Base Change of Hecke Characters revisited

Dinakar Ramakrishnan

Here we answer a nice question raised by Peter Sarnak as to whether we can give a new proof of any instance of the well known base change result for GL(1) by using trace inequalities.

Fix a finite Galois extension M/L of CM fields of degree m, and denote by E, resp. F, the totally real subfield M^+, resp. L^+, of M, resp. L. For any number field K, write $\mathbb{A}_K = K_\infty \times \mathbb{A}_K^f$ for the adele ring of k, Γ_K for the absolute Galois group, and C_K the idele class group $\mathbb{A}_K^*/K^*$. For $K \in \{M, L$, set, for a fixed tuple $k = (k_v)_{v|\infty}$ with each k_v in $\mathbb{Z}$,

$$X_K(k) := \{\chi \in \mathrm{Hom}_{\mathrm{cont}}(C_K, S^1) \mid \chi_v(z) = (z/|z|)^{k_v}, \forall v \mid \infty,\ \chi|_{C_{K^+,+}} = 1\},$$

where $C_{K^+,+}$ denotes the image of $\mathbb{A}^*_{K^+,+}$, the subgroup of $\mathbb{A}^*_{K^+}$ consisting of totally positive elements. Any χ in $X_K(k)$ is a unitary Hecke character of weight k of K, which is anti-cyclotomic when trivial on C_{K^+}.

Fix a finite set R of primes in F, and denote by R_M the set of primes of M above R. Let $m = (m_v)_{v\in R}$ be a tuple of positive integers m_v. Put

$$q(R_M, m) := \prod_{v\in R} \frac{\prod_{w|v} q_w^{m_v-1}(q_w-1)}{q_v^{m_v-1}(q_v-1)} \geq \prod_{v\in R} q_v^{m_v([M_w:L_v]-1)}.$$

The object here is to construct the *base change* of $\chi \in X_L(k)$ of conductor dividing $\prod_{v\in R} P_v^{m_v}$ to M without knowing *a priori* the answer as being $\chi \circ N_{M/L}$.

We will choose, as we may, (R, m) *large enough* so that the following holds:

$$(0) \qquad q(R_M, m) \geq C := (6\pi)^{[M:\mathbb{Q}]-[L:\mathbb{Q}]} \left(\frac{d_L d_E}{d_M d_F}\right)^{1/2} \frac{L(1,\nu_{L/F})}{L(1,\nu_{M/E})},$$

where d_K denotes, for any number field K, the absolute value of the discriminant of K, and $\nu_{L/F}$, resp. $\nu_{M/K}$ is the quadratic character of F, resp. E, attached to L, resp. M.

For the choice of measures we will take below,

$$C = (6\pi)^{[M:\mathbb{Q}]-[L:\mathbb{Q}]} \frac{\mathrm{vol}(\mathrm{L}^*\mathbb{A}^*_{\mathrm{F},+}\backslash\mathbb{A}^*_{\mathrm{L}})}{\mathrm{vol}(\mathrm{M}^*\mathbb{A}^*_{\mathrm{E},+}\backslash\mathbb{A}^*_{\mathrm{M}})}.$$

Theorem *Let M/L be a finite Galois extension of CM fields. Fix a weight $k = (k_v)$. Then there exists a set Y of positive density, consisting of degree one primes w in M such that for every χ in $X_L(k)$ $\exists\chi' \in$*

$X_M(k')$ for $k' = (k'_w)$ with $k'_w \in \{0, \pm k_w\}$ for every $w \mid \infty$, such that for all but finitely many $w \in Y$, $\chi'_w = \chi_v$ if v is the place of L lying below w.

This is what the trace inequality argument below will give, and when $k \neq 0$, a modification of the argument will even allow us to take Y to be of density one. In any case, to go further and obtain the full base change, we may appeal to a theorem of Hecke, which will assert that χ' is necessarily of the form $(\chi \circ N_{M/L})\nu$ for a finite order character ν of M as χ' and $\chi \circ N_{M/L}$ agree on a set of primes of positive density; in particular, $k' = k$. Thus the desired base change arises by setting $\chi_M = \chi'\nu^{-1}$.

To approach this without knowing about $\chi \circ N_{M/L}$, note that by Serre every χ, resp. χ', is attached, for ℓ a prime, to an ℓ-adic character χ_ℓ, resp. χ'_ℓ of Γ_L, resp. Γ_M with the same L-function as χ, resp. χ'. Then for every w in the set Y of degree one primes given by Theorem, the restrictions of χ_ℓ to Γ_{L_v} and χ'_ℓ to M_w are the same modulo the identification of M_w with L_v (when $w \mid v$). This forces the quotient $\nu_\ell := \chi'_\ell/\chi_\ell|_{\Gamma_M}$ and χ'_ℓ to have trivial restrictions to Γ_{M_w} for w is a set of positive density. This forces their quotient ν_ℓ to have finite order. If we accept the existence of a finite order idele class character ν attached to ν_ℓ, we again see that $\chi_M := \chi'\nu^{-1}$ is the desired base change, with

$$L(s, \chi_M) \;=\; L(s, \chi_\ell|_{\Gamma_M}).$$

Proof of Theorem:

Denote by Σ^1_M the (density one) set of finite places w of M which are unramified and of degree one over $\mathbb{Q}$. Let S be any finite set of places in Σ^1_M with the property that no pair of elements of S has the same norm (in $\mathbb{Q}$). The key is to prove the following:

Proposition A *For every $\chi \in X_L(k)$ and for every S as above which does not meet the (finite) set $R(\chi)$ of primes where χ is ramified, there is a character $\chi^{M,S} \in X_M(k')$, for some $k' \in \{0, \pm k\}$, such that for every $w \in S$, $\chi^{M,S}_w = \chi_v$ if the place v of L lies below w.*

Proof of Proposition A

Fix a character χ_0 in $X_L(k)$. If χ_0 is the trivial character, then we can take $\chi_0^{M,S}$ to be trivial as well, so we may, and we will, assume that χ_0 is non-trivial. At each finite place v, let $m_v = m_v(\chi_0)$ denote the exponent of the conductor $U_v^{(m_v)}$ of χ_0, where $U_v^{(n)}$ is $\mathcal{O}_v^*$ if $n = 0$ and $1 + P_v^n$ if $n > 0$. (Here P_v denotes the prime ideal at v.) Put $R = R(\chi_0)$, which is the finite set of v where $m_v > 0$. Write R_M for the set of primes of M above R.

For any number field k, assign the product measure $d^\times x = \otimes_v d^\times x_v$ on the idele group $\mathbb{A}_k^*$ such that at any finite place v, $d^\times x_v$ gives volume 1 to $\mathcal{O}_{k,v}^*$, at real v the measure $d^\times x_v = dx_v/|x_v|$ (with dx_v being Lebesgue), and for v complex, $d^\times x_v = \frac{dr}{r}d\theta$ (in polar coordinates $x_v = re^{i\theta}$), $r \in \mathbb{R}_+^*, \theta \in [0, 2\pi]$. We also get an induced measure on the relevant quotients of $\mathbb{A}_K^*$).

At any finite place v of L, call a compactly supported, locally constant function f_v on L_v^* *strongly positive* if it is of the form $\sum_{j=1}^r c_j \xi_{j,v} * \xi_{j,v}^\vee$ such that for every $j \leq r$, $c_j \geq 0$, and $\xi_{j,v} * \xi_{j,v}^\vee$ takes values in $\mathbb{R}_+$. (Here, as usual, $\xi_v^\vee$ denotes the function $x \mapsto \overline{\xi_v(x^{-1})}$.) Thus such an f_v is of positive type and takes non-negative values.

Write S_0 for the set of places of L below S, and define a (factorizable) function $f = \otimes_v f_v$ on $\mathbb{A}_L^*$ such that

(a) f_v is is the characteristic function of $U_v^{(m_v)}$, if v is a finite place outside S_0,
(b) If v is archimedean, let $f_v(re^{i\theta})$ equal 1, resp. $3 + 2\cos(k\theta)$, if $k = 0$, resp. $k \neq 0$;
(c) If $v \in S_0$, it is in the (closed) positive cone spanned by functions of the form $\xi * \xi^\vee$, with $\xi = \sum_{n=1}^r a_n ch_{\varpi_v^n \mathcal{O}_v^*}$, $a_n \geq 0$.

Here ch denotes the characteristic function, and ϖ_v the uniformizer at v.

Note that each f_v, and hence f, takes non-negative values on $\mathbb{A}_L^*$. Moreover, f_v is invariant under $U_v = \mathcal{O}_v^*$ for each $v \in S_0$, and f is invariant under $\prod_{v \notin R} U_v \times \prod_{v \in R} U_v^{(m_v)}$. If v is archimedean, f_v is by construction invariant under $\mathbb{R}_+^*$.

Associate to f a function $\phi = \otimes_w \phi_w$ on $\mathbb{A}_M^*$ by defining ϕ_w to be the characteristic function of $U_w'^{(m_v)}$ if $w \notin S$, and the function f_v if $w \in S$, for $w \mid v$. (This makes sense as $M_w = L_v$ for $w \in S$.)

For $K \in \{L, M\}$, put

$$\mathcal{V}_K := L^2(K^* \mathbb{A}_{K^+,+}^* \backslash \mathbb{A}_K^*),$$

which is a unitary representation under the translation action r_K of $\mathbb{A}_K^*$. Then, by the compactness of $K^* \mathbb{A}_{K^+,+}^* \backslash \mathbb{A}_K^*$, we have the decomposition as the Hilbert sum

$$\mathcal{V}_K \simeq \oplus_{n \in \mathbb{Z}^{[K:\mathbb{Q}]}} \oplus_{\chi \in X_K(n)} H_\chi,$$

where each H_χ is the χ-isotypic component, which is one-dimensional.

If we denote by $\mathrm{tr}\,\mathrm{r_K(h)}$ for the trace of any smooth, compactly supported function h on $\mathbb{A}_K^*$, then for h factorizable,

$$\mathrm{tr}\,\mathrm{r_K(h)} = \sum_{\mathrm{n}} \sum_{\chi} \mathrm{tr}\,\chi(\mathrm{h}) = \sum_{\mathrm{n}} \sum_{\chi} \prod_{\mathrm{v}} \mathrm{tr}\,\chi_{\mathrm{v}}(\mathrm{h_v}), \tag{1}$$

with

$$\mathrm{tr}\,\chi_{\mathrm{v}}(\mathrm{h_v}) \;=\; \hat{\mathrm{h}}_{\mathrm{v}}(\chi_{\mathrm{v}}) := \int_{\mathrm{K_v^*}} \chi_{\mathrm{v}}(\mathrm{t_v})\mathrm{h_v}(\mathrm{t_v})\mathrm{d}^{\times}\mathrm{t_v}.$$

On the right hand side expression of (1), the product inside is over all the places v of K, and since for all but a finite number of places χ_v is unramified and h_v is the characteristic function of $\mathcal{O}_v$, $\mathrm{tr}\,\chi_{\mathrm{v}}(\mathrm{h_v})$ equals, with our choice of measures, 1 for almost all v, and so the product is finite. Note that at any v, if h_v is of the form $\xi_v * \xi_v^\vee$, then

$$\mathrm{tr}\,\chi_{\mathrm{v}}(\mathrm{h_v}) \;=\; |\mathrm{tr}\,\chi_{\mathrm{v}}(\xi_{\mathrm{v}})|^2 \;\geq\; 0. \tag{2}$$

which is well known, but let us indicate why. Since $h_v(x) = \int_{L_v^*} \xi_v(xy)\overline{\xi}_v(y)dy$, $\mathrm{tr}\,\chi_{\mathrm{v}}(\mathrm{h_v})$ equals

$$\int_{L_v^*} \overline{\xi}_v(y)d^{\times}y \int_{L_v^*} \xi_v(xy)\chi_v(x)d^{\times}x \;=\; \mathrm{tr}\,\chi_{\mathrm{v}}(\xi_{\mathrm{v}}) \int_{\mathrm{L^*}} \overline{\xi}_{\mathrm{v}}(\mathrm{y})\chi_{\mathrm{v}}(\mathrm{y}^{-1})\mathrm{dy},$$

where we have made the change of variables $x \mapsto xy^{-1}$ to get the expression on the right. Now (2) follows as χ_v is unitary.

Now let v be an archimedean place. Then if $h_v = 3 + 2\cos(k_v\theta_v)$, we have for $\chi_v(re^{\theta}) = e^{in_v\theta}$,

$$\mathrm{tr}\,(\chi_{\mathrm{v}}(\mathrm{h_v})) = \int_0^{2\pi} \left(3\mathrm{e}^{\mathrm{in_v}\theta} + \mathrm{e}^{\mathrm{i(n_v-k_v)}\theta} + \mathrm{e}^{\mathrm{i(n_v+k_v)}\theta}\right)\mathrm{d}\theta \;=\; 2\pi\langle \mathrm{n_v}\rangle, \tag{3}$$

where $\langle n_v\rangle$ is 0, resp. 3, resp. 1, if $n_v \notin \{0, \pm k_v\}$, resp. $n_v = 0$, resp. $n_v = \pm k_v$. In any case, it is always ≥ 0 when h_v is of this special form.

Now let K be L, with h being our choice of $f = \otimes_v f_v$. Then

$$\mathrm{tr}\,\mathrm{r_L(f)} \;=\; {\sum_{\mathrm{n}}}' \sum_{\chi\in \mathrm{X_L(n)}\,|\,\mathrm{m}(\chi)\leq \mathrm{m}} \mathrm{b(n)} \left(\prod_{\mathrm{v}\in\mathrm{R}} [\mathcal{O}_{\mathrm{v}}^* : \mathrm{U_v^{(m_v)}}]\right) \prod_{\mathrm{v}\in\mathrm{S}} |\mathrm{tr}\,\chi_{\mathrm{v}}(\xi_{\mathrm{v}})|^2 \;\geq\; 0, \tag{4}$$

where $m(\chi) \leq m$ means $m(\chi_v) \leq m_v$ at each finite v, the first sum is over $\{n \,|\, n_v \in \{0, \pm k_v\}, \forall v\}$, and $b(n) = (b(n_w))_{w|\infty}$ is s.t. $b(n_w)$ is 6π, resp. 2π when n_w is 0, resp. $\pm k_w$.

Indeed, suppose that for some $\chi \in X_L$, $m(\chi_v) > m_v$ at some finite place v, so that χ_v is non-trivial on $U_v^{m_v}$. Then, since f_v is invariant under $U_v^{m_v}$,

$$\mathrm{tr}\,\chi_{\mathrm{v}}(\mathrm{f_v}) \;=\; \sum_{\mathrm{j}\in\mathbb{Z}} \chi(\varpi^{\mathrm{j}}) \int_{\mathrm{U_v^{m_v}}} \mathrm{f_v}(\varpi_{\mathrm{v}}^{\mathrm{j}}\mathrm{u})\chi_{\mathrm{v}}(\mathrm{u})\mathrm{du},$$

with the inside integral being

$$\sum_{\overline{u}\in U_v/U_v^{(m_v)}} f_v(\varpi_v^j\overline{u})\chi_v(\overline{u}) \int_{U_v^{m_v}} \chi_v(u_1)du_1,$$

which is zero by the orthogonality of characters of the compact group $U_v^{(m_v)}$.

It is important that the sums over n and χ in the expression (4) are both finite, which is because there are only a finite number of possible infinity types and the conductor of χ is restricted.

Similarly, for $K = M$ and h being our ϕ attached to f (and S),
(5)

$$\mathrm{tr}\,\mathrm{r_M}(\phi) \;=\; \sum_{\mathrm{n}'} \sum_{\chi'\in\mathrm{X_M}(\mathrm{n}')\,|\,\mathrm{m}(\chi')\leq\mathrm{m}} \mathrm{b}(\mathrm{n}') \left(\prod_{\mathrm{w}\in\mathrm{R_M}} [\mathcal{O}_{\mathrm{w}}^* : \mathrm{U_w^{(m_{v(w)})}}]^{-1}\right) \prod_{\mathrm{v}\in\mathrm{S}} |\mathrm{tr}\,\chi'_{\sigma(\mathrm{v})}(\xi_{\mathrm{v}})|^2,$$

which is again non-negative, with $v(w)$ denoting the place of L below w. As with $\mathrm{tr}\,\mathrm{r_L}(\mathrm{f})$, the only tuples $n' = (n'_w)_{w|\infty}$ which contribute are those with $n'_w \in \{0, \pm k_{v(w)}\}$, which restricts the χ' to lie in a finite set (since its conductor is also restricted).

Now define, for $K \in \{L, M\}$. the $\mathbb{A}_K^*$-stable subspace $\mathcal{V}_K^0$ of $\mathcal{V}_K$ by

$$\mathcal{V}_K \;=\; \mathcal{V}_K^0 \oplus \mathbb{C},$$

so that the characters χ appearing in V_K^0 are all the non-trivial ones (which are 1 on $K^*\mathbb{A}_{K+,+}^*$). Let r_K^0 denote the restriction of r_K to $\mathcal{V}_K^0$.

Comparing the geometric sides of the two trace formulae below, we will obtain

Proposition B *Let $(M/L, \chi, R, S, f, \phi)$ be as above, with $\chi \neq 1$. Then*

(a) $\mathrm{tr}(r_L(f)) \;\leq\; C'\mathrm{tr}(r_M^0(\phi))$, *where*

$$C' \;=\; (6\pi)^{[L:\mathbb{Q}]-[M:\mathbb{Q}]} \left(\prod_{v\in R} \frac{[\mathcal{O}_v^* : U_v^{(m_v)}]}{\prod_{w|v}[\mathcal{O}_w{}^* : U_w{}^{(m_v)}]}\right) \frac{\mathrm{vol}(L^*\mathbb{A}_{F,+}^*\backslash\mathbb{A}_L^*)}{\mathrm{vol}(M^*\mathbb{A}_{E,+}^*\backslash\mathbb{A}_M^*)} \;>\; 0.$$

(b) *Suppose* (0) *holds, with C given by the expression there. Then*

$$\mathrm{tr}(r_L^0(f)) \;\leq\; C\mathrm{tr}(r_M^0(\phi)).$$

Proof of Proposition B Since the group is commutative, the conjugacy classes are singletons. Noting that $\mathbb{A}_{F,+}^* \cap L^* = F_+^*$, we get

$$\text{(6)} \qquad \operatorname{tr}(r_L(f)) = \operatorname{vol}(L^*\mathbb{A}_{F,+}^*\backslash\mathbb{A}_L^*) \sum_{\alpha\in F_+^*\backslash L^*} f(\alpha) \geq 0,$$

using the fact that f takes non-negative values.

We may, by replacing each α by its multiple by $\prod_{v\in S} p_v^{\ell_v}$ for suitable $\ell_v \in \mathbb{Z}$, $p_v = N(v)$, we may choose a representative for α in L^* which is a unit at every $v \in S$. We have

$$f(\alpha) = \prod_v f_v(\alpha),$$

where at all the finite places v outside S, as f_v is the characteristic function of $U_v^{(m_v)}$, α_v needs to be a unit for $f_v(\alpha)$ to be non-zero. Furthermore, since every v in S is of degree one over $\mathbb{Q}$, say with norm p_v, we may also modify α by such that the modified α has valuation 0 at each $v \in S$. (Since the places v in S have distinct norms, the modification at any $v \in S$ does not affect the situation at another place (in S). Thus, for $f(\alpha)$ to be non-zero, it is necessary for α to be a global unit with

$$\alpha \in U_v^{m_v}, \ \forall\, v < \infty.$$

Similarly, we have, over M, the geometric expression

$$\operatorname{tr}(r_M(\phi) = \operatorname{vol}(M^*\mathbb{A}_{E,+}^*\backslash\mathbb{A}_M^*) \sum_{\beta\in\mathbb{Q}^*\backslash M^*} \phi(\beta) \geq 0.$$

Since we have a natural injection $F^*\backslash L^* \hookrightarrow E^*\backslash M^*$, we get a lower bound

$$\text{(7)} \qquad \operatorname{tr}(r_M(\phi) \geq \operatorname{vol}(M^*\mathbb{A}_{E,+}^*\backslash\mathbb{A}_M^*) \sum_{\alpha\in F_+^*\backslash L^*} \phi(\alpha) \geq 0,$$

and we call the right hand side the **L-hereditary part** of the trace of ϕ.

Lemma C *For every α in L^* lying in $U_v^{(m_v)}$ for all $v < \infty$,*

$$f(\alpha) \leq \phi(\alpha).$$

Proof of Lemma C It suffices to check that for each place v of L,

$$f_v(\alpha) \leq \prod_{w|v} \phi_w(\alpha).$$

When $v \notin R \cup S$, each side is 1. When $v \in R$, the left hand side is 1 if $\alpha \in U_v^{(m_v)}$ and 0 otherwise, and in the former case, α is also in ${U'_w}^{(m_v)}$ for each w, and by our choice of ϕ_w, the value there is 1. So now consider when v is in S. If w is above v but is not $\sigma(v)$, then ϕ_w is the characteristic function of the unit group at w, and so $\phi_w(\alpha) = 1$ for such w. Finally, $\phi_{\sigma(v)} = f_v$, and so takes on the same vale at α as $f_v(\alpha)$. Hence the Lemma.

Back to the proof of Proposition B

Thanks to Lemma C, we now have, applying (7):

$$\text{(8)} \qquad \mathrm{tr}\,(r_M(\phi) \;\geq\; \mathrm{vol}(M^*\mathbb{A}_{E,+}^*\backslash\mathbb{A}_M^*) \sum_{\alpha \in F_+^*\backslash L^*} f(\alpha) \;\geq\; 0,$$

Comparing this with (6), we then obtain

$$\text{(9)} \qquad \mathrm{tr}\,(\mathrm{r_L}(\mathrm{f})) \;\leq\; \mathrm{B}\,\mathrm{tr}\,(\mathrm{r_M}(\phi)), \text{ with } \mathrm{B} = \frac{\mathrm{vol}(\mathrm{L}^*\mathrm{F}_+^*\backslash\mathbb{A}_\mathrm{L}^*)}{\mathrm{vol}(\mathrm{M}^*\mathbb{A}_{\mathrm{E},+}^*\backslash\mathbb{A}_\mathrm{M}^*)},$$

which proves part (a) of Proposition B.

On the other hand, we have by (4),

$$\text{(10)} \qquad \mathrm{tr}\,(\mathrm{r_L}(\mathrm{f})) = \mathrm{tr}\,(\mathrm{r_L^0}(\mathrm{f})) + (6\pi)^{[\mathrm{L}:\mathbb{Q}]} \left(\prod_{\mathrm{v}\in\mathrm{R}} \mathrm{vol}(\mathrm{U_v^{m_v}})\right) \prod_{\mathrm{v}\in\mathrm{S}} \hat{\mathrm{f}}_\mathrm{v}(\underline{1}_\mathrm{v}).$$

Similarly by (5), using $\phi_{\sigma(v)} = f_v$,

$$\text{(11)} \quad \mathrm{tr}\,(\mathrm{r_M}(\phi)) = \mathrm{tr}\,(\mathrm{r_M^0}(\phi)) + (6\pi)^{[\mathrm{M}:\mathbb{Q}]} \left(\prod_{\mathrm{v}\in\mathrm{R}} \prod_{\mathrm{w}|\mathrm{v}} \mathrm{vol}(\mathrm{U_w^{m_v}})\right) \prod_{\mathrm{v}\in\mathrm{S}} \hat{\mathrm{f}}_\mathrm{v}(\underline{1}_\mathrm{v}).$$

Consequently, using (9),

(12)

$$\mathrm{tr}\,(\mathrm{r_L^0}(\mathrm{f})) + (6\pi)^{[\mathrm{L}:\mathbb{Q}]} \prod_{\mathrm{v}\in\mathrm{S}} \hat{\mathrm{f}}_\mathrm{v}(\underline{1}_\mathrm{v}) \;\leq\; \mathrm{B}\mathrm{tr}\,(\mathrm{r_M^0}(\phi)) + (6\pi)^{[\mathrm{M}:\mathbb{Q}]}\mathrm{B} \left(\prod_{\mathrm{v}\in\mathrm{R}} \frac{\prod_{\mathrm{w}|\mathrm{v}} \mathrm{vol}(\mathrm{U_w^{m_v}})}{\mathrm{vol}(\mathrm{U_v^{m_v}})}\right) \prod_{\mathrm{v}\in\mathrm{S}} \hat{\mathrm{f}}_\mathrm{v}(\underline{1}_\mathrm{v})$$

Note that by our normalization of measures which gives volume 1 to U_v and U_w,

$$\prod_{v\in R} \frac{\prod_{w|v} \mathrm{vol}(\mathrm{U_w^{m_v}})}{\mathrm{vol}(\mathrm{U_v^{m_v}})} \;=\; q(R_M, m).$$

The following can be deduced, as remarked by H. Jacquet, for our choice of measures, by imitating the computations in the book, *Basic Number theory*, by A. Weil.

Lemma D *For $K \in \{L, M\}$, we have*

$$\mathrm{vol}\left(\mathrm{K}^*\mathbb{A}_{\mathrm{K}^+,+}^*\backslash\mathbb{A}_\mathrm{K}^*\right) \;=\; \left(\frac{\mathrm{d_K}}{\mathrm{d_{K^+}}}\right)^{1/2} \mathrm{L}(1, \chi_{\mathrm{K}/\mathrm{K}^+}).$$

Thus (0) implies that

$$(6\pi)^{[M:\mathbb{Q}]}B\left(\prod_{v\in R}\frac{\prod_{w|v}\mathrm{vol}(\mathrm{U_w^{m_v}})}{\mathrm{vol}(\mathrm{U_v^{m_v}})}\right)\prod_{v\in S}\hat{f}_v(\underline{1}_v)\ \leq\ (6\pi)^{[L:\mathbb{Q}]}\prod_{v\in S}\hat{f}_v(\underline{1}_v),$$

which, when used in conjunction with (12), yields part (b) of Proposition B.

(In fact, we didn't really need Lemma D, as (0) could have been phrased directly in terms of the toric volumes.)

Back to the proof of Proposition A

We now have a positive constant C, independent of S, f, such that

$$C\mathrm{tr}\,\mathrm{r}^0_{\mathrm{M}}(\phi) - \mathrm{tr}\,\mathrm{r}^0_{\mathrm{L}}(\mathrm{f})\ \geq\ 0.$$

Recall that we started with a character χ_0 in $X_L(k)$ of conductor $\prod_{v\in R}P_v^{m_v}$, so $\mathrm{tr}\,\mathrm{r}^0_{\mathrm{L}}(\mathrm{f}) > 0$. Spectrally expanding, we get, using the strong positivity of f and ϕ,

$$\mathrm{tr}\,\chi_0(\mathrm{f})\ \leq\ \mathrm{C}\sum_{\chi'}\mathrm{tr}\,\chi'(\phi),$$

where χ' runs over Hecke characters in $\oplus_{k'=(k'_w)}X^0_M(k')$, with each $k'_w\in\{0,\pm k_v\}$ if $w\mid v$, and with conductor $\prod_{w\in R_M}P_w^{m'_w}$ with $m'_w\leq m_v$. In any case the number of χ' which intervene are finite in number.

Note that $\mathrm{tr}\,\chi_{\mathrm{v}}(\mathrm{f_v})$ is 1 when v is not in the set S_0 below S, and similarly, $\mathrm{tr}\,\chi'_{\mathrm{w}}(\phi_{\mathrm{w}}) = 1$ when $w\notin S$. At any archimedean place, the trace is 2π or 6π from (3) (as we have reduced to considering only those n_v, resp. n'_w, which are in $\{0,\pm k_v\}$. Putting these together, we obtain, using $\phi_{\sigma(v)} = f_v$ for $w=\sigma(v)\in S$,

(13)
$$\prod_{v\in S_0}\hat{f}_v(\chi_v)\ \leq\ A\sum_{\chi'}\prod_{v\in S_0}\hat{f}_v(\chi'_\sigma(v)),\ \ A=(2\pi)^{[M:\mathbb{Q}]-[L:\mathbb{Q}]}3^{[M:\mathbb{Q}]}C>0.$$

Since our f_v is, for each $v\in S_0$, of the form $\xi_v * \xi_v^\vee$, with $\xi_v = \sum_m b_m\mathrm{ch}_{\varpi_v^m U_v}$ with $b_m\geq 0$ (and 0 for all bt a finite number of $m\in\mathbb{Z}$), we get for any unramified character λ of F_v^*, using $\mathrm{vol}(\mathrm{U_v})=1$,

$$\hat{f}_v(\lambda) = |\hat{\xi}_v(\lambda)|^2 = |\sum_m b_m q_v^{-m}\lambda(\varpi_v)^m|^2. \tag{14}$$

We have freedom in choosing the b_m, as long as they are non-negative and are non-zero only for a finite number of m.

We have to show that for some χ', $\chi'(\varpi_v)$ equals $\chi(\varpi_v)$ at every v in S_0. Suppose not. Then there is a non-empty subset S_1 of S_0 such that

$$\chi'(\varpi_v)\ \neq\ \chi(\varpi_v),\ \forall\, v\in S_1. \tag{15}$$

And (13) becomes

$$\prod_{v\in S_1} |\hat{\xi}_v(\chi_v)|^2 \leq A \sum_{\chi'} \prod_{v\in S_1} |\hat{\xi}_v(\chi'_\sigma(v))|^2. \tag{16}$$

We get a contradiction once we prove the following

Lemma E *For every $v \in S_1$, and for any pair of positive real numbers α, β with $\alpha > \beta$, there exists ξ_v such that*

$$|\hat{\xi}_v(\chi_v)| > \alpha, \text{ and } \hat{\xi}_v(\chi'_v) < \beta, \ \forall \chi'.$$

Fix any $v \in S_1$, and enumerate the χ' in the relevant finite set as $\chi'_1, \ldots, \chi'_r$. Since χ_v, χ'_v are unitary, we may write $\chi_v(\varpi_v) = q_v^{it_0}$ and $\chi'_{j,v}(\varpi_v) = q_v^{it_j}$, $1 \leq j \leq r$. Clearly, t_0, t_j are uniquely defined only up to translation by $2\pi(\log q_v)^{\mathbb{Z}}$. Write $\theta_n = t_n(\log q_v)/2\pi$, for every $n = 0, 1, \ldots, r$. By hypothesis, $\theta_j \neq \theta_0$ for any $j \geq 1$. Moreover, by our choice of S, no chi'_v takes the value -1 on ϖ_v, so if we think of θ_n as lying in $[-\pi, \pi]$, it never takes the value 0 for any $n \geq 1$. If $\xi_v = \sum_m b_m \mathrm{ch}_{\varpi^m U_v}$ with $b_m \geq 0$, writing χ_v as $\chi'_{0,v}$ and putting $c_m = b_m q_v^{-m}$, we then have,

$$\hat{\xi}_v(\chi'_{n,v}) = g(\theta_n) := 2\sum_{m\geq 0} c_m e^{i\theta_n}, \ \forall n = 0, 1, \ldots, r. \tag{17}$$

Such a finite Fourier series $g(\theta)$ necessarily takes its maximum value at $\theta = 0$, as $c_m \geq 0$. The Lemma is easy to check wehn $\theta_0 = 0$, so we may assume that there is an $\varepsilon > 0$ such that θ_n lies in the set $J := ([-\pi, \pi] - (-\varepsilon, \varepsilon))$, for every $n = 0, 1, \ldots, r$. Choose an even, smooth, positive function h on $[-\pi, \pi]$ such that its absolute value satisfies the requisite inequalities at the θ_j, $0 \leq j \leq r$. We can also make h a positive definite function by choosing it to be sufficiently large at 0. Then the coefficients c_m are non-negative for the Fourier series representing h, which converges in $\mathcal{C}^\infty$-norm, implying uniform convergence of the series and the derivatives. So we may uniformly approximate h by a positive sum of characters. Since the inequalities satisfied by it form an open condition, the approximating positive character sum also satisfies the same inequalities at the θ'_j and at θ_0. □

The passage from S to a set of positive density

Fix $k = (k_v)$, $R, m = (m_v)$ as above. Let Y be the set of primes w of M which are of degree one over $\mathbb{Q}$ and where no character χ' of M of level dividing $\mathfrak{a} := \prod_{v\in R}\prod_{w|v} P_w^{m_v}$ of weight $k' = (k'_w)$ with $k'_w \in \{0, \pm k_{v(w)}\}$ has the value 1 on ϖ_w.

Lemma F *Y has positive density. If $k \neq 0$, it has density one.*

Proof of Lemma E First look at χ' of weight 0, when it defines a ray class character of conductor dividing $\mathfrak{a}$, and if the corresponding ray class group is a product $C_1 \times \cdots \times C_n$, with each C_j cyclic of generator x_j, Tchebotarev assures of the existence of a set Z_o, say, of primes P_w of positive density δ whose Frobenius elements are $(x_1, \ldots, x_n)$. (δ is at least $1/h_M(\mathfrak{a})$.) Now suppose χ' is of weight $k' \neq 0$. We claim that the set Z of primes P_w where $\chi'(\varpi_w) = 1$ has density zero. Indeed, if Z hs positive density, as χ' and the trivial character agree on Z, their ratio will need to be of finite order, hence of weight 0, contradicting the fact that $k' \neq 0$. The Lemma now follows because there are only a finite number of relevant k' and hence χ' (because of bounded ramification). □

Let Y_0 denote the set of rational primes below Y, and write elements of Y_0 as $p_1, p_2, \ldots, p_n, \ldots$ with $p_i < p_j$ if $i < j$. Then enumerate the primes of M lying in Y as $\{v_{i,j} \mid N(w_{i,j}) = p_i\}$. Given a finite set S_0 of Y_0 and a section $\sigma : S_0 \to Y$, i.e., a choice of a place $w = \sigma(p)$ for every $p \in S_0$, it defines uniquely a finite set S of the type we have considered above, and every such S arises this way. Let $\chi'(S)$ $(= \chi'(S_0, \sigma))$ be the character of M attached above to χ.

Lemma G *There exists a (unique) character χ_M from among the family of characters $\{\chi'(S_0, \sigma)\}$ indexed by (S_0, σ), such that for all place w in Y but a finite number, we have*

$$\chi_{M,w} = \chi_{v(w)}.$$

Proof of Lemma G

Let N denote the cardinality of the finite set of anti-cyclotomic Hecke characters χ' of M of conductor diviing $\mathfrak{a}$ and weight $k' = (k'_w)$ such that for each archimedean w above v, $k'_w \in \{0, \pm k_v\}$. Put

$$\nu := \frac{\log N}{\log\left(1 + (1/(d-1))\right)}$$

Suppose the Proposition is false. Then there exists some integer $m > \nu$ such that, for any π_E^s, $(*)$ fails to hold at $\nu + 1$ (or more) places v_j with $j \leq m$. Fix such an m, and define an equivalence relation on the finite set $\mathcal{S}_m$ of restrictions of sections, again denoted by s, to $\{v_1, \ldots, v_m\}$, by setting

$$s \sim s' \iff \pi^s_{E,s(v_j)} \simeq \pi^{s'}_{E,s'(v_j)}.$$

Then the set $\mathcal{S}_m/\sim$ of equivalence classes $[s]$ in $\mathcal{S}_m$ has cardinality bounded above by N. Of course, $|\mathcal{S}_m| = d^m$. Then, by the pigeon-hole principle, at least one class $[s]$, call it $\mathcal{O}_m$, must have at least d^m/N members in $\mathcal{S}_m$.

Note that if $(*)$ fails to hold at r places in $\mathcal{S}_m$, then the cardinality of $\mathcal{O}_m$ can be at most $(d-1)^r d^{m-r} = \left(\frac{d-1}{d}\right)^r d^m$. Since $\mathcal{O}_m \geq \frac{d^m}{N}$, it follows that

$$\frac{1}{N} \geq \left(\frac{d-1}{d}\right)^r,$$

or equivalently,

$$r \leq \frac{\log N}{\log\left(1 + (1/(d-1))\right)} = \nu.$$

Since $r \geq \nu + 1$, we get the desired contradiction.

□

This completes the proof of the main Theorem.

October 26, 2016

Dinakar Ramakrishnan
253-37 Caltech
Pasadena, CA 91125
(visiting Princeton, NJ - Fall 2016 - on a Simons Fellowship)