Representation theory of compact groups -
Homework 04

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March 8, 2017

Write solutions to all of the following problems. Due date: 03/17/17.
Please ask me if you are not sure about the terminology or anything else in
the problems.

1. Let $G = SU(n)$ and $T \subset G$ the subgroup of diagonal matrices.

   (a) Give a concrete parametrization of the set of roots $R \subset \mathfrak{t}^*$, a concrete
description of the Weyl group $W$, and a concrete description of the
action of $W$ on $R$.

   (b) Describe a choice of positive roots $R^+ \subset R$, and show what is the
corresponding set of simple roots $S \subset R^+$.

   (c) Describe concretely, in terms of your concrete description of $W$, what
is the element $w_0 \in W$ which satisfies $\ell(w_0) = |R^+|$ (the longest
element).

2. Let $G = SU(n)$ and $T \subset G$ the subgroup of diagonal matrices. Construct
a bijection between $G/T$ and the set of flags in $\mathbb{C}^n$, i.e. $(n+1)$-tuples
of subvectorspaces of $\mathbb{C}^n$, $(V_0, V_1, \ldots, V_n)$, such that $V_i \subset V_{i+1}$ for all
$0 \leq i \leq n - 1$ and $\dim V_i = i$ for all $0 \leq i \leq n$ (you should prove that the
map you constructed is indeed a bijection).

3. Let $G$ be a connected compact Lie group. Suppose that $G$ is not abelian.

   (a) Show that $G$ has irreducible representations of arbitrarily high dimen-
sion.

   (b) Show that if $G$ is semisimple (i.e. the center $Z(G)$ is finite), then $G$
has only finitely many non-isomorphic irreducible representations of
any given dimension.

   (c) Show that if $G$ is not semisimple, then $G$ has infinitely many non-
isomorphic irreducible representations of dimension 1.