Galois Symbols on the square of an elliptic curve

by

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Abstract

We prove some theorems concerning the Galois symbol map

$$s_p : T_F(E \times E)/p \to H^2(F, E[p]^{\otimes 2}),$$

where F is a local or a number field, E a *semistable* elliptic curve over F, p a prime, $E[p]$ the Galois module of p-division points in $E(\overline{F})$, and $T_F(E \times E)$ the Albanese kernel over F. Explicitly, $T_F(E \times E)$ is generated by norms from finite extensions K/F of *symbols* $\langle P, Q\rangle$ representing the linear equivalence class of $(P,Q) - (P,0) - (0,Q) + (0,0)$, with $P, Q \in E(K)$.

When F is a number field, our global results include the following:

(i) *The image of s_p is zero for any odd prime p which is unramified in F;*

(ii) *If p is an odd prime such that E has good ordinary reduction at a place of F above p (or if $p = 2$ with F having a real place), then for a suitable quadratic extension K of $F([E[p])$ there exists an element θ in $T_K(E \times E)/p$ such that $s_p(\theta)$ is non-zero.*

On the way we also prove (for p odd) that if E has ordinary or multiplicative reduction, then s_p is *injective*, and we describe the image precisely. When E has supersingular reduction, we get sufficient conditions for the image to be zero. Over archimedean fields, s_p is injective for all p, and the image is simple to describe.

In order to tie together the local results and get the global one, we establish the following *local-global result*:

Let Σ denote the set of all places v of F. Then the natural map

$$H^2(F, E[p]^{\otimes 2}) \to \prod_{v \in \Sigma} H^2(F_v, E[p]^{\otimes 2})$$

is injective.

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