1. The operator $\nabla$ and the gradient:

Recall that the gradient of a differentiable scalar field $\phi$ on an open set $D$ in $\mathbb{R}^n$ is given by the formula:

\[
\nabla \phi = \left( \frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \ldots, \frac{\partial \phi}{\partial x_n} \right).
\]

It is often convenient to define formally the differential operator in vector form as:

\[
\nabla = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \right).
\]

Then we may view the gradient of $\phi$, as the notation $\nabla \phi$ suggests, as the result of multiplying the vector $\nabla$ by the scalar field $\phi$. Note that the order of multiplication matters, i.e., $\frac{\partial \phi}{\partial x_j}$ is not $\phi \frac{\partial}{\partial x_j}$.

Let us now review a couple of facts about the gradient. For any $j \leq n$, $\frac{\partial \phi}{\partial x_j}$ is identically zero on $D$ iff $\phi(x_1, x_2, \ldots, x_n)$ is independent of $x_j$. Consequently,

\[
\nabla \phi = 0 \text{ on } D \iff \phi = \text{constant}.
\]

Moreover, for any scalar $c$, we have:

\[
\nabla \phi \text{ is normal to the level set } L_c(\phi).
\]

Thus $\nabla \phi$ gives the direction of steepest change of $\phi$.

2. Divergence

Let $F : \mathcal{D} \rightarrow \mathbb{R}^n$, $\mathcal{D} \subset \mathbb{R}^n$, be a differentiable vector field. (Note that both spaces are $n$-dimensional.) Let $F_1, F_2, \ldots, F_n$ be the component (scalar) fields of $f$. The divergence of $F$ is defined to be
DIV, GRAD, AND CURL

(5) \[ \text{div}(F) = \nabla \cdot F = \sum_{j=1}^{n} \frac{\partial F_j}{\partial x_j}. \]

This can be reexpressed symbolically in terms of the dot product as

(6) \[ \nabla \cdot F = \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right) \cdot (F_1, \ldots, F_n). \]

Note that div($F$) is a scalar field.

Given any $n \times n$ matrix $A = (a_{ij})$, its trace is defined to be:

\[ \text{tr}(A) = \sum_{i=1}^{n} a_{ii}. \]

Then it is easy to see that, if $DF$ denotes the Jacobian matrix of $F$, i.e., the $n \times n$-matrix $(\partial F_i/\partial x_j)$, $1 \leq i, j \leq n$, then

(7) \[ \nabla \cdot F = \text{tr}(DF). \]

Let $\varphi$ be a twice differentiable scalar field. Then its Laplacian is defined to be

(8) \[ \nabla^2 \varphi = \nabla \cdot (\nabla \varphi). \]

It follows from (1),(5),(6) that

(9) \[ \nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 \varphi}{\partial x_2^2} + \cdots + \frac{\partial^2 \varphi}{\partial x_n^2}. \]

One says that $\varphi$ is harmonic iff $\nabla^2 \varphi = 0$. Note that we can formally consider the dot product

(10) \[ \nabla \cdot \nabla = \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right) \cdot \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right) = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}. \]

Then we have

(11) \[ \nabla^2 \varphi = (\nabla \cdot \nabla) \varphi. \]
Examples of harmonic functions:

(i) \( \mathcal{D} = \mathbb{R}^2; \varphi(x, y) = e^x \cos y. \)

Then \( \frac{\partial \varphi}{\partial x} = e^x \cos y, \frac{\partial \varphi}{\partial y} = -e^x \sin y, \)

and \( \frac{\partial^2 \varphi}{\partial x^2} = e^x \cos y, \frac{\partial^2 \varphi}{\partial y^2} = -e^x \cos y. \) So, \( \nabla^2 \varphi = 0. \)

(ii) \( \mathcal{D} = \mathbb{R}^2 - \{0\}. \varphi(x, y) = \log(x^2 + y^2) = 2 \log(r). \)

Then \( \frac{\partial \varphi}{\partial x} = \frac{2x}{x^2 + y^2}, \frac{\partial \varphi}{\partial y} = \frac{2y}{x^2 + y^2}, \frac{\partial^2 \varphi}{\partial x^2} = \frac{2(2x^2 + y^2) - 2x(2x)}{(x^2 + y^2)^2} = -\frac{2(x^2 - y^2)}{(x^2 + y^2)^2}, \) and

\( \frac{\partial^2 \varphi}{\partial y^2} = \frac{2(x^2 + y^2) - 2y(2y)}{(x^2 + y^2)^2} = \frac{2(x^2 - y^2)}{(x^2 + y^2)^2}. \) So, \( \nabla^2 \varphi = 0. \)

(iii) \( \mathcal{D} = \mathbb{R}^n - \{0\}. \varphi(x_1, x_2, \ldots, x_n) = (x_1^2 + x_2^2 + \cdots + x_n^2)^{\alpha/2} = r^\alpha \) for some fixed \( \alpha \in \mathbb{R}. \)

Then \( \frac{\partial \varphi}{\partial x_i} = \alpha r^{\alpha - 1} x_i \frac{x_i}{r} = \alpha r^{\alpha - 2} x_i, \) and

\( \frac{\partial^2 \varphi}{\partial x_i^2} = \alpha(\alpha - 2) r^{\alpha - 4} x_i \cdot x_i + \alpha r^{\alpha - 2} \cdot 1. \)

Hence \( \nabla^2 \varphi = \sum_{i=1}^{n} (\alpha(\alpha - 2) r^{\alpha - 4} x_i^2 + \alpha r^{\alpha - 2}) = \alpha(\alpha - 2 + n) r^{\alpha - 2}. \)

So \( \phi \) is harmonic for \( \alpha = 0 \) or \( \alpha = 2 - n (\alpha = -1 \) for \( n = 3). \)

3. Cross product in \( \mathbb{R}^3 \)

The three-dimensional space is very special in that it admits a vector product, often called the cross product. Let \( \textbf{i}, \textbf{j}, \textbf{k} \) denote the standard basis of \( \mathbb{R}^3. \) Then, for all pairs of vectors \( v = x\textbf{i} + y\textbf{j} + z\textbf{k} \) and \( v' = x'\textbf{i} + y'\textbf{j} + z'\textbf{k}, \) the cross product is defined by

\[ (12) \ v \times v' = \text{det} \begin{pmatrix} \textbf{i} & \textbf{j} & \textbf{k} \\ x & y & z \\ x' & y' & z' \end{pmatrix} = (yz' - y'z)\textbf{i} - (xz' - x'z)\textbf{j} + (xy' - x'y)\textbf{k}. \]

Lemma 1. (a) \( v \times v' = -v' \times v \) (anti-commutativity)

(b) \( \textbf{i} \times \textbf{j} = \textbf{k}, \textbf{j} \times \textbf{k} = \textbf{i}, \textbf{k} \times \textbf{i} = \textbf{j} \)

(c) \( v \cdot (v \times v') = v' \cdot (v \times v') = 0. \)

Corollary: \( v \times v = 0. \)

Proof of Lemma (a) \( v' \times v \) is obtained by interchanging the second and third rows of the matrix whose determinant gives \( v \times v'. \) Thus \( v' \times v = -v \times v'. \)

(b) \( \textbf{i} \times \textbf{j} = \text{det} \begin{pmatrix} \textbf{i} \textbf{j} \textbf{k} \\ 1 \ 0 \ 0 \\ 0 \ 1 \ 0 \end{pmatrix}, \) which is \( \textbf{k} \) as asserted. The other two identities are similar.

(c) \( v \cdot (v \times v') = x(yz' - y'z) - y(xz' - x'z) + z(xy' - x'y) = 0. \) Similarly for \( v' \cdot (v \times v'). \)

Geometrically, \( v \times v' \) can, thanks to the Lemma, be interpreted as follows. Consider the plane \( P \) in \( \mathbb{R}^3 \) defined by \( v, v'. \) Then \( v \times v' \) will lie along the normal to this plane at the origin, and its orientation is
given as follows. Imagine a corkscrew perpendicular to \( P \) with its tip at the origin, such that it turns clockwise when we rotate the line \( Ov \) towards \( Ov' \) in the plane \( P \). Then \( v \times v' \) will point in the direction in which the corkscrew moves perpendicular to \( P \).

Finally the length \( ||v \times v'|| \) is equal to the area of the parallelogram spanned by \( v \) and \( v' \). Indeed this area is equal to the volume of the parallelepiped spanned by \( v \), \( v' \) and a unit vector \( u = (u_x, u_y, u_z) \) orthonormal to \( v \) and \( v' \). We can take \( u = v \times v'/||v \times v'|| \) and the (signed) volume equals

\[
\det \begin{pmatrix} u_x & u_y & u_z \\ x & y & z \\ x' & y' & z' \end{pmatrix} = u_x(yz' - y'z) - u_y(xz' - x'z) + u_z(xy' - x'y) \\
= ||v \times v'|| \cdot (u_x^2 + u_y^2 + u_z^2) = ||v \times v'||.
\]

4. Curl of vector fields in \( \mathbb{R}^3 \)

Let \( F : \mathcal{D} \to \mathbb{R}^3 \), \( \mathcal{D} \subset \mathbb{R}^3 \) be a differentiable vector field. Denote by \( P, Q, R \) its coordinate scalar fields, so that \( F = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k} \). Then the curl of \( F \) is defined to be:

\[
\text{curl}(F) = \nabla \times F = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{pmatrix}.
\]

Note that it makes sense to denote it \( \nabla \times F \), as it is formally the cross product of \( \nabla \) with \( f \). Explicitly we have

\[
\nabla \times F = (\partial R/\partial y - \partial Q/\partial z) \mathbf{i} - (\partial R/\partial x - \partial P/\partial z) \mathbf{j} + (\partial Q/\partial x - \partial P/\partial y) \mathbf{k}
\]

If the vector field \( F \) represents the flow of a fluid, then the curl measures how the flow rotates the vectors, whence its name.

**Proposition 1.** Let \( h \) (resp. \( F \)) be a \( \mathcal{C}^2 \) scalar (resp. vector) field. Then

(a): \( \nabla \times (\nabla h) = 0 \).

(b): \( \nabla \cdot (\nabla \times F) = 0 \).

**Proof:** (a) By definition of gradient and curl,

\[
\nabla \times (\nabla h) = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{pmatrix} = \left( \frac{\partial^2 h}{\partial y \partial z} - \frac{\partial^2 h}{\partial z \partial y} \right) \mathbf{i} + \left( \frac{\partial^2 h}{\partial z \partial x} - \frac{\partial^2 h}{\partial x \partial z} \right) \mathbf{j} + \left( \frac{\partial^2 h}{\partial x \partial y} - \frac{\partial^2 h}{\partial y \partial x} \right) \mathbf{k}.
\]
Since \( h \) is \( C^2 \), its second mixed partial derivatives are independent of the order in which the partial derivatives are computed. Thus, \( \nabla \times (\nabla fh) = 0 \).

(b) By the definition of divergence and curl,
\[
\nabla \cdot (\nabla \times F) = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left( \frac{\partial^2 R}{\partial y \partial x} - \frac{\partial^2 Q}{\partial x \partial z}, -\frac{\partial^2 R}{\partial y \partial x} + \frac{\partial^2 P}{\partial y \partial z}, \frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 P}{\partial z \partial y} \right).
\]
Again, since \( F \) is \( C^2 \), \( \frac{\partial^2 R}{\partial x \partial y} = \frac{\partial^2 R}{\partial y \partial x} \), etc., and we get the assertion. \( \square \)

**Warning:** There exist twice differentiable scalar (resp. vector) fields \( h \) (resp. \( F \)), which are not \( C^2 \), for which (a) (resp. (b)) does not hold.

When the vector field \( F \) represents fluid flow, it is often called **irrotational** when its curl is 0. If this flow describes the movement of water in a stream, for example, to be irrotational means that a small boat being pulled by the flow will not rotate about its axis. We will see later that the condition \( \nabla \times F = 0 \) occurs naturally in a purely mathematical setting as well.

**Examples:** (i) Let \( D = \mathbb{R}^3 - \{0\} \) and \( F(x, y, z) = \frac{y}{x^2 + y^2} \mathbf{i} - \frac{x}{x^2 + y^2} \mathbf{j} \).
Show that \( F \) is irrotational. Indeed, by the definition of curl,
\[
\nabla \times F = \text{det} \begin{pmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
\frac{x}{x^2 + y^2} & \frac{y}{x^2 + y^2} & 0
\end{pmatrix}
= \frac{\partial}{\partial z} \left( \frac{x}{x^2 + y^2} \right) \mathbf{i} + \frac{\partial}{\partial z} \left( \frac{y}{x^2 + y^2} \right) \mathbf{j} + \left( \frac{\partial}{\partial x} \left( \frac{-x}{x^2 + y^2} \right) - \frac{\partial}{\partial y} \left( \frac{y}{x^2 + y^2} \right) \right) \mathbf{k}
= \left[ \frac{-(x^2 + y^2) + 2x^2}{(x^2 + y^2)^2} - \frac{(x^2 + y^2) - 2y^2}{(x^2 + y^2)^2} \right] \mathbf{k} = 0.
\]

(ii) Let \( m \) be any integer \( \neq 3 \), \( D = \mathbb{R}^3 - \{0\} \), and \( F(x, y, z) = \frac{1}{r^m}(x \mathbf{i} + y \mathbf{j} + z \mathbf{k}) \), where \( r = \sqrt{x^2 + y^2 + z^2} \). Show that \( F \) is not the curl of another vector field. Indeed, suppose \( F = \nabla \times G \).
Then, since \( F \) is \( C^1 \), \( G \) will be \( C^2 \), and by the Proposition proved above, \( \nabla \cdot (\nabla \times G) \) would be zero. But,
\[
\nabla \cdot F = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left( \frac{x}{r^{m+1}}, \frac{y}{r^{m+1}}, \frac{z}{r^{m+1}} \right)
\]
\[ r^m - 2x^2(\frac{m}{2})r^{m-2} \] \[ + \frac{r^m - 2y^2(\frac{m}{2})r^{m-2}}{r^{2m}} \] \[ + \frac{r^m - 2z^2(\frac{m}{2})r^{m-2}}{r^{2m}} \] \[ = \frac{1}{r^{2m}} (3r^m - m(x^2 + y^2 + z^2)r^{m-2}) = \frac{1}{r^m} (3 - m). \]

This is non-zero as \( m \neq 3 \). So \( F \) is not a curl.

**Warning:** It may be true that the divergence of \( F \) is zero, but \( F \) is still not a curl. In fact this happens in example (ii) above if we allow \( m = 3 \). We cannot treat this case, however, without establishing Stoke’s theorem.

5. **An interpretation of Green’s theorem via the curl**

Recall that Green’s theorem for a plane region \( \Phi \) with boundary a piecewise \( C^1 \) Jordan curve \( C \) says that, given any \( C^1 \) vector field \( G = (P, Q) \) on an open set \( D \) containing \( \Phi \), we have:

\[ \int \int_{\Phi} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy = \oint_{C} P \, dx + Q \, dy. \]  

We will now interpret the term \( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \). To do that, we think of the plane as sitting in \( \mathbb{R}^3 \) as \( \{ z = 0 \} \), and define a \( C^1 \) vector field \( F \) on \( \tilde{D} := \{ (x, y, z) \in \mathbb{R}^3 | (x, y) \in D \} \) by setting \( F(x, y, z) = G(x, y) = Pi + Qj \). We can interpret this as taking values in \( \mathbb{R}^3 \) by thinking of its value as \( Pi + Qj + 0k \). Then \( \nabla \times F = det \left( \begin{array}{ccc} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ P & Q & 0 \end{array} \right) = \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) k \), because \( \frac{\partial P}{\partial z} = \frac{\partial Q}{\partial z} = 0 \). Thus we get:

\[ (\nabla \times F) \cdot k = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}. \]

And Green’s theorem becomes:

**Theorem 1.** \( \int \int_{\Phi} (\nabla \times F) \cdot k \, dx \, dy = \oint_{C} P \, dx + Q \, dy \)

6. **A criterion for being conservative via the curl**

A consequence of the reformulation above of Green’s theorem using the curl is the following:

**Proposition 1.** Let \( G : D \to \mathbb{R}^2 \), \( D \subset \mathbb{R}^2 \) open and simply connected, \( G = (P, Q) \), be a \( C^1 \) vector field. Set \( F(x, y, z) = G(x, y) \), for all \( (x, y, z) \in \mathbb{R}^3 \) with \( (x, y) \in D \). Suppose \( \nabla \times F = 0 \). Then \( G \) is conservative on \( D \).
**Proof:** Since $\nabla \times F = 0$, the reformulation in section 5 of Green’s theorem implies that $\oint_C P \, dx + Q \, dy = 0$ for all Jordan curves $C$ contained in $\mathcal{D}$. QED

**Example:** $\mathcal{D} = \mathbb{R}^2 - \{(x,0) \in \mathbb{R}^2 \mid x \leq 0\}$, $G(x,y) = \frac{y}{x^2+y^2} \mathbf{i} - \frac{x}{x^2+y^2} \mathbf{j}$. Determine if $G$ is conservative on $\mathcal{D}$:

Again, define $F(x,y,z)$ to be $G(x,y)$ for all $(x,y,z)$ in $\mathbb{R}^3$ such that $(x,y) \in \mathcal{D}$. Since $G$ is evidently $\mathcal{C}^1$, $F$ will be $\mathcal{C}^1$ as well. By the Proposition above, it will suffice to check if $F$ is irrotational, i.e., $\nabla \times F = 0$, on $\mathcal{D} \times \mathbb{R}$. This was already shown in Example (i) of section 4 of this chapter. So $G$ is conservative.