Lecture 11

Dinakar Ramakrishnan

(The odd numbered lectures are given by D. Ramakrishnan, and the even ones by R. Tanner.)

0.1 A quadratic approximation at $s = s_0$

This section is not delicate as the previous ones, since the $s$-derivatives of $v'p/ds$ are all well defined and easily calculated at $s_0$ (unlike at $s = 0$). Nevertheless, the formulae below are useful in the following section. As before, we will write $s'$ for $ds/dt$, $c' = dc/dt$, etc.

Lemma. For the point $s = s_0$, the following values hold:

(a) $s' = -k_2s_0e_0$, $c' = k_1s_0e_0$, and $e' = k_1s_0e_0$.

(b) $dv/ds = -k_3$.

(c) $d^2v/ds^2 = -k_3^2/k_1s_0e_0$.

Consequently, the quadratic Taylor approximation to $v$ near $s = s_0$ is given by

$$v = -k_3(s - s_0) - \frac{k_3^2}{2k_1s_0e_0}(s - s_0)^2 + O((s - s_0)^3).$$

Proof. (a): This follows directly from the basic differential equations by evaluation at $s_0$.

(b): We saw in the proof of Lemma 2.1 that

$$\frac{dc}{ds} = -1 - k_3 \frac{c}{s'}.$$

Since $v = k_3c$ and $c = c_0 - e$ is zero at $s_0$, we get $dv/ds = -k_3$.

(c): Differentiating relative to $t$,

$$\frac{d}{dt} \left( \frac{dc}{ds} \right) = -k_3 \frac{s'c' - s''c}{(s')^2}.$$
0.2 Approximations to $s_p$

Now that we have expansions for $v$ at 0 and at $s_0$, we can find a series of approximations $s_{p,n}$ to $s_p$, which will be good for small $s_0$, by equating the $n$-th order terms of the respective expansions.

**Proposition**

(a) $s_{p,1} = \frac{k_3 s_0}{m + k_3}$;

(b) $s_{p,2}$ satisfies a quadratic equation:

$$AX^2 + BX + C = 0,$$

with

$$A = \left( \frac{k_3^2}{2k_1 s_0 e_0} - \frac{(k_3 + m) k_1 m}{k_3^2 + k_3 + m k_2} \right),$$

$$B = \left( \frac{k_3}{2k_1 e_0} - m - k_3 \right),$$

and

$$C = \left( \frac{k_3^2}{2k_1 e_0} - k_3 s_0 \right) \left( \frac{k_3}{2k_1 s_0 e_0} - 1 \right).$$

Note that $s_{p,1}$ corresponds to the $s$-coordinate of the point obtained by intersecting the tangent lines to the $(v, s)$-curve at $s = 0$ and $s = s_0$. On the other hand, $s_{p,2}$ denotes the $s$-coordinate of the meeting of the quadratic approximations to the $(v, s)$-curve at 0 and $s_0$, which provides a better approximation.