1. Let $k$ be a field of characteristic $p$, and let $t, u$ be algebraically independent over $k$. Show that

(a) $k(t, u)$ has degree $p^2$ over $k(t^p, u^p)$.

(b) If $v \in k(t, u) \setminus k(t^p, u^p)$, then $k(t^p, u^p, v)$ has degree $p$ over $k(t^p, u^p)$

Conclude that the finite extension $k(t, u)/k(t^p, u^p)$ cannot be generated by a single element, and that there are infinitely many intermediate extensions.

2. Show if $k$ is a finite field of order $p^n$, then every element of $k$ has a unique $p$-th root.

3. Let $L \cong \mathbb{Q}[\alpha]/(\alpha^4 + 20)$, and let $K \subset L$ be the subfield generated by $\alpha^2$. Show that $L/K$ and $K/\mathbb{Q}$ are normal, but $L/\mathbb{Q}$ is not.

4. Let $E = F(x)$ where $x$ is transcendental over $F$.

(a) Let $K$ be a subfield of $E$ strictly containing $F$. Show that $E/K$ is an algebraic extension.

(b) Let $\frac{f(x)}{g(x)} \in E \setminus F$ with $\gcd(f, g) = 1$. Show that $E$ is algebraic over $F(\frac{f(x)}{g(x)})$ of degree $\max(\deg(f), \deg(g))$.

5. Let $f \in k[x]$ be an irreducible polynomial of prime degree $p$ such that $K \cong k[x]/f(x)$ is a separable extension. Show that if $f$ has more than one root in $K$, then $f$ splits and $K/k$ is Galois.

6. Let $k$ be a field of characteristic not 2, and suppose $E/F/k$ is a tower of extensions given by $F = k(\sqrt{c})$, $E = F(\sqrt{a + b\sqrt{c}})$ for $a, b, c \in k$. Show that the following are equivalent:

- $E/k$ is Galois
- There is an element $\beta \in E$ such that $\beta^2 = a - b\sqrt{c}$.
- $a^2 - cb^2 \in k^2$ or $c(a^2 - cb^2) \in k^2$.

In the two cases corresponding to the last condition, compute the Galois group of $E/k$. If $E/k$ is not Galois, what is the Galois group of its Galois closure?