Math 120b Galois Theory
Homework Set 3
Due Friday 2/24/17 at 5pm

In the problems below, an extension of $G$ by $A$ is a group fitting in a short exact sequence

$$1 \to A \to H \to G \to 1.$$ 

Two such extensions are isomorphic if there is a map of short exact sequences agreeing with the identity on $A$ and $G$.

1. (a) Show that for any finite field, $H^l(F_q; G_m) = 0$ for $l > 0$. (b) Conclude that any finite division ring is a field. (Here you may assume that every division algebra is a central simple algebra as we have defined it.)

2. Here we show that extensions of a group $G$ by an abelian group $A$ with fixed (outer) action are classified by $H^2(G; A)$, and construct the connecting map.

   (a) There is a natural bijection between $Z^2(G; A)$ and extensions equipped with a set map $G \to H$ taking $e$ to $e$.

   (b) Changing the set map changes the cocycle by an arbitrary coboundary, and thus two extensions are isomorphic iff the cohomology classes agree.

   (c) Although extensions are classified by $H^2(G; A)$ up to isomorphism, they are not in general classified up to unique isomorphism. Show that any element of $Z^1(G; A)$ induces an automorphism of $H$ as an extension of $G$ by $A$.

   (d) Given two extensions $H$, $H'$, there is an extension of $G$ by $A \times A$ given as the group of pairs $(h, h')$ mapping to the diagonal in $G$. If we quotient by the diagonal, we get a new extension of $G$ by $A$. How does this operation act on $H^2(G; A)$?

   (e) Let $B$ be a group with a $G$-action such that $A \subset Z(B)$ (as $G$-groups), and consider subgroups $H \subset B \rtimes G$ such that $H \cap B = A$ and the induced map from $H \to G$ is surjective. Show that the set of such subgroups is naturally bijective with $Z^1(G; B/A)$, and show that the induced map $Z^1(G; B/A) \to H^2(G; A)$ (viewing $H$ as an extension of $G$ by $A$) is precisely the connecting map.

3. Suppose $A$ is an abelian group and $G$ a group of order prime to $|A|$. Show that any extension of $G$ by $A$ splits.