1. Let $F$ be a local field (assume that $F = \mathbb{R}$ if you wish), and $B$ the group of invertible $2 \times 2$ upper-triangular matrices with determinant 1 over the field $F$. The group $B$ is a locally compact group. Calculate the modulus function $\Delta_B : B \rightarrow \mathbb{R}_{>0}$.

**Answer:** The answer is $\Delta_B(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}) = |a|^2$.

2. For $n \in \mathbb{Z}_{\geq 0}$, let $V_n$ be an irreducible $(n + 1)$-dimensional representation of $SU(2)$.

(a) Show that $V_2$ is isomorphic to the complexified adjoint representation of $SU(2)$.

(b) Show that there exists a unique, up to scalar, non-zero bilinear $SU(2)$-invariant form $B$ on $V_n$. In addition, show that this form is non-degenerate.

(c) Show that the form $B$ is either symmetric, or anti-symmetric.

(d) Show that if $n$ is even, $B$ is symmetric.

(e) *(not for handing in)* Show that if $n$ is odd, $B$ is anti-symmetric.

**Solution:**

(a) All we have to show is that the complexified adjoint representation of $SU(2)$ is irreducible. This can be done by a concrete calculation.

(b) $G$-invariant bilinear forms are identified with elements of $\text{Hom}_G(V_n, V_n^*)$. This space is one-dimensional by Schur’s lemma, since $V_n, V_n^*$ are irreducible of the same dimension, hence isomorphic by our classification of irreducibles of $G$. Moreover, a non-zero element in this $\text{Hom}_G$-space is an isomorphism by Schur’s lemma. This translates to the corresponding bilinear form being non-degenerate.

(c) We assume that $B$ is non-zero of course. We can write $B = C + D$ where $C$ is the symmetrization of $B$ and $D$ is the anti-symmetrization of $B$. Since by the previous item $C, D$ are proportional to $D$, and since a
symmetric and anti-symmetric form is zero, we get that either $C$ or $D$ is zero, so $B = C$ or $B = D$, meaning $B$ is symmetric or $B$ is anti-symmetric.

(d) There are no non-degenerate anti-symmetric bilinear forms on even-dimensional vector spaces.

3. Show that the exponential map $\exp$ for the Lie group $SL_2(\mathbb{R})$ is not surjective.

**Solution:** Recall that the Lie algebra of $SL_2(\mathbb{R})$ is the subalgebra of $M_2(\mathbb{R})$ consisting of matrices with trace zero. If a such a matrix $A$ is triangulizable over $\mathbb{R}$, then $\exp(A)$ has positive eigenvalues. If $A$ is not triangulizable, then it’s eigenvalues are $ib, -ib$ for some $0 \neq b \in \mathbb{R}$, so $A$ is diagonalizable over $\mathbb{C}$, hence $\exp(A)$ is diagonalizable over $\mathbb{C}$. So, considering the matrix $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \in SL_2(\mathbb{R})$, which is neither with positive eigenvalues, nor is diagonalizable over $\mathbb{C}$, we see that it is not in the image of $\exp$.

4. For $n \in \mathbb{Z}_{\geq 0}$, Let $V_n$ be an irreducible $(n+1)$-dimensional representation of $SU(2)$. Show that

$$V_n \otimes V_m \cong \bigoplus_{j=0}^{\min(n,m)} V_{n+m-2j}$$

(as representations of $SU(2)$).

**Solution:** We can work with the characters restricted to $T$. Denote by $\zeta$ a character generating the character lattice $X^*(T)$. Then $\chi_n$, the character of $V_n$ restricted to $T$, is equal to:

$$\chi_n = \zeta^n + \zeta^{n-2} + \ldots + \zeta^{-n} = \sum_{0 \leq i \leq n} \zeta^{n-2i}.$$ 

Let us say $m \leq n$. Then

$$\chi_n \chi_m = \sum_{0 \leq i \leq n} \zeta^{n-2i} \sum_{0 \leq j \leq m} \zeta^{m-2j} = \sum_{0 \leq i \leq n, 0 \leq j \leq m} \zeta^{n+m-2(i+j)} = \ldots$$

one proceeds combinatorially (imagining a grid, and ordering the points in it in a specific fashion); see for example the book by Brocker and Dieck, chapter II, section 5, proposition (5.5) (page 87).