HOMEWORK QUESTIONS FOR MA 110B

Homeowrk 1 (due by Friday Feb 3)

1. Suppose \( f \) is an entire function, and
\[
|f(z)| \leq A + B|z|^k,
\]
for every \( z \in \mathbb{C} \), where \( A, B \) and \( k \) are positive numbers. Prove that \( f \) is a polynomial.

2. Suppose \( f_n \) is a uniformly bounded sequence of holomorphic functions in a domain \( \Omega \) such that \( f_n(z) \) converges for every fixed \( z \in \Omega \). Prove that the convergence is uniform on every compact subset of \( \Omega \).

3. Let \( \gamma \) denote the positively oriented unit circle. Compute
\[
\frac{1}{2\pi i} \int_{\gamma} \frac{e^z - e^{-z}}{z^4} \, dz.
\]

4. Suppose \( f_n \) are holomorphic functions on a domain \( \Omega \), and that none of the functions \( f_n \) has a zero in \( \Omega \). If \( f_n \) converges to a function \( f \) uniformly on compact sets in \( \Omega \), prove that either \( f \) has no zeroes or \( f(z) = 0 \) for every \( z \in \Omega \).

5. Suppose a domain \( \Omega \) contains the unit disc. Let \( f \) be holomorphic on \( \Omega \) and \( |f(z)| < 1 \) when \( |z| = 1 \). How many fixed points must \( f \) have in the unit disc (that is, how many solutions \( f(z) = z \) there are)?

6. Suppose \( f \) is holomorphic on \( \Omega \) and that \( \Omega \) contains the unit disc. Assume in addition that \( |f(z)| > 2 \) when \( |z| = 1 \) and \( f(0) = 1 \). Does \( f \) need to have a zero in the unit disc.

7. Denote by \( \mathbb{H} \) the upper half plane. Let \( f : \mathbb{H} \to \mathbb{C} \) be holomorphic and \( |f(z)| \leq 1 \) for \( z \in \mathbb{H} \). How large can \( |f'(i)| \) be. Find the extremal functions (namely those where \( |f'(i)| \) is maximized).
8. Suppose $\Omega$ is a bounded domain and $f_n$ a sequence of holomorphic functions defined on some neighborhood of $\overline{\Omega}$. Prove that if $f_n$ converges uniformly on the boundary of $\Omega$, then $f_n$ converges uniformly on $\overline{\Omega}$.

Homeowork 2 (due by Friday Feb 17)

1. Prove that if $f$ is holomorphic and has no zeroes on a domain $\Omega$, then the function $u(z) = \log |f(z)|$ is harmonic on $\Omega$.

2. Suppose that $u(z)$ is a positive harmonic function on the unit disc $\mathbb{D}$ and $u(0) = 1$. How small can $u\left(\frac{1}{2}\right)$ be? How large can $u\left(\frac{1}{2}\right)$ be?

3. Let $\Omega = \{z \in \mathbb{C} : \text{Re}(z) > 0\}$. Let $f$ be holomorphic on $\Omega$ and continuous on the closure $\overline{\Omega}$. Suppose that there are constants $A > 0$ and $0 < \alpha < 1$ such that

$$|f(z)| \leq Ae^{|z|^\alpha},$$

for every $z \in \Omega$, and $|f(iy)| \leq 1$ for every $y \in \mathbb{R}$. Show $|f(z)| \leq 1$ for every $z \in \Omega$.

4. Let $f$ be holomorphic on the unit disc $\mathbb{D}$. Find a sequence $z_n \in \mathbb{D}$ such that

$$\lim_{n \to \infty} |z_n| = 1,$$

and

$$|f(z_n)| < M,$$

for every $n \in \mathbb{N}$ and some $M > 0$.

5. Is there a sequence of polynomials $P_n(z)$ such that

$$\lim_{n \to \infty} P_n(z) = \psi(z), \quad z \in \mathbb{C},$$

where $\psi(z) = 1$ for $\text{Re}(z) > 0$, $\psi(z) = -1$ for $\text{Re}(z) < 0$, and $\psi(z) = 0$ for $\text{Re}(z) = 0$. 

6. Let $P_n(z)$ be a sequence of polynomials such that
\[
\lim_{n \to \infty} P_n(z) = \psi(z), \quad \text{for every } z \in \mathbb{C},
\]
for some function $\psi: \mathbb{C} \to \mathbb{C}$. Show that $\psi$ is holomorphic on an open and dense subset of $\mathbb{C}$.

7. Let $z_1, z_2 \in \mathbb{D}$ and set $\Omega = \mathbb{D} \setminus \{z_1, z_2\}$. Describe the group $\text{Aut}(\Omega)$.

8. Let $f: \mathbb{D} \to \mathbb{D}$ be a Möbius transformation. Let
\[
f_n(z) = (f \circ \cdots \circ f)(z), \quad (n \text{ times}).
\]
Does the limit
\[
\lim_{n \to \infty} f_n(z),
\]
exist for every $z \in \mathbb{D}$? Characterize all such $f$ so that the above limit exists for every $z \in \mathbb{D}$.

**Homework 3 (due by Friday March 3)**

1. If $f, g \in \text{PSL}(2, \mathbb{C})$ are different than the identity, then $f$ is conjugate to $g$ in $\text{PSL}(2, \mathbb{C})$ if and only if $\text{trace}(f) = \text{trace}(g)$.

2. [This questions carries double credit] Suppose $R$ is a rational function such that $|R(z)| = 1$ when $|z| = 1$. Prove that
\[
R(z) = C z^m \prod \frac{z - a_n}{1 - \overline{a}_n z},
\]
where $c$ is a constant, $m$ an integer, and $a_n$ are complex numbers different than zero and $|a_n| \neq 1$. Obtain an analogous description of those rational functions which are positive on the unit circle $\partial \mathbb{D}$.

3. Let $\Omega \subset \mathbb{C}$ be a region whose boundary consists of two non-intersecting circles (not necessarily concentric). Prove that there exists a conformal map $f: \Omega \to A(R)$, for some $A(R) = \{1 < |z| < R\}$, $R > 0$.

4. Suppose $\Omega$ is a convex region, $f: \Omega \to \mathbb{C}$ holomorphic, and $\text{Re}[f'(z)] > 0$, for all $z \in \Omega$. Prove that $f$ is injective.
5. Suppose $f_n, f: \Omega \to \mathbb{C}$ are holomorphic and $f_n \to f$ uniformly on compact sets in $\Omega$ and that $f$ is injective. Does it follow that for each compact set $K \subset \Omega$ there exists an integer $N(K) > 0$ such that the mappings $f_n$ are injective on $K$ for all $n > N(K)$?

6. Suppose $\Omega = \{x + iy : -1 < y < 1\}$ and $f: \Omega \to \mathbb{D}$ is holomorphic such that $f(x) \to 0$ when $x \to \infty$. Prove that
\[
\lim_{x \to \infty} f(x + iy) = 0,
\]
for every $y \in (-1, 1)$.

7. Recall that $f \in S$ if $f: \mathbb{D} \to \mathbb{C}$, 
\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\]
and $f$ is injective on $\mathbb{D}$. Find all $f \in S$ so that $\mathbb{D} \subset f(\mathbb{D})$. Also find all $f \in S$ such that $|a_2| = 2$.

Homeowork 4 (due by Friday March 17)

1. For $z \in \overline{\mathbb{D}}$ and $z^2 \neq 1$, we let 
\[
f(z) = e^{\frac{i\pi z}{1-z}},
\]
by choosing a branch of log that has log 1 = 0. Describe the set $f(E)$ if $E$ is one of the following: $E = \mathbb{D}$; $E$ is the upper half of the circle $\{|z| = 1\}$; $E$ is the lower half of the circle $\{|z| = 1\}$; $E$ is any disc $\{z : |z - r| < 1 - r\}$, for $0 < r < 1$.

2. Let $\Omega \subset \mathbb{C}$ be a simply connected region and $u: \Omega \to \mathbb{R}$ a harmonic function. Prove that there exists a holomorphic function $f: \Omega \to \mathbb{C}$ such that the real part of $f$ is equal to $u$. Show that this fails in every region that is not simply connected.

3. Prove that the Möbius transformations $z \to z + 1$ and $z \to -\frac{1}{z}$, generate the group $\text{PLS}(2, \mathbb{Z})$.

4. (This question carries double credit) Let $E \subset \mathbb{R}$ be a compact set of positive 1-dimensional Lebesgue measure and $\Omega = \mathbb{C} \setminus E$. Set 
\[
f(z) = \int_{E} \frac{dt}{t - z}
\]
for $z \in \Omega$. Answer the following questions:
(1) Is \( f \) constant?
(2) Can \( f \) be extended to an entire function?
(3) Does
\[
\lim_{z \to \infty} zf(z)
\]
exists. If so, what is it?
(4) What is
\[
\int_{\gamma} f(z) \, dz
\]
if \( \gamma \subset \Omega \) is positively oriented circle which has \( E \) in its interior?
(5) Does there exists a bounded holomorphic function on \( \Omega \) which is non-constant?

5. We say that an entire function \( f \) is of finite order if the inequality
\[
|f(z)| \leq e^{\lambda |z|},
\]
holds for all \( z \in \mathbb{C} \) when \( |z| \) is large enough. Let \( z \in A \) if
\[
e^{e^z} = 1.
\]
Show that no (non-constant) entire function that has zeroes at all points in \( A \) can be of finite order.

6. Suppose \( f \) is an entire function, \( f(0) \neq 0 \), and
\[
|f(z)| \leq e^{p |z|},
\]
when \( |z| \) is large enough. Denote by \( z_n \) the zeroes of \( f \) (counted according to their multiplicities). Prove that the series
\[
\sum_{n=1}^{\infty} |z_n|^{-p-\epsilon}
\]
converges for every \( \epsilon > 0 \).

7. Set \( \alpha_n = 1 - n^{-2} \), \( n \in \mathbb{N} \), and let \( B(z) \) denote the Blaschke product with zeroes at the points \( \alpha_n \). Prove
\[
\lim_{r \to 1} B(r) = 0,
\]
where \( 0 < r < 1 \).