Then the restriction of τ_1, and hence τ, to Γ_K is reducible. This implies that β_K is not supercuspidal. □

Remark: We may realize $\mathrm{SO}(4,\mathbb{C})$ as a quotient of $\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})$ by $\{\pm 1\}$. But $H^2(\Gamma_E, \{\pm 1\}) = {}_2\mathrm{Br}(E)$ is not trivial, and so we cannot hope to write τ as $\tau_1 \otimes \tau_2$ with τ_1, τ_2 of determinant 1, unless $W(\tau) = 1$.

Now let π be as in the theorem, and fix a place v of F. By Lemma 2 of section 3, it is self-dual. We will see later (cf. Remark following Theorem 5.2) that it has trivial central character as well. By Lemma 4 we may then assume that π_v is *allowable* at *every* finite v. The assertion of Theorem 5.3 will be a consequence, for v finite, of putting together the following three propositions. The archimedean case will use a different, but simpler argument.

Proposition 5.1. *Suppose v is finite. Then the following are equivalent:*

1. *σ_v is symplectic.*
2. *$L(s, \Lambda^2(\sigma_v))$ has a pole at $s = 0$.*

Proof. This is clear when σ_v is a true representation of the Weil group W_{F_v}, as $\Lambda^2(\sigma_v)$ admits a trivial summand iff its L-function has a pole at $s = 0$. Indeed, if β is an irreducible non-trivial summand, then $L(s, \beta) = 1$ if β has dimension > 1 or is ramified of dimension 1; otherwise, there is a non-trivial, unramified quasi-character ν of F_v^* such that $L(s, \beta) = (1-\nu(\varpi_v)Nv^{-s})^{-1}$, which is holomorphic at $s = 0$. The converse direction is also clear. (Here ϖ_v denotes a uniformizer at v, and Nv the norm.) By linearity, we are then reduced to treating representations of W'_{F_v} of the following form: $\sigma_v = \beta \otimes \mathrm{sp}(m)$, where β is an irreducible, unitary representation of W_{F_v} pulled back to W'_{F_v}. (See [De2] for the definition and properties of the indecomposable module $\mathrm{sp}(m)$, which corresponds to the symmetric $(m-1)$th power representation of the standard representation of $\mathrm{SL}(2,\mathbb{C})$ and, in the local

Langlands correspondence, to the Steinberg module of $\mathrm{GL}(m, F_v)$.) In such a case, we have

$$\Lambda^2(\sigma_v) \simeq \mathrm{Sym}^2(\beta) \otimes \Lambda^2(\mathrm{sp}(m)) \oplus \Lambda^2(\beta) \otimes \mathrm{Sym}^2(\mathrm{sp}(m)).$$

Since $\mathrm{sp}(m)$ is symplectic (resp. orthogonal) iff m is even (resp. odd), we see that σ_v is symplectic iff β is orthogonal (resp. symplectic) when m is even (resp. odd). On the other hand, if δ is an irreducible summand of $\beta \otimes \beta = \mathrm{Sym}^2(\beta) \oplus \Lambda^2(\beta)$, we have

$$L(s, \delta \otimes \mathrm{sp}(k)) = L(s, \delta \otimes |.|^{\frac{k-1}{2}}),$$

for any k. Since δ is unitary, this L-function has a pole at $s = 0$ iff $k = 1$ and $\delta = 1$. The Proposition now follows as any irreducible summand of the tensor square of $\mathrm{sp}(m)$ is of the form $\mathrm{sp}(k)$ for some k.

□

Let $L_1(s, \pi_v, \Lambda^2)$ denote the local factor associated to π_v and the exterior square representation of ${}^LG = \mathrm{GL}(4, \mathbb{C})$ by the theory of Eisenstein series relative to the realization of GL(4) as a Levi subgroup of SO(8) ([Sh1]). To be precise, he associates a factor $\gamma(s, \pi_v, \Lambda^2)$ to this situation, which occurs naturally in the functional equation, and one has

$$\gamma(s, \pi_v, \Lambda^2) = \epsilon_1(s, \pi_v, \Lambda^2) \frac{L_1(1 - s, \pi_v^\vee, \Lambda^2)}{L_1(s, \pi_v, \Lambda^2)}. \tag{5.2}$$

The ε_1-factor is invertible, and the L_1-factors have the usual shape, in particular having no zeros. It should also be noted that by definition there are no common factors between the numerator and the denominator. They need not be the root number and the L-factor appearing in the definition of $\gamma(s, \Lambda^2(\sigma_v)$ as dictated by the parametrization problem, if π_v is not tempered. If v is archimedean, one knows ([Sh5]) that $\gamma(s, \pi_v, \Lambda^2)$ equals $\gamma(s, \Lambda^2(\sigma_v))$.

Proposition 5.2. *Let v be finite. Then the following are equivalent:*

1. *$L_1(s, \pi_v, \Lambda^2)$ has a pole at $s = 0$.*
2. *$L(s, \Lambda^2(\sigma_v))$ has a pole at $s = 0$.*

Proof. First consider the case when π_v is supercuspidal. Then, since π_v is allowable by hypothesis, we can find, by Prop. 4.1, a quadratic extension K/F_v and a supercuspidal λ of $\mathrm{GL}(2, K)$ such that π_v is $I(\lambda)$ (local automorphic induction). Let τ be the associated irreducible two dimensional W_K-module. Then σ_v is the induction of τ to W_{F_v}. Arguing as in the proof of Proposition 4.2, we can find an irreducible two dimensional representation α of the Weil group of a quadratic extenson N/F with local extension K/F_v such that τ is its restriction to W_K. Let β be the corresponding automorphically induced cuspidal of $\mathrm{GL}(4, \mathbb{A}_F)$ with local component π_v, corresponding globally to the induction $I(\alpha)$ of α to W_F. Denote by $L(s, \Lambda^2(I(\alpha)))$ the (completed) Artin L-function of the exterior square of $I(\alpha)$, and let $L_1(s, \beta, \Lambda^2)$ be the gobal exterior square L-function of β considered by Shahidi ([Sh1]). It is known that both these functions have meromorphic continuations and admit functional equations of the standard type. If $f(s), g(s)$ are two functions whose quotient is invertible, we will write $f(s) \equiv g(s)$; this is clearly an equivalence relation. At the places v where the representations are unramified, one knows that Shahidi's local factors coincide with the Langlands factors and hence with those of

$\Lambda^2(I(\alpha))$. Moreover, the factors also agree at the archimedean places ([Sh5]). This leads to the following equivalence:

$$\prod_{u\in S} L_u(s,\Lambda^2(I(\alpha)))L_1(1-s,\beta_u^\vee,\Lambda^2) \equiv \prod_{u\in S} L_u(1-s,\Lambda^2(I(\alpha)^\vee))L_1(s,\beta_u,\Lambda^2).$$

Suppose that v lies in S, for otherwise there is nothing to prove. By Brauer's theorem, $L(s,\Lambda^2(I(\alpha)))$ is a ratio of abelian L-functions, and this allows us to find a linear character μ of W_N such that (i) $\mu_v = 1$, and (ii) $L_u(s,\Lambda^2(I(\alpha)\otimes\mu_u)) = 1$, at each $u \in S - \{v\}$; for this, we just have to make μ_u sufficiently ramified. (We will, by abuse of notation, use μ to denote also the idele class character defined by class field theory.) One also has the following automorphic analog:

Lemma 5. *(Shahidi) Let u be any finite place. Then there exists a positive number $C = C(u,\beta)$ such that, for every idele class character μ whose local component μ_u is non-quadratic and has conductor larger than C, $L_1(s,\beta_u\otimes\mu_u,\Lambda^2) = 1$.*

Since this is not in print, we indicate how to prove it. Let μ be any idele class character with μ_u non-quadratic. Viewing M =GL(4) as a Levi subgroup of $G =$ SO(8), Shahidi realizes the exterior square L-function of $\beta\otimes\mu$ via the Eisenstein series on $G(\mathbb{A}_F)$ defined by inducing $\beta\otimes\mu$. We need to analyze the poles of $L_1(s,\beta_u\otimes\mu_u,\Lambda^2)$. These are contained in the set of poles of the corresponding (local) intertwining operator, and this set is empty if the local induced representation is irreducible, for any unramified twist of π_u. By inductivity and the factorization formula ([Sh3, Sh4]) of Shahidi, we may assume that the inducing representations are supercuspidal. If w_0 denotes the longest root in the Weyl group of G modulo that of M, reducibility can happen only if $w_0(\beta_u\otimes\mu_u)\simeq\beta_u\otimes\mu_u\nu$, for some unramified character ν. (This is a folklore assertion, discussed in [Sh1]; see also [Si], whose main theorem says that the commuting algebra of the induced representation has dimension 1 otherwise, leading to irreducibility.) Suppose such an identity holds for $\beta_u\otimes\mu_u$ and for $\beta_u\otimes\mu'_u$, for a second character μ'. Then the criterion above implies that there exists an unramified character λ such that the following holds:

$$\beta_u \simeq \beta_u\otimes\mu_u{\mu'_u}^{-1}w_0(\mu_u^{-1}\mu'_u)\lambda.$$

In other words, β_u admits a self-twist. Note that, since w_0 is an involution, it cannot fix any non-quadratic character. Once we fix μ', we see then that, even with variable χ, there are only a finite number of *ramified* non-quadratic μ_u for which this can happen. We are done by choosing C to be larger than the conductors of all such exceptional μ_u.

□

Now applying this lemma and the earlier remark dealing with the Galois side, we can easily find a unitary idele class character μ such that

$$L_1(s,\beta_u\otimes\mu_u,\Lambda^2) = L_1(s,\beta_u^\vee\otimes\overline{\mu}_u,\Lambda^2) = L(s,\Lambda^2(I(\alpha)\otimes\mu)_u) = L(s,\Lambda^2(I(\alpha)\otimes\mu)_u^\vee) = 1,$$

for every u in $S - \{v\}$.

Implicit at this point, and later on in similar situations, is the assertion that a relevant identity of L-factors at an unramified place continues to hold after twisting by a possibly ramified character. This is known to be true for the functions we consider.

Consequently, remembering that $\beta_v = \pi_v$ and $I(\alpha)_v = \sigma_v$ by construction, we deduce the equivalence
(5.3)
$$L_1(s, \pi_v \otimes \mu_v, \Lambda^2)L(1-s, \Lambda^2(\sigma_v^\vee \otimes \overline{\mu}_v)) \equiv L(s, \Lambda^2(\sigma_v \otimes \mu_v))L_1(1-s, \pi_v^\vee \otimes \overline{\mu}_v, \Lambda^2).$$
None of these L-factors can have zeros. Moreover, the poles of $L(1-s, \Lambda^2(\sigma_v^\vee \otimes \overline{\mu}_v))$ all occur on the line $\Re(s) = 1$, since the corresponding inverse roots have absolute value 1. We do not have such a strong result in general on the automorphic side, but for supercuspidals we do by Shahidi ([Sh1]). (In fact, all we need is that $L_1(1-s, \pi_v^\vee \otimes \overline{\mu}_v, \Lambda^2)$ has no pole at $s = 0$, which works in general, see *loc. cit.*) The assertion of Proposition 5.2 thus follows for π_v supercuspidal.

Next suppose that π_v is not supercuspidal. Then one knows that there is a maximal parabolic subgroup P' of M =GL(4) with Levi M' =GL$(a)\times$GL(b), $a \geq b \geq 1$, $a+b = 4$, and a generic representation $\eta = \eta_1 \otimes \eta_2$ of $M(F_v)$ such that π_v is a subrepresentation of the representation of $M(F_v)$ parabolically induced by η. (π_v is the unique generic constituent of this induced module.) An important factorization result due to Shahidi (see [Sh4], last formula on p.284) gives the following identity:

$$\gamma(s, \pi_v, \Lambda^2) = \gamma(s, \eta_1, \Lambda^2)\gamma(s, \eta_2, \Lambda^2)\gamma(s, \eta_1 \times \eta_2), \tag{5.4}$$

where the first two factors on the right are the ones associated to the exterior square representations of GL$(a, \mathbb{C})$ and GL$(b, \mathbb{C})$ respectively, and the last one is the Rankin-Selberg factor on GL$(a)\times$GL(b). (If $b = 1$, the middle factor is taken to be 1. It may also be useful to note that there may be cancellations, up to invertible factors, in this deceptively uniform formula when π_v is not the full induced representation, for example when it is the Steinberg module.)

Let τ_1 (resp. τ_2) be the representation of W'_{F_v} of dimension a (resp. b) associated to η_1 (resp. η_2) by the local Langlands correspondence, which we can apply as $a, b \leq 3$. Then it is easy to see that there is a similar factorization on the Galois side, with π_v replaced by σ_v and η_j by τ_j, $j = 1, 2$. Note that we have $\gamma(s, \eta_1 \times \eta_2) = \gamma(s, \tau_1 \otimes \tau_2)$ because the epsilon and L-factors of these pairs agree. (For this we use [He1] and Prop. 4.2, which we can apply as $a + b = 4$.) On the other hand, the exterior square representation of GL(3) identifies with the map $g \to g(\det g)^{-1}$. Consequently, if $a = 3$, then we have

$$\gamma(s, \eta_1, \Lambda^2) = \gamma(s, \eta_1 \otimes \omega_1^{-1}) = \gamma(s, \tau_1 \otimes \omega_1^{-1}) = \gamma(s, \Lambda^2(\tau_1)),$$

where ω_j denotes the central character of η_j, identified with the determinant of τ_j. When $a = b = 2$, $L(s, \eta_j, \Lambda^2)$ is none other than $L(s, \omega_j)$, agreeing with $L(s, \Lambda^2(\tau_j))$. One also knows by the Rankin-Selberg theory that $L(s, \eta_1 \times \eta_2) = L(s, \tau_1 \otimes \tau_2)$. So in either case we get the equality of $\gamma(s, \pi_v, \Lambda^2)$ and $\gamma(s, \Lambda^2(\sigma_v))$. This once again leads to the equivalence (5.3), and the remainder of the argument is as in the supercuspidal case.

□

The next step is to make use of the following

Theorem 5.2. *(Jacquet, Piatetski-Shapiro and Shalika) Let β be a cuspidal automorphic representation such that, for some unitary idele class character ν and a finite set S of places, $L^S(s, \beta, \Lambda^2 \otimes \nu^{-1})$ has a pole at $s = 1$. Then there exists a globally generic, cuspidal automorphic representation Π of $GSp(4, \mathbb{A}_F)$ of central character ν. Moreover, one has the following properties:*

1. *At every unramified place v, the Langlands classes $A_v(\Pi)$ and $A_v(\pi)$ agree under the natural embedding $r_4 : {}^L GSp(4) = GSp(4, \mathbb{C}) \hookrightarrow GL(4, \mathbb{C})$.*
2. *At every archimedean place v, the representation of W_{F_v} with values in $GSp(4, \mathbb{C})$ associated to Π_v by Langlands agrees with σ_v under r_4.*

This beautiful result is unfortunately unpublished still, though a key step is achieved in [JS1] and the strategy is sketched in [So]. Efforts are under way to remedy this situation, and we ask the reader's indulgence on this part. (It should perhaps be mentioned that some people, including one of the authors, have gone through the details of proof.)

We will apply this theorem to π and its twists. One immediate consequence of this result is that σ_v is symplectic if v is archimedean. Indeed, σ_v lands in $\mathrm{GSp}(4, \mathbb{C})$ by the above, and moreover, its polarization identifies, under the archimedean Langlands correspondence, with the central character of Π_v, which is trivial as $\nu = 1$ for $\beta = \pi$. Thus we may, and we will, assume henceforth that v is finite.

Remark: Let v be a finite place where π is unramified. Since π is of symplectic type, the central character of Π is trivial. Thus we see that the Langlands class of π_v is, thanks to the Theorem 5.2, represented by a diagonal matrix of the form $[a_v, b_v, a_v^{-1}, b_v^{-1}]$, which has determinant 1; so π_v has trivial central character. Now, if ω is the central character of π, it is an idele class character which is trivial at almost all places, hence must be trivial by the "strong multiplicity one" for GL(1) due to Hecke. (In fact, Hecke's theorem says that ω is trivial if $\omega_v = 1$ for all v in a set of primes of density $> \frac{1}{2}$.)

Now let r_5 denote the natural five dimensional representation of $\mathrm{GSp}(4, \mathbb{C})$. Then, as is well known, we have the identity

$$\Lambda^2 \circ r_4 \;=\; r_5 \oplus \lambda, \tag{5.5}$$

where λ is the polarization (symplectic similitude) character. Denote by $L_1(s, \Pi, r_5)$ the global L-function associated to (Π, r_5) by Shahidi's theory, by using the embedding of GSp(4) in GSp(6). Again, it has meromorphic continuation and functional equation, with the archimedean and unramified factors agreeing with the recipe of Langlands. (This L-function, some times called the standard L-function of GSp(4), can also be approached via the integral representations of Rallis and Piatetski-Shapiro [PS-R], but we do not seem to have good information at the ramified places.)

Let μ be a unitary idele class character. Apply Theorem 5.2 with $\beta = \pi \otimes \mu$, and write the corresponding representation of $\mathrm{GSp}(4)/F$ as $\Pi(\mu)$, which has central character μ^2. Appealing to (5.5), we then get the following identity at every finite place u (with norm Nu) where $\pi \otimes \mu$ is *unramified*:

$$L_1(s, \pi_u \otimes \mu_u, \Lambda^2) \;=\; L_1(s, \Pi(\mu)_u, r_5) L(s, \mu_u^2). \tag{5.6}$$

Such an identity also holds at archimedean places. Thus, in view of Propositions 5.1 and 5.2, Theorem 5.1 (and hence Theorem A') will follow once we prove the following

Proposition 5.3. *Write $\Pi = \Pi(1)$. Then $L_1(s, \pi_v, \Lambda^2)$ has a pole at $s = 0$.*

Proof. For any μ, associate the following global L-function:

$$L_6(s, \Pi(\mu)) \;=\; L_1(s, \Pi(\mu), r_5)L(s, \mu^2).$$

Then it has a functional equation, with almost all the factors agreeing with those of $L(s, \pi \otimes \mu, \Lambda^2)$, which has its own functional equation. Noting that π and Π are self-dual, this leads to the equivalence

(5.7)
$$\prod_{u \in S} L_6(s, \Pi(\mu)_u)L_1(1-s, \pi_u \otimes \overline{\mu}_u, \Lambda^2) \;\equiv\; \prod_{u \in S} L_6(1-s, \Pi(\overline{\mu}))L_1(s, \pi_u \otimes \mu_u, \Lambda^2),$$

where S is the set of finite places where $\pi \otimes \mu$ is ramified.

We may suppose that π is ramified at v, as otherwise the assertion is clear. Arguing as in Lemma 5, we can find a unitary idele class character μ, trivial at v, such that, at each u in $S - \{v\}$, μ_u is sufficiently ramified that the factors $L_1(s, \pi_u \otimes \mu_u, \Lambda^2)$, $L_1(s, \pi_u \otimes \overline{\mu}_u, \Lambda^2)$, $L_1(s, \Pi(\mu)_u, r_5)$, $L_1(s, \Pi(\overline{\mu}), r_5)$, $L(s, \mu_u^2)$, and $L(s, \overline{\mu}_u^2)$ are all 1. Since $\mu_v = 1$, this leads to the equivalence

(5.8)
$$L_1(s, \Pi_v, r_5)L(s, 1_v)L_1(1-s, \pi_v, \Lambda^2) \;\equiv\; L_1(1-s, \Pi, r_5)L(1-s, 1_v)L_1(s, \pi_v, \Lambda^2).$$

Since $L(s, 1_v) = (1 - (Nv)^{-s})^{-1}$, it has a unique pole at $s = 0$. So the right hand side (of (5.8)) also has a pole at $s = 0$, which could not be accounted for by $L(1-s, 1_v)$. Furthermore, it is known that at any place u, $L_1(s, \Pi_u, r_5)$ has no pole at $s = 1$. (This is because its poles must come from those of the corresponding intertwining operator, which by [Sh1], Theorem 5.2, is holomorphic in $\Re(s) \geq 1$.) So the pole at $s = 0$ of the expression on the right of (5.8) must come from that of $L_1(s, \pi_v, \Lambda^2)$.

□

6. Proof of Theorem B

Let π be a cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_F)$ of trivial central character. Then π is self-dual and of symplectic type. Let $L(s, \pi, \mathrm{Sym}^2)$ be the L-function associated ([GJ]) to the symmetric square lift $\mathrm{Sym}^2(\pi)$, which is an automorphic representation of $\mathrm{GL}(3, \mathbb{A}_F)$. Then one knows that there is a factorization

$$L(s, \pi \times \pi) \;=\; L(s, \pi, \mathrm{Sym}^2)L(s, 1), \tag{6.1}$$

where $L(s, 1)$ denotes the Dedekind Zeta function of F with the archimedean factors added. It should be noted that from the published proofs, one gets this factorization at almost all places. But then we can use the fact that both sides have functional equations, and that (by using [JS3]), given any place u, we can twist by a highly ramified character to trivialize the L and ϵ-factors at u. The exact identity (6.1) then follows by checking it at the ramified places, one by one, as in sections 4, 5.

Thus we get

$$W(\pi, \mathrm{Sym}^2) \;=\; W(\pi \times \pi)/W(1).$$

It is well known that $W(1) = 1$, and moreover, $W(\pi \times \pi) = 1$ by Theorem A. This proves the triviality of $W(\pi, \mathrm{Sym}^2)$.

Now consider the symmetric fourth power root number of π. One defines the symmetric fourth power L-function as follows:

$$L(s,\pi,\mathrm{Sym}^4) \,=\, L(s,\mathrm{Sym}^2(\pi)\times\mathrm{Sym}^2(\pi))/L(s,\mathrm{Sym}^2(\pi))L(s,1).$$

We have just seen that $W(\mathrm{Sym}^2(\pi)) \,=\, W(1) \,=\, 1$. So the desired triviality of $W(\pi,\mathrm{Sym}^4)$ will result from the following

Proposition 6.1. $W(\mathrm{Sym}^2(\pi)\times\mathrm{Sym}^2(\pi)) \,=\, 1.$

Note that this cannot be proved by appealing to Theorem A as $\mathrm{Sym}^2(\pi)$ is not of symplectic type.

Proof. At any place v, let σ_v denote the representation of W'_{F_v} associated to π_v by the local Langlands correspondence. It has determinant 1 as π has trivial central character. Under the local correspondence for GL(3), $\mathrm{Sym}^2(\sigma_v)$ is associated to $\mathrm{Sym}^2(\pi_v)$. We claim that, at any place v, we have

$$L(s,\mathrm{Sym}^2(\pi_v)\times\mathrm{Sym}^2(\pi_v)) \,=\, L(s,\mathrm{Sym}^2(\sigma_v)\otimes\mathrm{Sym}^2(\sigma_v)) \tag{6.2}$$

and

$$\epsilon(s,\mathrm{Sym}^2(\pi_v)\times\mathrm{Sym}^2(\pi_v)) \,=\, \epsilon(s,\mathrm{Sym}^2(\sigma_v)\otimes\mathrm{Sym}^2(\sigma_v)).$$

Indeed, applying Theorem 4.1 of [He], we get the equality of the associated gamma factors. Then, twisting by a highly ramified character, and arguing as in section 4.2 of *loc. cit.*, we get the assertion of the claim.

Consequently, it suffices to prove that for each v,

$$W(\mathrm{Sym}^2(\sigma_v)\otimes\mathrm{Sym}^2(\sigma_v)) \,=\, 1. \tag{6.3}$$

Write τ_v for $\mathrm{Sym}^2(\sigma_v)\otimes\mathrm{Sym}^2(\sigma_v))$. Then τ_v is the composite

$$W'_{F_v} \,\to \mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C}) \,\to \mathrm{GL}(9,\mathbb{C}),$$

where the first arrow is (σ_v,σ_v), and the second arrow, r say, is the composition of taking the symmetric square of each factor, landing in $\mathrm{GL}(3,\mathbb{C})\times$ $\mathrm{GL}(3,\mathbb{C})$, and then taking the tensor product. Since $\mathrm{Sym}^2(\sigma_v)$ is orthogonal, so is its tensor square. Thus r takes values in $\mathrm{O}(9,\mathbb{C})$. Since $\mathrm{SL}(2,\mathbb{C})\times$ $\mathrm{SL}(2,\mathbb{C})$ is simply connected, we may apply the discussion following Lemma 1 of section 2 and obtain (6.3).

□

Bibliography

[AC]: J. Arthur and L. Clozel, *Simple Algebras, Base Change and the Advanced Theory of the Trace Formula*, Ann. Math. Studies **120** (1989), Princeton, NJ.

[BF]: D. Bump, S. Friedberg, The exterior square automorphic L-functions on GL(n), in *Festschrift in honour of I.I.Piatetski-Shapiro*,part II, Israel Math. Conf. Proc. **3** (1990), 47-65.

[Ch]: J.-P.Jeff Chen, *Local Factors, Central Characters, and Representations of the General Linear Group over Non-archimedean Fields*, Doctoral dissertation, Yale University (1996).

[C]: L. Clozel, Base Change for GL(n), Proc. ICM, Berkeley (1986), 791-797.

[Co-S]: J. Coates and C.-G. Schmidt, Iwasawa theorey for symmetric square of an ellptic curve, J. Reine Angew. Math. **375/ 376** (1987), 104-156.

[De1]: P. Deligne, Les constantes locales de l'équation fonctionnelle de la fonction L d'Artin d'une représentation orthogonale, Inventiones Math. **35** (1976), 299-316.

[De2]: P. Deligne, Les constantes des équations fonctionnelles des fonctions L, in *Modular functions of one variable* II, Springer Lecture Notes **349** (1973), 501-597.

[Fr-Q]: A. Fröhlich and J. Queyrut, On the functional equation of the Artin L-function for characters of real representations, Inventiones Math. **20** (1973), 125-138.

[GJ]: S. Gelbart and H. Jacquet, A relation between automorphic representations of GL(2) and GL(3), Ann, Scient. ENS (4) **11** (1979), 471–542.

[Ha]: M. Harris, The local Langlands conjecture for GL(n) over a p-adic field, $n < p$, preprint (1997).

[He1]: G. Henniart, La conjecture locale pour GL(3), Mémoires S.M.F. (nouvelle série), no. 11/12 (1984).

[He2]: G. Henniart, On the local Langlands conjecture for GL(n), the cyclic case, Ann. Math. **123** (1986), 145-203.

[He3]: G. Henniart, Caractérisation de la correspondence de Langlands locale pour les facteurs ε de paires, Inventiones Math. **113** (1993), 339-350.

[HH]: G. Henniart and R. Herb, Automorphic induction for GL(n) (over local non-Archimedean fields), Duke Math. Journal **78** (1995), no. 1, 131-192.

[JS1]: H. Jacquet and J.A. Shalika, Exterior square L-functions, in *Automorphic Forms, Shimura Varieties, and L-functions*, vol. 2, ed. by L. Clozel and J. Milne, Academic Press, NY (1990), 143-226.

[JS2]: H. Jacquet and J.A. Shalika, Euler products and the classification of automorphic forms I & II, Amer. J of Math. **103** (1981), 499–558 & 777–815.

[JS3]: H. Jacquet and J.A. Shalika, A lemma on highly ramified epsilon factors, Math. Annalen **271** (1985), no. 3, 319-332

[JPSS]: H. Jacquet, I. Piatetski-Shapiro and J. Shalika, Rankin-Selberg convolutions, Amer. J of Math. **105** (1983), 367–464.

[Ku]: P. Kutzko, The Langlands conjecture for GL(2) over a local field, Ann. Math. **112** (1980), 381-412.

[La1]: R.P.Langlands, Problems in the theory of Automorphic Forms, in "Lectures in Modern Analysis and Applications", Springer Lecture Notes **170** (1970), 18–61.

[La2]: R.P. Langlands, *Base Change for GL*(2), Ann. Math. Studies **96** (1980), Princeton, NJ.

[La3]: R.P. Langlands, Automorphic representations, Shimura varieties and motives. Ein Märchen, Proc. Symp. Pure Math. **33**, ed. by A. Borel and W. Casselman, part 2, AMS (1979), 205–246.

[PS-R]: I. Piatetski-Shapiro and S. Rallis, A new way to get Euler products, J. Reine Angew. Math. **392** (1988), 110-124.

Ro]: D. Rohrlich, Galois theory, Elliptic curves, and Root numbers, Compositio Mathematica **100** (1996), 311-349.

[Sa]: T. Saito, The sign of the functional equation of the L-function of an orthogonal motive, Inventiones Math. **120**, 119-142 (1995).

[Se]: J.-P. Serre, Modular forms of weight one and Galois representations, in *Algebraic Number Fields*, ed. by A. Fröhlich, Acad. Press (1977), 193-268.

[Sh1]: F. Shahidi, On the Ramanujan conjecture and the finiteness of poles for certain L-functions, Ann. of Math. (2) **127** (1988), 547–584.

[Sh2]: F. Shahidi, Symmetric power L-functions for GL(2), CRM Proceedings & Lecture Notes **4**, ed. by M.R. Murty, Montréal (1994).

[Sh3]: F. Shahidi, A proof of the Langlands conjecture on Plancherel measures; Complementary series for p-adic groups, Ann. of Math. **132** (1990), 273-330.

[Sh4]: F. Shahidi, On multiplicativity of local factors, in *Festschrift in honour of I.I.Piatetski-Shapiro*, Israel Math. Conf. Proc. **3** (1990), 279-289.

[Sh5]: F. Shahidi, Local coefficients as Artin factors for real groups, Duke Math. Journal **52** (1985), 973-1007.

[Si]: A. Silberger, The Knapp-Stein dimension theorem for p-adic groups, Proc. AMS **68** (1978), 243-246.

[So]: D. Soudry, Automorphic forms on GSp(4), in *Festschrift in honour of I.I.Piatetski-Shapiro*, Israel Math. Conf. Proc. **3** (1990), 291-303.

[Ta]: J. Tate, Les conjectures de Stark sur les Fonctions L d'Artin en $s = 0$, Progress in Math. **47**, Birkhäuser, Boston.

[Tu]: J. Tunnell, Artin's conjecture for repreesentations of octahedral type, Bulletin AMS **5**, no. 2 (1981), 173-175.

Dipendra Prasad
Mehta Research Institute
10 Kasturba Marg, Allahabad 21102, India.
dprasad@mri.ernet.in

Dinakar Ramakrishnan
Department of Mathematics
California Institute of Technology, Pasadena, CA 91125.
dinakar@cco.caltech.edu