INTRODUCTION

For any integer $n > 1$, let $\mathcal{H}_C^n$ be the $n$-dimensional complex hyperbolic space, represented by the unit ball in $\mathbb{C}^n$ equipped with the Bergman metric of constant holomorphic sectional curvature $-4/(n + 1)$, on which the real Lie group $U(n, 1)$ acts in a natural way. Given a lattice $\Gamma$ in $U(n, 1)$, we consider the quotient $Y_\Gamma := \Gamma \backslash \mathcal{H}_C^n$.

Let $M$ be a CM quadratic extension of a totally real number field $F$ of degree $d$ and ring of integers $\mathcal{O}_F$, and let $G$ be a reductive group over $F$ defined by a hermitian form on $M^3$ of signature $(n, 1)$ at one infinite place $i$ and $(n + 1, 0)$ at the others. A subgroup $\Gamma \subset G(F)$ is arithmetic, if it is commensurable with $G(\mathcal{O}_F)$, and we will use the same notation for its image for its image $\iota(\Gamma) \subset G(F_i) = U(n, 1)$, which is by definition an arithmetic lattice. By the Baily-Borel theorem the arithmetic quotient $Y_\Gamma$ has a structure of a normal, quasi-projective variety, in fact defined over a number field. It is projective if, and only if, $G$ is anisotropic.

We will prove below three results for arithmetic $Y_\Gamma$, the first one concerning the compact case, the second dealing with the general non-compact case, and the third giving precise and explicit conclusions in the case of (non-compact) surfaces.

A quasi-projective variety $X$ defined over a number field $k$ is said to be Mordellic if, and only if, for any finite extension $k'$ of $k$, the set $X(k')$ of $k'$-rational points of $X$ is finite. Lang conjectured in [L, Conjecture VIII.1.2] (see also [T, p.xviii]) that any hyperbolic, projective $X$ is Mordellic; this is consistent with the general philosophy of Vojta.

**Theorem 0.1.** If $\Gamma$ is an arithmetic cocompact subgroup of $U(2, 1)$ such that all its torsion elements are scalar, then the projective surface $Y_\Gamma$ is Mordellic.

Note that even though this theorem only concerns arithmetic subgroups, because $F$ and $M$ can vary, it can be applied to infinitely many pairwise non-commensurable cocompact discrete subgroups in $U(2, 1)$.

Our results do not apply, however, to the analogous case of (a cocompact discrete subgroup of) a unitary group defined by a division algebra of dimension 9 over an imaginary quadratic field $M$ with an involution of the second kind. In that case it is known that the
Albanese of $Y_\Gamma$ is zero for any congruence subgroup $\Gamma$; this was proved by Rapoport and Zink [RZ] under a ramification hypothesis, and later by Rogawski [R1] using a different method, without the hypothesis.

After an earlier version of this paper was written, we came to know of Ullmo’s work [U] on Shimura varieties of abelian type which contains Theorem 0.1 for suitable covers since Picard modular surfaces of congruence type are in this class. We hope that our (quite different) approach is nevertheless of interest since we do not make use of the Shafarevich conjecture. Instead we use certain key theorems of Rogawski [R1, R2] and of Faltings [F1, F2] (see Propositions 2.4 and 3.2). We have also come to know recently of some related work of Yeung [Ye] on rational points on ball quotients.

Consider now the more complicated case when $G$ is isotropic, which necessarily implies that $F = \mathbb{Q}$ and $M$ is imaginary quadratic. Its toroidal compactification $X_\Gamma$ is not hyperbolic even if $\Gamma$ is torsion free (for example, if $n = 2$ then $X_\Gamma$ is a union of $Y_\Gamma$ with a finite number of elliptic curves indexed by the cusps). However, by a result of Mumford [M, Proposition 4.2], for $\Gamma$ neat, $X_\Gamma$ is always of log general type. This implies that for $\Gamma$ sufficiently small, $X_\Gamma$ is of general type. The Bombieri-Lang conjecture predicts that the points of $X_\Gamma$ over a given number field are not Zariski dense. This we establish in Proposition 3.2 using a deep Theorem due to Faltings, when $X_\Gamma$ is smooth and does not admit a dominant map to its Albanese variety, in particular when its irregularity is $> n$. This allows us to solve an alternative of Ullmo and Yafaev [UY] regarding the Lang locus.

**Theorem 0.2.** For all $\Gamma \subset U(n, 1)$ arithmetic and sufficiently small, $Y_\Gamma$ is Mordellic.

Keeping the assumption that $M$ is imaginary quadratic, say of fundamental discriminant $-D$, and supposing in addition that $n = 2$, we restrict our attention to the corresponding locally symmetric spaces $Y_\Gamma$, called Picard modular surfaces. For every ideal $\mathfrak{N} \subset \mathcal{O}_M$ we consider the congruence subgroups $\Gamma(\mathfrak{N})$ and $\Gamma_1(\mathfrak{N})$ (see Definition 1.5). We state now our main theorem.

**Theorem 0.3.** Let $\mathfrak{D} | D$ be the conductor of a simplest Hecke character over $M$ (see (12)).

(i) Let $\Gamma = \begin{cases} \Gamma(\mathfrak{D}) & , \text{if } D \notin \{3, 4, 7, 8, 11, 15, 19, 20, 23, 24, 31, 39, 43, 47, 67, 71, 163\}, \\ \Gamma(\mathfrak{D}^2) & , \text{if } D \in \{8, 15, 20, 23, 24, 31, 39, 47, 71\}, \\ \Gamma(\mathfrak{D})^2 & , \text{if } D \in \{3, 4, 7, 11, 19, 43, 67, 163\}. \end{cases}$

Then $Y_\Gamma$ is Mordellic, while $X_\Gamma$ is a minimal surface of general type.

(ii) Let $N > 2$ be a prime inert in $M$ and not equal to 3 when $D = 4$. Then $Y_{\Gamma(N)} \cap \Gamma_1(\mathfrak{D})$ is Mordellic, while $X_{\Gamma(N)} \cap \Gamma_1(\mathfrak{D})$ is a minimal surface of general type.

At the heart of our proof are some arithmetic computations yielding, for each imaginary quadratic field $M$, explicit congruence subgroups $\Gamma$ such that $X_\Gamma$ does not admit a dominant
map to its Albanese variety. A geometric ingredient of the proof is a result of Holzapfel et al that $X_\Gamma$ is of general type, implying by a theorem of Nadel [N] that $Y_\Gamma$ does not contain curves of genus $\leq 1$.

In a related paper [DR], we will investigate further the structure of the Albanese variety of Picard modular surfaces $X_\Gamma$ for specific families of congruence subgroups $\Gamma$, and exhibit the abelian varieties $B(p)$ of Gross as Albanese quotients (up to isogeny) and also explain their relationship to the elliptic curves over the cusps. In addition, we will exploit the Albanese quotients coming from residual automorphic forms, and present an alternate method to deal with the possible curves on genus $\leq 1$ by means of a formal immersion at special points on the curves at infinity by analyzing the Fourier-Jacobi coefficients. This leads to another approach for proving finiteness of rational points on the open surface. In a third paper we will analyze the situation over $\text{Spec}(\mathcal{O}_M)$ with a view towards showing the paucity, even lack, of rational points on the open surface for a suitable infinite class of congruence subgroups. The ultimate aim of our program is to establish a weaker analogue of Mazur’s theorem on modular curves, with a consequence for the boundedness of torsion for principally polarized abelian 3-folds with multiplication by $\mathcal{O}_M$.

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1. Basics: Lattices, general type and neatness

Definition 1.1. Given a discrete subgroup $\Gamma \subset U(n, 1)$ we let $\bar{\Gamma} = \Gamma/\Gamma \cap U(1)$ denote its image in the adjoint group $\text{PU}(n, 1) = U(n, 1)/U(1)$, where $U(1)$ is centrally embedded in $U(n, 1)$. We put $Y_\Gamma := Y_\Gamma = \Gamma \backslash \mathcal{H}_P^n$.

Conversely any discrete subgroup $\bar{\Gamma} \subset \text{PU}(n, 1) = \text{PSU}(n, 1)$ is the image of a discrete subgroup of $U(n, 1)$, namely $U(1)\bar{\Gamma} \cap \text{SU}(n, 1)$.

Lemma 1.2. Let $\Gamma$ be a lattice in $U(n, 1)$.

(i) The analytic variety $Y_\Gamma$ is an orbifold and one has the following implications:

$\Gamma$ neat $\Rightarrow \Gamma$ torsion free $\Rightarrow \bar{\Gamma}$ torsion free $\Rightarrow Y_\Gamma$ is a hyperbolic manifold.
(ii) Assume that $\bar{\Gamma}$ is torsion free. Then the natural projection $\mathcal{H}_C^n \rightarrow Y_\Gamma$ is an étale covering with deck transformation group $\bar{\Gamma}$. Moreover for every finite index normal subgroup $\Gamma'$ of $\Gamma$ the natural morphism $Y_{\Gamma'} \rightarrow Y_\Gamma$ is an étale covering of group $\bar{\Gamma}/\bar{\Gamma}'$.

Proof. The stabilizer in $U(n,1)$ of any point of $\mathcal{H}_C^n$ is a compact group, hence its intersection with the discrete subgroup $\Gamma$ is finite, showing that $Y_\Gamma$ is an orbifold. If $\Gamma$ is neat, then no element of it has a non trivial root of unity as an eigenvalue, in particular $\Gamma$ is torsion free. Since $\Gamma \cap U(1)$ is finite, this implies that $\bar{\Gamma}$ is torsion free too. Under the latter assumption, $\Gamma \cap U(1)$ acts trivially on $\mathcal{H}_C^n$, and $\bar{\Gamma}$ acts freely and properly discontinuously on it, hence $Y_\Gamma$ is a manifold. Since $\mathcal{H}_C^n$ is simply connected, it is a universal covering space of $Y_\Gamma$ with group $\bar{\Gamma}$. In particular, $Y_\Gamma$ is hyperbolic. The last claim follows from the exact sequence: $1 \rightarrow \bar{\Gamma}' \rightarrow \bar{\Gamma} \rightarrow \bar{\Gamma}/\bar{\Gamma}' \rightarrow 1$.  

Proposition 1.3. Assume that $\bar{\Gamma}$ is cocompact and torsion free. Then the projective variety $Y_\Gamma$ is of general type and can be defined over a number field.

Proof. The existence of the positive Bergman metric on $\mathcal{H}_C^n$ implies by the Kodaira embedding theorem that $Y_\Gamma$ has ample canonical bundle, which results in $Y_\Gamma$ being of general type; it even implies that any subvariety is of general type. For surfaces one may alternately use the hyperbolicity of $Y_\Gamma$ to rule out all the cases in the Enriques-Kodaira classification where the Kodaira dimension is less that 2, thus showing that $Y_\Gamma$ is of general type.

Calabi and Vesentini [CV] have proved that $Y_\Gamma$ is locally rigid, hence by Shimura [Sh1] it can be defined over a number field. For the convenience of the reader, we will provide a second, more direct proof when $n = 2$ and $\Gamma$ is arithmetic, based on Yau’s algebra-geometric characterization of compact Kähler surfaces covered by $\mathcal{H}_C^2$. Since $Y_\Gamma$ has an ample canonical bundle it can be embedded in some projective space, hence is algebraic over $\mathbb{C}$ by Chow. Since $Y_\Gamma$ is uniformized by $\mathcal{H}_C^2$, the Chern numbers $c_1, c_2$ of its complex tangent bundle satisfy the relation $c_1^2 = 3c_2$. Since everything can be defined algebraically, for any automorphism $\sigma$ of $\mathbb{C}$, the variety $Y_{\Gamma}^\sigma$ also has ample canonical bundle and $c_1^\sigma = 3c_2^\sigma$. By a famous result of Yau [Y, Theorem 4], this is equivalent to the fact that $Y_{\Gamma}^\sigma$ may be realized as $\bar{\Gamma}/\mathcal{H}_C^2$ for some cocompact discrete irreducible torsion free subgroup $\bar{\Gamma}^\sigma$.

Since $\bar{\Gamma}$ is arithmetic, it has infinite index in its commensurator in $PU(2,1)$, denoted $\text{Comm}(\bar{\Gamma})$. For every element $g \in \text{Comm}(\bar{\Gamma})$ there is a Hecke correspondence

$Y_\Gamma \leftarrow Y_{\Gamma \cap g^{-1} \Gamma g} \xrightarrow{\sim} Y_{g^{-1} \Gamma g^{-1} \Gamma} \rightarrow Y_\Gamma$

(1)

and the correspondences for $g$ and $g'$ differ by an isomorphism $Y_{g^{-1} \Gamma g^{-1} \Gamma} \xrightarrow{\sim} Y_{g^{-1} \Gamma g^{-1} \Gamma}$ over $Y_\Gamma$ if, and only if, $g' \in \Gamma g$. By Chow (1) is defined algebraically, hence yields a
correspondence on $Y^\sigma_\Gamma = Y^\sigma_{\bar{\Gamma}}$:

$$Y^\sigma_{\bar{\Gamma}} \leftarrow Y^\sigma_{\bar{\Gamma}_1} \sim Y^\sigma_{\bar{\Gamma}_2} \rightarrow Y^\sigma_{\bar{\Gamma}},$$

for some finite index subgroups $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ of $\bar{\Gamma}$. By the universal property of the covering space $H^2_C$, the middle isomorphism is given by an element of $g \in \text{PU}(2,1) \simeq \text{Aut}(H^2_C)$. Since $\text{Aut}(H^2_C) \simeq \bar{\Gamma}_i (i = 1, 2)$, it easily follows that $\bar{\Gamma}_2 = g \bar{\Gamma}_1 g^{-1}$, and by applying $g^{-1}$ one sees that $\bar{\Gamma}_1 = \bar{\Gamma}^\sigma \cap g^{-1} \bar{\Gamma}^\sigma g$. It follows that $g \in \text{Comm}_\Gamma(\bar{\Gamma}^\sigma)$ and one can check that $g' \in \bar{\Gamma}^\sigma g$ if, and only if, $g' \in \bar{\Gamma} g$. Therefore $\text{Comm}(\bar{\Gamma}^\sigma)/\bar{\Gamma}^\sigma \simeq \text{Comm}(\bar{\Gamma})/\bar{\Gamma}$ is infinite too, which by a major theorem of Margulis implies that $\Gamma^\sigma$ is arithmetic, providing an alternative proof of a result of Kazhdan.

Consider now the action of $\text{Aut}(\mathbb{C})$ on the set of equivalence classes of cocompact arithmetic subgroups $\Gamma$ (modulo their center and up to conjugation by an element of $U(2,1)$). The group $U(2,1)$ has only countably many $\mathbb{Q}$-forms, classified by central simple algebras of dimension 9 over a CM field, endowed with an involution of a second kind and verifying some conditions at infinity (see [PR, pp. 87-88]). Finally, there are only countably many arithmetic subgroups for a given $\mathbb{Q}$-form, since those are all finitely generated and contained in their common commensurator, which is countable. It follows that $\Gamma$ is fixed by an open subgroup of $\text{Aut}(\mathbb{C})$, allowing one to conclude that $Y^\Gamma$ is defined over a number field.

It is a well known fact that any compact orbifold admits a finite cover which is a manifold. We will now provide such a cover explicitly for arithmetic quotients. Recall that an arithmetic subgroup $\Gamma \subset G(F)$ is a congruence subgroup if there exists an integer $N$ such that $\Gamma$ contains the principal congruence subgroup of level $N$, defined as:

$$\Gamma(N) = \ker(G(O_F) \to G(O_F/NO_F)).$$

The following lemma is well-known (see [H1, Lemma 4.3]).

**Lemma 1.4.** For any integer $N > 2$ the group $\Gamma(N)$ is neat.

Since $G$ is defined by a hermitian form on $M^{n+1}$, we have an embedding $G(O_F) \hookrightarrow \text{GL}(n+1, O_M)$, through which we may view elements of $G(F)$ as $(n+1) \times (n+1)$ matrices. This allows us to define the following finer congruence subgroups.

**Definition 1.5.** For every ideal $\mathfrak{N} \subset O_M$ we define the congruence subgroup $\Gamma(\mathfrak{N})$ (resp. $\Gamma_1(\mathfrak{N})$) as the kernel (resp. the inverse image of upper unipotent matrices) of the composed homomorphism:

$$G(O_F) \hookrightarrow \text{GL}(n+1, O_M) \to \text{GL}(n+1, O_M/O_M).$$

**Lemma 1.6.** Suppose that $n = 2$ and that $M$ is an imaginary quadratic field of fundamental discriminant $-D \notin \{-3, -4, -7, -8, -24\}$. Then $\Gamma_1(\sqrt{-D})$ is neat.
Proof. Suppose that the subgroup of \( C^\times \) generated by the eigenvalues of some \( \gamma \in \Gamma_1(\sqrt{-D}) \) contains a non-trivial root of unity \( \zeta \). If \( \gamma \) is elliptic then it is necessarily of finite order. Otherwise \( \gamma \) fixes a boundary point of \( \mathcal{H}_C^2 \subset \mathbb{P}^2(\mathbb{C}) \) hence is conjugated in \( \text{GL}(3, \mathbb{C}) \) to a matrix of the form \( \begin{pmatrix} 0 & \beta & \ast \\ \ast & 0 & \ast \\ 0 & 0 & \alpha^{-1} \end{pmatrix} \), where \( \beta \) is necessarily a root of unity. Hence, it both cases, one may assume that \( \zeta \) is an eigenvalue of \( \gamma \).

By the Cayley-Hamilton theorem we have \( [M(\zeta) : M] \leq 3 \) and since \( D \neq 7 \) we may assume (after possibly lifting \( \gamma \) to some power) that \( \zeta \) has order 2 or 3. By the congruence condition, each prime \( p \) dividing \( D \) has to divide also the norm of \( \zeta - 1 \), hence \( D \) can be only divisible by the primes 2 or 3. Hence \( D \in \{3, 4, 8, 24\} \) leading to a contradiction. \( \Box \)

2. Irregularity of arithmetic surfaces

Non-vanishing (and unboundedness) of irregularity for sufficiently small arithmetic subgroups has been known since the works of Kazhdan [K] and Shimura [Sh2, Theorem 8.1]. The starting point for the arithmetic application of this paper was our knowledge that Rogawski’s classification [R1, R2] of cohomological automorphic forms on \( G \) contributing to \( H^1(Y, \mathbb{C}) \) allowed one to prove such result for explicit congruence subgroups \( \Gamma \) which can be chosen in various ways.

Throughout this section we assume that \( n = 2 \).

2.1. Automorphic forms contributing to the irregularity. Given a neat congruence subgroup \( \Gamma \), we denote by \( q(Y) \) the irregularity of \( Y \), given by the dimension of \( H^0(Y, \Omega^1_{Y/\mathbb{C}}) \).

Fixing a maximal compact subgroup \( K_\infty \simeq (U(2) \times U(1)) \times U(3)^{d-1} \) of the reductive Lie group \( G_\infty = G(F \otimes_{\mathbb{Q}} \mathbb{R}) \simeq U(2,1) \times U(3)^{d-1} \), we obtain a decomposition:

\[
H^1(Y, \mathbb{C}) \simeq \bigoplus_{\pi_\infty} H^1(\text{Lie}(G_\infty), K_\infty; \pi_\infty)^{\oplus m(\pi_\infty, \Gamma)},
\]

where \( \pi_\infty \) runs over irreducible unitary representations of \( G_\infty \) occurring with multiplicity \( m(\pi_\infty, \Gamma) \) in the discrete spectrum of \( L^2(\Gamma \backslash G_\infty) \). When \( \Gamma \) is cocompact, the entire \( L^2 \)-spectrum is discrete and this decomposition follows from [BW, Chap.XIII]. When \( \Gamma \) is a non-cocompact lattice, one gets such a decomposition \( a \text{ priori} \) only for the \( L^2 \)-cohomology of \( Y \). However, it is known for Picard modular surfaces that \( H^1(Y, \mathbb{C}) \) is isomorphic to the middle intersection cohomology (in degree 1) of the Baily-Borel compactification of \( Y \), which is in turn isomorphic to the \( L^2 \)-cohomology \( H^1_{(2)}(Y, \mathbb{C}) \) (see [MR, §1]). In addition, \( H^1_{(2)}(Y, \mathbb{C}) \) is isomorphic to \( H^1(X, \mathbb{C}) \), where \( X \) is a smooth toroidal compactification of \( Y \) (see loc. cit.). In particular, \( H^1(Y, \mathbb{C}) \) admits a pure Hodge structure and its dimension is given by \( 2q(Y) \).
At the distinguished Archimedean place \( \nu \) where \( G(F_\nu) = U(2,1) \) there are exactly two irreducible non-tempered unitary representations of \( G(F_\nu) \), denoted \( \pi_\nu^+ \) and \( \pi_\nu^- \), with non-zero relative Lie algebra cohomology in degree 1, while at the remaining infinite places, the only irreducible unitary representation with non-zero relative Lie algebra cohomology in degree 0 is the trivial one.

The restrictions to \( \mathbb{C}^\times \) of the Langlands parameters of \( \pi_\nu^+ \) and \( \pi_\nu^- \) are given by:

\[
z \mapsto \begin{pmatrix}
\bar{z} & 0 & 0 \\
0 & z/\bar{z} & 0 \\
0 & 0 & z^{-1}
\end{pmatrix} \in lG^0 = \text{GL}_3(\mathbb{C}),
\]

and its complex conjugate (see \([\text{La}, \text{p.62}]\)).

We will now introduce the adelic setting which is better suited for computing the irregularity. For \( K \) a neat open compact subgroup of \( G(A_F; f) \), where \( A_F; f \) denotes the ring of finite adeles of \( F \), we consider the adelic quotient

\[
Y_K := G(F) \setminus G(\mathbb{A}_F)/K_{\infty}K.
\]

Since \( G^1 := \ker(\det: G \to M^1) \) is simply connected and \( G^1_{\infty} \) is non-compact, by strong approximation (see \([\text{PR}, \text{Theorem 7.12}]\)) \( G^1(F) \) is dense in \( G^1(\mathbb{A}_F; f) \). It follows that \( Y_K \) is a finite disjoint union, indexed by the class group

\[
\pi_0(Y_K) \simeq A_M^1/M^1 \text{det}(K)M^1_{\infty},
\]

of surfaces \( Y_\Gamma \) for some neat congruence subgroups \( \Gamma \subset G(F) \) and (2) can be rewritten as:

\[
H^1(Y_K; \mathbb{C}) \simeq \bigoplus_\pi H^1(\text{Lie}(G_{\infty}), K_{\infty}; \pi_\infty) \hat{\otimes} m(\pi_f, K),
\]

where \( \pi = \pi_\infty \otimes \pi_f \) runs over all automorphic representation of \( G(\mathbb{A}_F) \). By the above description of \( \pi_\infty \), and Rogawski’s multiplicity one theorem \([\text{R2}]\) one deduces the formula:

\[
q(Y_K) = \sum_{\pi \text{ automorphic}} \dim(\pi^K_F) = \sum_{\pi \text{ automorphic}} \dim(\pi^K_f),
\]

where \( 1 \otimes \ldots \otimes 1 \) denotes the trivial representation of \( U(3)^{d-1} \).

2.2. Rogawski’s theory. Rogawski \([\text{R1}, \text{R2}]\) gives an explicit description, in terms of global Arthur packets, of the automorphic representations \( \pi \) of \( G(\mathbb{A}) \) such that \( \pi_\nu \simeq \pi_\nu^+ \) and \( \pi_\nu = 1 \) at all Archimedean places \( v \neq \nu \), which we will now present.

Let \( G' \) denote the quasi-split unitary group associated to \( M/F \), so that \( G \) is an inner form of \( G' \) (note that \( G \simeq G' \) only for \( d = 1 \)).

Let \( \lambda \) be a unitary Hecke character of \( M \) whose restriction to \( F \) is the quadratic character associated to \( M/F \), and let \( \nu \) be a unitary character of \( \mathbb{A}_M^1/M^1 \).

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At a place $v$ of $F$ which does not split in $M$, the local Arthur packet $\Pi'(\lambda_v, \nu_v)$ consists of a square-integrable representation $\pi_s(\lambda_v, \nu_v)$ and a non-tempered representation $\pi_n(\lambda_v, \nu_v)$ of $G'$. Those can be described (see [R1, §12.2]) as the unique sub-representation and the corresponding (Langlands) quotient of the induction of the character on the standard upper-triangular Borel subgroup $B(F_v)$ which is trivial on the unipotent subgroup and given on the diagonal torus $T(F_v)$ by:

$$ (\bar{\alpha}, \beta, \alpha^{-1}) \mapsto \lambda_v(\bar{\alpha})|\alpha|_{M_v}^{3/2} \nu_v(\beta), \text{ where } \alpha \in M_v^x, \beta \in M_v^1. $$

If one considers unitary induction, then one has to divide the above character by the square root of the modular character of $B(F_v)$, that is to say by $(\bar{\alpha}, \beta, \alpha^{-1}) \mapsto |\alpha|_{M_v}$.

At a place $v$ of $F$ which splits in $M$, $G'(F_v)$ (resp. $(M \otimes F_v)\times$) can be identified with $GL(3, M_v)$ (resp. $M_v^x$) where $w$ is a place of $M$ dividing $v$. The local Arthur packet $\Pi'(\lambda_v, \nu_v)$ has a unique element $\pi_n(\lambda_v, \nu_v)$ which is induced from the character:

$$ \left( \begin{array}{ccc} h_2 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & h_1 \end{array} \right) \mapsto \lambda_w(\det(h_2))|\det(h_2)|_{M_v}^{3/2} \nu_w(h_1) $$

of a maximal parabolic of $G'(F_v)$ (see [R2, §1]).

For almost all $v$, $\pi_n(\lambda_v, \nu_v)$ is necessarily unramified. We set

$$ \Pi'(\lambda, \nu) = \{ \otimes_v \pi_v | \pi_v \in \Pi'(\lambda_v, \nu_v) \text{ for all } v \text{, and } \pi_v = \pi_n(\lambda_v, \nu_v) \text{ for almost all } v \} . $$

Recall that a CM type of $M$ is the choice, for each Archimedean places $v$ of $F$, of an isomorphism $M \otimes_{F_v} \mathbb{R} \simeq \mathbb{C}$. Suppose that $\lambda$ (resp. $\nu$) is algebraic of weight 1 (resp. $-1$) relatively to a CM type $\Phi$ of $M$, in the sense that:

$$ \lambda_\infty(z) = \prod_{v \in \Phi} \overline{z_v}, \text{ for all } z \in M_\infty \left( \text{resp. } \nu_\infty(z) = \prod_{v \in \Phi} z_v, \text{ for all } z \in M_\infty^x \right). $$

Denote by $\Xi$ the set of such pairs $(\lambda, \nu)$.

**Theorem 2.1** (Rogaswski [R1, R2]).

(i) For every $(\lambda, \nu) \in \Xi$, $\Pi'(\lambda, \nu)$ is a global Arthur packet for $G'$ such that for all infinite $v$, $\pi_n(\lambda_v, \nu_v) = \pi_n^+$ or $\pi_n^-$.

(ii) $\Pi'(\lambda, \nu)$ can be transferred to an Arthur packet $\Pi(\lambda, \nu)$ on $G$ such that $\Pi(\lambda_v, \nu_v) = \{ 1 \}$ at all Archimedean places $v \neq v$, and $\Pi(\lambda_v, \nu_v) = \Pi'(\lambda_v, \nu_v)$ at the remaining places.

(iii) Denote by $W(\lambda \nu_M) \in \{ \pm 1 \}$ the root number of the weight 3 algebraic Hecke character $\lambda \nu_M$, where $\nu_M(z) = \nu(z/z)$ is the base change character, and by $s(\pi)$ the number of finite places $v$ such that $\pi_v \simeq \pi_s(\lambda_v, \nu_v)$. Then

$$ \pi \in \Pi(\lambda, \nu) \text{ is automorphic if, and only if, } W(\lambda \nu_M) = (-1)^{d-1+s(\pi)}. $$
(iv) Any automorphic representation $\pi$ of $G(\mathbb{A})$ such that $\pi_v \simeq \pi_n^\pm$ and $\pi_v = 1$ at all Archimedean places $v \neq \iota$, belongs to $\Pi(\lambda, \nu)$ for some $(\lambda, \nu) \in \Xi$.

Proof. Let $H = U(2) \times U(1)$ be the unique elliptic endoscopic group, shared by $G'$ and all its inner forms over $F$. The embedding of L-groups $LH \hookrightarrow LG = LG'$ depends on the choice of a Hecke character $\mu$ of $M$, whose restriction to $F$ is the quadratic character associated to $M/F$, and allows one to transfer discrete L-packets on $H$ to automorphic L-packets on $G$ (see [R2, §13.3]). The character $\mu$ being fixed, any couple of characters $(\lambda, \nu) \in \Xi$ uniquely determine a (one-dimensional) character of $H$, whose endoscopic transfer is $\Pi'(\lambda, \nu)$ (see [R2, §1]).

Denote by $W_F$ (resp. $W_M$) the global Weil group of $F$ (resp. $M$). By loc.cit., the restriction to $W_M$ of the global Arthur parameter

$$W_F \times \text{SL}(2, \mathbb{C}) \to LG = \text{GL}(3, \mathbb{C}) \times \text{Gal}(M/F)$$

of $\Pi(\lambda, \nu)$ is given by the 3-dimensional representation $(\lambda \otimes \text{St}) \oplus (\nu_M \otimes 1)$, where St (resp. 1) is the standard 2-dimensional (resp. trivial) representation of $\text{SL}(2, \mathbb{C})$. Comparing this with the local parameter at infinity (3) yields $\pi_n^s(\lambda_v, \nu_v) = \pi_n^\pm$ at each infinite place $v$. It follows that for every archimedean place $v$, $\Pi'(\lambda_v, \nu_v)$ is a packet containing a discrete series representation of $G'_v$, and hence by [R1, §14.4], there will be a corresponding Arthur packet $\Pi(\lambda, \nu)$ of representations of $G(\mathbb{A}_F)$ such that at any archimedean place $v \neq \iota$, $\Pi(\lambda_v, \nu_v)$ is a singleton consisting of a finite-dimensional representation of the compact real group $G(F_v) = U(3)$. In the notation of [R2, p.397] the representations $\pi_n^+$ and $\pi_n^-$ have parameters $(r, s) = (1, -1)$ and $(r, s) = (0, 1)$, respectively, and hence, by the recipe on the same page, the highest weight of the associated finite-dimensional representation equals $(1, 0, -1)$. Therefore at every Archimedean $v \neq \iota$ we have $\Pi(\lambda_v, \nu_v) = \{1\}$.

So far we have established (i) and (ii), while (iii) is the content of [R2, Theorem 1.1].

Conversely, any $\pi$ as in (iv) is discrete, hence belongs to a Arthur packet $\Pi$ on $G$, which can be transferred to an Arthur packet $\Pi'$ on $G'$ (see [R1, Proposition 14.6.2] and [R1, §14.4]). By definition $\Pi_v = \Pi'_v$ at $v = \iota$ and at all the finite places $v$. In particular $\pi_n^\pm \in \Pi_v = \Pi'_v$, hence $\Pi'$ arises by endoscopy from $H$, that is to say equals $\Pi'(\lambda_v, \nu_v)$ for some unitary Hecke character $\lambda$ of $M$ whose restriction to $F$ is the quadratic character associated to $M/F$, and some unitary character $\nu$ of $A^1_M/M^1$ (see [R1, Theorem 13.3.6]). Since $\Pi_v = \{1\}$ for all Archimedean places $v \neq \iota$, by the above mentioned recipe $\Pi(\lambda_v, \nu_v)$ contains either $\pi_n^+$ or $\pi_n^-$, implying that $\lambda$ (resp. $\nu$) is algebraic of weight 1 (resp. $-1$) relative to a unique choice of a CM type $\Phi$ of $M$. \hfill $\Box$

2.3. Levels of induced representations. Let $p$ be a prime of $F$ divisible by a unique prime $\mathfrak{p}$ of $M$ and let $\mathbb{F}_q$ be the residue field $\mathcal{O}_F/\mathfrak{p}$. In this section we exhibit open compact
subgroups \( K \) of \( G(F_p) \) for which \( \pi_n(\lambda_p, \nu_p) \) (resp. \( \pi_s(\lambda_p, \nu_p) \)) admit a non-zero \( K \)-invariant subspace, and compute in some cases the exact dimension of this space.

For every integer \( n \geq 1 \), we define the open compact subgroup \( K_1(\mathfrak{F}^n) \) (resp. \( K_1(\mathfrak{F}^n)_\mathfrak{M} \)) of \( G(F_p) \) as the kernel (resp. the inverse image of upper unipotent matrices) of the composed homomorphism:

\[
G(O_{F_p}) \hookrightarrow GL(n+1, O_{M, \mathfrak{M}}) \rightarrow GL(n+1, O_M/\mathfrak{F}^n).
\]

**Lemma 2.2.** Let \( n \geq 1 \) be an integer such that the character \((7)\) is trivial on \( K_1(\mathfrak{F}^n) \cap T(F_p) \). Then both \( \pi_n(\lambda_p, \nu_p) \) and \( \pi_s(\lambda_p, \nu_p) \) have non-zero fixed vectors under \( K_1(\mathfrak{F}^n) \).

**Proof.** Let \( J \) denote the Jacquet functor sending admissible \( G(F_p) \)-representations to admissible \( T(F_p) \)-representations. The Jacquet functor is exact and its basic properties imply:

\[
\begin{align*}
J(\pi_s(\lambda_p, \nu_p)) : (\bar{\alpha}, \beta, \alpha^{-1}) \rightarrow \lambda_p(\bar{\alpha})\nu_p(\beta)|\alpha|_{M_p}^{3/2} = \lambda_p(\bar{\alpha})\nu_p(\beta)|\alpha|_{M_p}^{1/2} \cdot |\alpha|_{M_p}, \\
J(\pi_n(\lambda_p, \nu_p)) : (\bar{\alpha}, \beta, \alpha^{-1}) \rightarrow \lambda_p(\bar{\alpha})\nu_p(\beta)|\alpha|_{M_p}^{1/2} = \lambda_p(\alpha^{-1})\nu_p(\beta)|\alpha|_{M_p}^{-1/2} \cdot |\alpha|_{M_p}.
\end{align*}
\]

One knows that the pro-p Iwahori subgroup \( K_1(\mathfrak{F}^n) \) admits an Iwahori decomposition:

\[
K_1(\mathfrak{F}^n) = (K_1(\mathfrak{F}^n) \cap N(F_p)) \cdot (K_1(\mathfrak{F}^n) \cap T(F_p)) \cdot (K_1(\mathfrak{F}^n) \cap \check{N}(F_p)),
\]

where \( N(F_p) \) (resp. \( \check{N}(F_p) \)) denotes the unipotent of the standard (resp. opposite) Borel containing \( T(F_p) \). This is proved for the principal congruence subgroup \( K(\mathfrak{F}^n) \) in [Cs, Proposition 1.4.4] and the extension to \( K_1(\mathfrak{F}^n) \) is an easy exercise. Now by [Cs, Proposition 3.3.6], given any admissible \( G(F_p) \)-representation \( V \), one has a canonical surjection:

\[
V^{K_1(\mathfrak{F}^n) \cap T(F_p)} \twoheadrightarrow J(V)^{K_1(\mathfrak{F}^n) \cap T(F_p)}.
\]

Since both characters in \((9)\) are trivial on \( K_1(\mathfrak{F}^n) \cap T(F_p) \), the claim follows. \( \square \)

**Lemma 2.3.** Suppose that \( p \) is inert in \( M \) and that \((\lambda_p, \nu_p)\) is unramified. Then the dimension of the \( K(p) \)-fixed subspace of \( \pi_s(\lambda_p, \nu_p) \) (resp. \( \pi_n(\lambda_p, \nu_p) \)) equals \( q^3 \) (resp. \( 1 \)).

**Proof.** Since \((\lambda_p, \nu_p)\) is unramified, restriction to the standard hyperspecial maximal compact subgroup \( K_p \) of \( G(F_p) \) yields, by Iwasawa decomposition, the following exact sequence:

\[
0 \rightarrow \pi_s(\lambda_p, \nu_p)|_{K_p} \rightarrow Ind_{B(F_p) \cap K_p}^{K_p}\pi(1) \rightarrow \pi_n(\lambda_p, \nu_p)|_{K_p} \rightarrow 0.
\]

The subspace of \( K(p) \)-invariant vectors in \( Ind_{B(F_p) \cap K_p}^{K_p}\pi(1) \) identifies naturally with the space of \( \mathbb{C} \)-valued functions on the set:

\[
(B(F_p) \cap K_p) \backslash K_p/K(p) \simeq B(\mathbb{F}_q) \backslash G(\mathbb{F}_q),
\]

on which \( K_p/K(p) = G(\mathbb{F}_q) \) acts by right translation. By Iwahori decomposition, since \( G(\mathbb{F}_q) \) has rank one, the representation \( Ind_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)}\pi(1) \) has exactly two irreducible constituents which are the trivial representation and the Steinberg representation. Since both \( \pi_n(\lambda_p, \nu_p)|_{K(p)} \)
and $\pi_s(\lambda_p, \nu_p)^K(p)$ are non-zero by Lemma 2.2, and since $\pi_s(\lambda_p, \nu_p)^K = 0$, it follows that $\pi_s(\lambda_p, \nu_p)^K(p)$ (resp. $\pi_s(\lambda_p, \nu_p)^K(p)$) is isomorphic to the trivial (resp. Steinberg) representation of $G(F_q)$, hence its dimension equals 1 (resp. $q^3$).

2.4. Irregularity growth. The positivity of $q(Y_{\Gamma})$ is an essential ingredient in the proof of our Diophantine results. Each step of the proof of Proposition 2.4 can be carried out explicitly providing a precise level $\Gamma$, depending on $M$, at which $q(Y_{\Gamma}) > r$. After the completion of the work on this paper, we learned of Marshall's interesting work [Ma] giving sharp asymptotic bounds for $q(Y_{\Gamma})$ when $\Gamma$ shrinks, also by using Rogawski's theory.

**Proposition 2.4.** For any arithmetic subgroup $\Gamma$ of $G(F)$ and for any $r > 0$, there is a finite index torsion free subgroup $\Gamma'$ of $\Gamma$ such that $q(Y_{\Gamma'}) > r$.

**Proof.** Note that it suffices to find a neat congruence subgroup $\Gamma'$ such that $q(Y_{\Gamma'}) > r$, since then, for any arithmetic $\Gamma$, the natural morphism $Y_{\Gamma \cap \Gamma'} \to Y_{\Gamma'}$ is finite and surjective, hence $q(Y_{\Gamma \cap \Gamma'}) \geq q(Y_{\Gamma'}) > r$.

**Lemma 2.5.** For any CM extension $M/F$ and any CM type $\Phi$, there exists an algebraic Hecke character $\lambda$ of weight 1 and CM type $\Phi$.

**Proof.** Consider the character on $M^\times_{\infty}$ given by $\lambda_{\infty}(z) = \prod_{v \in \Phi} \frac{\bar{z}_v}{|z_v|}$. Since $M/F$ is a CM extension, the index $m$ of $(O_F^X)^2$ in $O_M^X$ is finite, and $\lambda_{\infty}$ is trivial on $(O_M^X)^m$. By [C, Théorème 1] there exists an open compact subgroup $U$ of $\mathbb{A}_{M, f}$ such that $U \cap O_M^X \subset (O_M^X)^m$, hence $\lambda_{\infty}$ can be extended (trivially) to $M^X UM_{\infty}^\times$. Finally, since $\mathbb{A}_M^X/M^X UM_{\infty}^\times$ is a finite abelian (class) group, there exists a character $\lambda$ of $\mathbb{A}_M^X/M^X$ extending $\lambda_{\infty}$. 

Let $\Pi(\lambda, \lambda_{M}^{-1, 1})$ be the global Arthur packet on $G$ associated to a character $\lambda$ as in the lemma, and let $v_0$ be a finite place of $F$ which does not split in $M$. Choose a open compact subgroup $K(\lambda) = \prod_v K(\lambda)_v$ of $G(\mathbb{A}_{F, f})$ such that $\pi_s^{K(\lambda), v_0} \neq 0$ and $\pi_{n, v}^{K(\lambda)} \neq 0$ for all finite places $v$, with $K(\lambda)_v$ being the standard hyperspecial maximal compact for all $v \neq v_0$ relatively prime to the conductor of $\lambda$.

Choose a finite place $p$ of $F$ inert in $M$ relatively prime to $v_0$ and to the conductor of $\lambda$, such that $q = |O_F/p| > r$. Let $K(\lambda, p)$ be the subgroup of $K(\lambda)$ with the maximal compact $K(\lambda)_p$ replaced by $K(p)$.

Consider an element $\pi = \otimes_v \pi_v \in \Pi(\lambda, \lambda_{M}^{-1, 1})$ such that $\pi_\iota = \pi_{n, v_0}^\iota$, $\pi_v = 1$ for every infinite place $v \neq \iota$, $\pi_p = \pi_s(\lambda_p, \nu_p)$, $\pi_v = \pi_{n, v}$ for every finite $v \neq p, v_0$, and finally:

\[
\pi_{v_0} = \begin{cases} 
\pi_{n, v_0}, & \text{if } W(\lambda^2) = (-1)^d, \text{ and} \\
\pi_{s, v_0}, & \text{if } W(\lambda^2) = (-1)^{d-1}. 
\end{cases}
\]
By Theorem 2.1(iii), $\pi$ is automorphic, and by Lemma 2.3 we have $\dim(\pi^K_{f(\lambda,p)}) \geq q^3$. By (5), for every character $\chi$ of the finite abelian group $\pi_0(Y_{K(\lambda,p)})$, one still has $\dim(\pi_f \otimes \chi)^K_{(\lambda,p)} \geq q^3$. By (6) we have $q(Y_{K(\lambda,p)}) \geq q^3|\pi_0(Y_{K(\lambda,p)})|$, hence there must exist a connected component $Y_{T'}$ of $Y_{K(\lambda,p)}$ such that $q(Y_{T'}) \geq q^3 > r$. \qed

2.5. Irregularity at low level. For the rest of this section $M$ is imaginary quadratic.

Let $\tilde{K}$ be an open compact subgroup of $\tilde{G}(A_{\mathbb{Q},f})$, where $\tilde{G} \supseteq G$ is the group of unitary similitudes, and let $K = \tilde{K} \cap G(A_{\mathbb{Q},f})$. Then Shimura variety

$$S_{\tilde{K}}(\mathbb{C}) = \tilde{G}(\mathbb{Q}) \backslash \mathcal{H}^2_{\mathbb{C}} \times \tilde{G}(A_{\mathbb{Q},f}) / \tilde{K}$$

has a canonical model $S_{\tilde{K}}$ over its reflex field $M$ and the set of geometrically connected components $\pi_0(S_{\tilde{K}} \times_M \tilde{M}) \simeq \pi_0(S_{\tilde{K}}(\mathbb{C}))$ is a principal homogeneous space under $\text{Gal}(M'/M)$ for some abelian extension $M'$ of $M$ (see [Go, §4]). Hence they are all Galois conjugates, and in particular share the same irregularity.

By [Go, Lemma 2.4] the identity component of $S_{\tilde{K}}(\mathbb{C})$ can be identified with $Y_{T'}$, where

$$\Gamma = \tilde{G}(\mathbb{Q}) \cap (\tilde{K} \cdot \tilde{G}(\mathbb{R})).$$

Since for any $g \in \tilde{G}(\mathbb{R}) = \text{GU}(2,1)$ we have $\nu(g)^3 = |\det(g)|^2 > 0$, implying $\nu(g) \in \mathbb{R}^+_3$, we deduce that $\nu(\Gamma) \subset \mathbb{Q}^\times \cap \tilde{\mathbb{Z}}^\times \mathbb{R}^+_3 = \{1\}$. Hence $\Gamma = G(\mathbb{Q}) \cap (K \cdot G(\mathbb{R}))$ and the identity component of $Y_{K}$ can also be identified with $Y_{T'}$. Therefore the connected components of $Y_{K}$ are a subset of those of $S_{\tilde{K}}(\mathbb{C})$ and thus share the same irregularity.

Suppose that $\Gamma$ is neat. Using Theorem 2.1(iii) one can easily transform (6) into:

$$2q(Y_{T'}) = \sum_{(\lambda,\nu) \in \Xi} \sum_{\pi_0(Y_{K})} \dim(\pi^K_f)(1 + W(\lambda\nu_M)(-1)^s(\pi)), \quad (11)$$

where a (finite order) character $\chi$ of $\pi_0(Y_{K})$ sends $(\lambda,\nu) \in \Xi$ to $(\lambda\chi_{M^1}^{-1},\nu\chi) \in \Xi$. Note that this action preserves the root number $W(\lambda\nu_M)$.

**Proposition 2.6.** Any $\Gamma$ as in Theorem 0.3 is neat and $q(Y_{T'}) > 2$.

**Proof.** There exists a compact open subgroup $\tilde{K}$ of $\tilde{G}(A_{\mathbb{Q},f})$ such that $\Gamma$ equals $\tilde{G}(\mathbb{Q}) \cap (\tilde{G}(\mathbb{R}) \cdot \tilde{K})$ as well as $G(\mathbb{Q}) \cap (G(\mathbb{R}) \cdot K)$, where $K = \tilde{K} \cap G(A_{\mathbb{Q},f})$.

Let $\lambda$ be a unitary simplest Hecke character as in [Ya, p.88] of conductor:

$$\mathfrak{D} = \begin{cases} \sqrt{-D} & \text{if } D \neq 3 \text{ is odd}, \\ 2\sqrt{-D} & \text{if } 8 \text{ divides } D, \\ 3 & \text{if } D = 3, \\ \sqrt{-2D} & \text{otherwise}. \end{cases}$$

(12)

In the first two cases those are the canonical characters studied in Rohrlich [Roh].
By definition, \((\lambda, \lambda_{M1}^{-1}) \in \Xi\) and is trivial on \(K_1(\mathfrak{O}) \cap T(\mathbb{Q}_f, f)\). Lemma 2.2 implies that:

\[
(13) \quad \pi_f^{K_1(\mathfrak{O})} \neq 0, \text{ for all } \pi \in \Pi(\lambda, \lambda_{M1}^{-1}).
\]

In case (ii) we fix a prime \(p\) dividing \(D\) and \(\pi = \otimes_v \pi_v \in \Pi(\lambda, \lambda_{M1}^{-1})\) such that \(\pi_v = \pi_n(\lambda_v, \lambda_{M1}^{-1})\) for all \(v \neq p, N\), \(\pi_N = \pi_s(\lambda_N, \lambda_{M1}^{-1})\) and

\[
\pi_p = \begin{cases} 
\pi_n(\lambda_p, \lambda_{M1}^{-1}), & \text{if } W(\lambda^3) = -1, \\
\pi_s(\lambda_p, \lambda_{M1}^{-1}), & \text{if } W(\lambda^3) = 1.
\end{cases}
\]

Since \(\Gamma(N)\) is neat by Lemma 1.4, we can apply (11) which combined with Lemma 2.3 yields

\[
q(Y_{\Gamma(N) \cap \Gamma_1(\mathfrak{O})}) \geq \dim(\pi_N^{K(N)}) \geq N^3 \geq 3.
\]

We now turn to case (i) and suppose first that \(M\) has class number \(h \geq 3\). For any class character \(\xi\) one has \((\lambda\xi, \lambda_{M1}^{-1}) \in \Xi\) giving \(h\) pairwise distinct elements in \(\Xi/\pi_0(Y_{K_1(\mathfrak{O})})\). Fix a prime \(p | D\) and consider \(\pi = \otimes_v \pi_v \in \Pi(\lambda\xi, \lambda_{M1}^{-1})\) such that \(\pi_v = \pi_n(\lambda_v \xi_v, \lambda_{M1}^{-1})\) for all \(v \neq p\) and

\[
\pi_p = \begin{cases} 
\pi_n(\lambda_p \xi_p, \lambda_{M1}^{-1}), & \text{if } W(\lambda^3) = 1, \\
\pi_s(\lambda_p \xi_p, \lambda_{M1}^{-1}), & \text{if } W(\lambda^3) = -1.
\end{cases}
\]

Since \(\Gamma_1(\mathfrak{O})\) is neat by Lemma 1.6, one can apply (11) which combined with (13) yields

\[
q(Y_{\Gamma(\mathfrak{O})}) \geq h \geq 3.
\]

If \(M\) is one of the 18 imaginary quadratic fields of class number 2, then its fundamental discriminant \(D\) has (exactly) two distinct prime divisors \(p < q\). For each simplest character \(\lambda\) on \(M\), consider \(\pi \in \Pi(\lambda, \lambda_{M1}^{-1})\) such that \(\pi_v = \pi_n(\lambda_v, \lambda_{M1}^{-1})\) for all \(v \neq p, q\) and

\[
(\pi_p, \pi_q) = \begin{cases} 
(\pi_n(\lambda_p, \lambda_{M1}^{-1}), \pi_n(\lambda_q, \lambda_{M1}^{-1})) & \text{or } (\pi_s(\lambda_p, \lambda_{M1}^{-1}), \pi_s(\lambda_q, \lambda_{M1}^{-1})) , \text{ if } W(\lambda^3) = 1, \\
(\pi_n(\lambda_p, \lambda_{M1}^{-1}), \pi_s(\lambda_q, \lambda_{M1}^{-1})) & \text{or } (\pi_s(\lambda_p, \lambda_{M1}^{-1}), \pi_n(\lambda_q, \lambda_{M1}^{-1})) , \text{ if } W(\lambda^3) = -1.
\end{cases}
\]

If \(D \neq 24\) then \(\Gamma_1(\mathfrak{O})\) is neat by Lemma 1.6 and (11) implies that \(q(Y_{\Gamma_1(\mathfrak{O})}) \geq 2 \cdot 2 = 4\). If \(D = 24\) then \(\Gamma(\mathfrak{O})\) is neat by Lemma 1.4, since 4 divides \(\mathfrak{O}\), and again \(q(Y_{\Gamma(\mathfrak{O})}) \geq 4\).

Finally, we consider the nine imaginary quadratic fields of class number 1.

For \(D \in \{7, 11, 19, 43, 67, 163\}\) there is a unique simplest character \(\lambda\) (the canonical one). Any character of \((1 + \sqrt{-D}\mathfrak{O}_M/1 + \mathfrak{O}_M) \simeq (\mathbb{Z}/D)\) lifts to a finite order Hecke character \(\xi\) of \(M\) with trivial restriction to \(Q\), hence \((\lambda\xi, \lambda_{M1}^{-1}) \in \Xi\). Let \(\pi = \otimes_v \pi_v \in \Pi(\lambda\xi, \lambda_{M1}^{-1})\) be such that \(\pi_v = \pi_n(\lambda_v \xi_v, \lambda_{M1}^{-1})\) for all \(v \neq D\) and

\[
\pi_D = \begin{cases} 
\pi_n(\lambda_D \xi_D, \lambda_{M1}^{-1}), & \text{if } W(\lambda^3) = 1, \\
\pi_s(\lambda_D \xi_D, \lambda_{M1}^{-1}), & \text{if } W(\lambda^3) = -1.
\end{cases}
\]
Since $\Gamma(D)$ is neat by Lemma 1.4, by (11) we get $q(Y_{\Gamma(D)}) \geq D \cdot \dim(\pi_D^{K(D)}) \geq D$.

For $D = 3$ the same argument with $D^2$ instead of $D$, implies that $q(Y_{\Gamma(9)}) \geq 3$.

For $D = 4$ (resp. $D = 8$) the group $\Gamma(8)$ (resp. $\Gamma(2\sqrt{-8})$) is neat by Lemma 1.4 and it is an easy exercise in class field theory to show that there are at least 3 weight one Hecke characters on $M$ whose restriction to $\mathbb{Q}$ is the quadratic character attached to $M$, and whose conductor divides 8 (resp. $2\sqrt{-8}$). It follows then from (11) and (13) that for $D = 4$ (resp. $D = 8$) one has $q(Y_{\Gamma(8)}) \geq 3$ (resp. $q(Y_{\Gamma(2\sqrt{-8})}) \geq 3$).

Remark 2.7.

(i) The computation of the smallest level $K$ for which there exists an automorphic representation $\pi \in \Pi(\lambda, \nu)$ such that $\pi_f^K \neq 0$ is analyzed in detail in [DR]. In particular, if $\lambda$ is a canonical character, we check that the level subgroup at any $p \mid D$ is precisely the one conjectured by B. Gross, namely the index 2 subgroup of the maximal parahoric subgroup with reductive quotient $PGL(2)$.

(ii) One consequence of Rogawski’s theory is that the Albanese variety is of CM type for any congruence subgroup (see [MR]). If $M$ is imaginary quadratic and $D$ is prime, we will show in [DR] that the factor of the Albanese corresponding to the canonical character $\lambda$ turns out (at an appropriate prime to $D$ level) to be isogenous to the CM abelian variety $B(D)$ defined by B. Gross in [G].

(iii) When $\Gamma$ is not a congruence subgroup, there are examples of C. Schoen where the Albanese is not of CM type (see [Sc]).

3. Mordellicity

We will deduce our main theorems from a more general proposition which is a consequence of the following powerful result of Faltings on the rational points of subvarieties of abelian varieties.

Theorem 3.1 (Faltings [F2], [V]). Suppose $A$ is an abelian variety over a number field $k$, $Z \subset A$ a closed subvariety. Then there are finitely many translates $Z_i$ of $k$-rational abelian subvarieties of $A$, such that $Z_i \subset Z$, and such that each $k$-rational point of $Z$ lies on one of the $Z_i$.

Proposition 3.2. Let $X$ be a smooth projective variety over a number field $k$ which is geometrically irreducible and does not admit a dominant map to its Albanese variety. Then $X(k)$ is not Zariski dense in $X$.

Proof of Proposition 3.2. If $X(k)$ is empty, there is nothing to prove. Otherwise, use a $k$-rational point of $X$ to define the Albanese map over $k$:

$$j : X \to Alb(X).$$
Then $Z = j(X)$ is a closed, irreducible subvariety of $\text{Alb}(X)$. Applying Theorem 3.1 with $A = \text{Alb}(X)$, we get a finite number, say $m \geq 1$, of $k$-rational translates $Z_i$ of abelian subvarieties of $\text{Alb}(X)$ such that

$$Z(k) \subset \bigcup_{i=1}^{m} Z_i(k) \quad \text{and} \quad Z_i \subset Z.$$

Since the Albanese map is defined over $k$, all the $k$-rational points of $X$ are contained in those $j^{-1}(Z_i)$. Finally each $j^{-1}(Z_i)$ is a proper closed sub-scheme of $X$ (possibly singular and reducible), since otherwise, the irreducibility of $X$ would imply that $Z = Z_i = \text{Alb}(X)$, contradicting the assumption that $X$ does not admit a dominant map to its Albanese variety.

**Remark 3.3.** When Lang originally made his conjecture on Mordellicity, his definition of a variety $X$ over $k \subset \mathbb{C}$ being hyperbolic required the Kobayashi semi-distance on $X(\mathbb{C})$ to be in fact a metric. Later it was established by R. Brody [B] that in the compact case this was equivalent to requiring that there is no non-constant holomorphic map from $\mathbb{C}$ to $X(\mathbb{C})$. It is expected that every smooth projective irreducible variety $X$ of general type over $\mathbb{C}$ containing no curve of genus $\leq 1$ is hyperbolic, and this is known if $X$ is a surface not admitting a dominant map to its Albanese variety.

**Proof of Theorem 0.1.** By Proposition 2.4 there exists a finite index subgroup $\Gamma'$ of $\Gamma$ such that $q(Y_{\Gamma'}) > 2$, which can be assumed to be normal. It follows that $Y_{\Gamma'}$ cannot admit a dominant map to its Albanese variety. Moreover $Y_{\Gamma'}$ is a geometrically irreducible projective surface, hence by Proposition 3.2 $Y_{\Gamma'}(k)$ is not Zariski dense in $Y_{\Gamma'}$. If $Y_{\Gamma'}(k)$ is infinite, then $Y_{\Gamma'}$ contains an irreducible curve $C$ defined over $k$ and such that $C(k)$ infinite. Since $C(k)$ is Zariski dense in $C$, the curve $C$ is geometrically irreducible and its geometric genus is at most one by Faltings’ celebrated proof of Mordell’s conjecture [F1]. Taking a uniformization of $C$ yields a non-constant holomorphic map from $\mathbb{C}$ to $Y_{\Gamma'}$, which is impossible since by Lemma 1.2(i), $Y_{\Gamma'}(\mathbb{C})$ is a smooth compact hyperbolic manifold. Therefore $Y_{\Gamma'}$ is Mordellic.

By Lemma 1.2(ii), the natural morphism $f : Y_{\Gamma'} \rightarrow Y_{\Gamma}$ is finite, etale and defined over a number field $k$. Denote $S$ the finite set of places of $k$ where $f$ ramifies. Then, for any given number field $k' \supset k$,

$$f^{-1}(Y_{\Gamma}(k')) \subset \bigcup_{k''} Y_{\Gamma'}(k''),$$

where $k''$ runs over all extensions of $k'$ of degree at most the degree of $f$ which are unramified outside $S$. Since $Y_{\Gamma'}$ is Mordellic and there are only finitely many such extensions $k''$ (Hermite-Minkowski), it follows that $Y_{\Gamma}(k')$ is finite, hence $Y_{\Gamma}$ is Mordellic. □

**Proof of Theorem 0.2.** The Lang locus of a quasi-projective irreducible variety $Z$ is defined as the Zariski closure of the union, over all number fields $k$, of irreducible components of
positive dimension of the Zariski closure of \( Z(k) \). It is clear that \( Z \) is Mordellic if, and only if, its Lang locus is empty. The main theorem in [UY] states that, for \( \Gamma \) sufficiently small, the Lang locus of a Baily-Borel compactification of \( Y_\Gamma \) is either empty or full, which implies immediately that the same statement holds for \( Y_\Gamma \) itself.

For \( \Gamma \) neat, \( Y_\Gamma \) admits a smooth toroidal compactification \( X_\Gamma \) defined over a number field and by [Sh2, Theorem 8.1] one can assume by further shrinking \( \Gamma \) that \( q(X_\Gamma) > n \). By Proposition 3.2 the Lang locus of \( X_\Gamma \) is not all, hence the Lang locus of \( Y_\Gamma \) is empty.

\textbf{Proof of Theorem 0.3.} Let us first show that \( X_\Gamma \) is of general type, hence its canonical divisor \( K_{X_\Gamma} \) is big (see [N, Definition 1.1]). Note that just like irregularity, the Kodaira dimension cannot decrease when going to a finite covering. By Holzapfel [H2, Theorem 5.4.15] and Feustel [Feu] the surface \( X_{\Gamma(D)} \) is of general type for all \( D \notin \{3, 4, 7, 8, 11, 15, 19, 20, 23, 24, 31, 39, 47, 71\} \). By the main theorem of Džambić [Dž], the surface \( X_{\Gamma(D)} \) is of general type for all \( D \in \{11, 15, 19, 20, 23, 31, 39, 47, 71\} \) and a careful inspection of his proof (using the prime above 3) shows that this is also true when \( D = 24 \).

The remaining varieties \( (D \in \{3, 4, 7, 8\}) \) are of general type by [H1, Proposition 4.13].

If \( g = \sum_{i,j=1}^{2} g_{ij} dz_i d\bar{z}_j \) denotes the Bergman metric of \( H_C^2 \) viewed as the unit ball \( \{z = (z_1, z_2) \in \mathbb{C}^2, |z| < 1\} \), normalized by requiring that
\[
\text{Ric}(g) = \sum_{i,j=1}^{2} \frac{\partial^2 \log(g_{11}g_{22} - g_{21}g_{12})}{\partial z_i \partial \bar{z}_j} dz_i d\bar{z}_j = -g,
\]
then the holomorphic sectional curvature \( h \) is constant and equals \(-4/3\) (see [GKK, §3.3], where \( g_{ij} = \frac{3((1-|z|^2)^{4}h_{ij} + \bar{z}_i z_j)}{(1-|z|^2)^2} \)).

Since by Proposition 2.6 we have that \( \Gamma \) is neat and \( q(X_\Gamma) = q(Y_\Gamma) > 2 \), Proposition 3.2 implies that \( X_\Gamma(k) \) is not Zariski dense in \( X_\Gamma \). Then \( X_\Gamma(k) \) is contained, up to a finite set, in a union of geometrically irreducible curves \( C \) in \( X_\Gamma \) which, by Faltings’ proof of Mordell’s conjecture, can be assumed to be of geometric genus at most one. Now applying a result of Nadel [N, Theorem 2.1] with \( \gamma = 1 \) (so that \(-\gamma \geq h = -4/3\)), we see that the bigness of \( K_X \) implies that each \( C \) is contained in the compactifying divisor, which is a finite union of elliptic curves indexed by the cusps. It follows that \( Y_\Gamma(k) \) is finite and that \( X_\Gamma \) does not contain any rational curves, hence it is a minimal surface of general type. □

\textbf{References}


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